Can. J. Math., Vol. XXXIX, No. 2, 1987 pp. 345-364

# THE $L$-THEORY OF TWISTED QUADRATIC EXTENSIONS

ANDREW RANICKI

**Introduction.** For surgery on codimension 1 submanifolds with non-trivial normal bundle the theory of Wall [**13**, Section 12C] has obstruction groups $LN_*(\pi' \to \pi)$, with $\pi$ a group and $\pi'$ a subgroup of index 2, such that there is defined an exact sequence involving the ordinary $L$-groups of rings with involution

$$\ldots \to LN_n(\pi' \to \pi) \to L_n(\mathbf{Z}[\pi]) \to L_{n+1}(\mathbf{Z}[\pi'] \to \mathbf{Z}[\pi]^w)$$

$$\to LN_{n-1}(\pi' \to \pi) \to \ldots$$

with the superscript $w$ signifying a different involution on $\mathbf{Z}[\pi]$. Geometry was used in [**13**] to identify

$$LN_n(\pi' \to \pi) = L_n(\mathbf{Z}[\pi'], \alpha, u),$$

with $(\alpha, u)$ an antistructure on $\mathbf{Z}[\pi']$ in the sense of Wall [**14**]. The main result of this paper is a purely algebraic version of this identification, for any twisted quadratic extension of a ring with antistructure.

The geometric applications of the $LN$-theory generalize to the non-simply-connected case the work of Browder and Livesay [**1**] and Lopez de Medrano [**9**] on free involutions on simply-connected manifolds. Ranicki [**12**, Section 7.6] contains a general account of these applications. The $LN$-groups have been used by Cappell and Shaneson [**2**], [**3**], Hambleton [**4**], Harsiladze [**6**], [**7**] and Hambleton, Taylor and Williams [**5**] for computations of the $L$-groups of finite groups, and for the detection of the closed manifold surgery obstructions.

On the purely algebraic side $LN$-theory is related to the work of Lewis [**8**] and Warshauer [**15**] on the $L$-theory of quadratic extensions of fields, as detailed in [**5**, Section 1]. Indeed, this paper was originally intended to serve as Appendix 4 to reference [*H-T-W*] of [**5**]. Accordingly, it uses the same terminology, with right modules and antistructures as first defined by Wall [**14**], rather than left modules and antistructures as in [**11**], [**12**].

The quadratic $L$-groups $L_*(R, \alpha, u)$ of a ring $R$ with antistructure $(\alpha, u)$ are defined in Section 1 using $(\alpha, u)$-quadratic Poincaré complexes over $R$, in the style of Ranicki [**11**].

Received June 10, 1985.

The brief Section 2 deals with scaling isomorphisms in the $L$-groups.

Given a ring $R$, a unit $a \in R$ and an automorphism $\rho: R \to R$ such that $\rho(a) = a$ and

$$\rho^2(x) = axa^{-1} \in R \quad (x \in R)$$

let $S = R_\rho[\sqrt{a}]$ be the $\rho$-twisted quadratic extension of $R$, the quotient of the $\rho$-twisted polynomial extension $R_\rho[t]$ $(tx = \rho(x)t)$ by the ideal $(t^2 - a)$. In Section 3 it is shown that an antistructure $(\alpha, u)$ on $S$ which restricts to an antistructure $(\alpha_0, u)$ on $R$ determines two distinct morphisms of rings with antistructure

$$i:(R, \alpha_0, u) \to (S, \alpha, u), \quad \tilde{i}:(R, \tilde{\alpha}_0, \tilde{u}) \to (S, \tilde{\alpha}, \tilde{u}),$$

in both cases defined by the inclusion of rings $R \to S$. There are defined induction and transfer maps $i_!$, $i^!$ in the $L$-groups and relative $L$-groups $L_*(i_!)$, $L_*(i^!)$ to fit into exact sequences

$$\ldots \to L_n(R, \alpha_0, u) \xrightarrow{i_!} L_n(S, \alpha, u) \to L_n(i_!) \to L_{n-1}(R, \alpha_0, u) \to \ldots$$

$$\ldots \to L_n(S\, \alpha, u) \xrightarrow{i^!} L_n(R, \alpha_0, u) \to L_n(i^!) \to L_{n-1}(R, \alpha_0, u) \to \ldots,$$

and similarly with $\tilde{i}$ in place of $i$.

In Section 4 the algebraic gluing operation of Ranicki [**12**] is used to define natural isomorphisms of relative $L$-groups

$$\Gamma_!: L_n(\tilde{i}_!) \to L_{n+1}(i_!)$$

$$\Gamma^!: L_n(i^!) \to L_{n+1}(\tilde{i}^!),$$

as required for the applications described in [**5**].

**1. The $L$-theory of a ring with antistructure.** Let $R$ be a ring with antistructure $(\alpha, u)$, that is an associative ring with 1 together with a function $\alpha: R \to R$ and a unit $u \in R$ such that

$$\alpha(a + b) = \alpha(a) + \alpha(b), \alpha(ab) = \alpha(b)\alpha(a), \alpha(1) = 1$$

$$\alpha(u) = u^{-1}, \alpha^2(a) = uau^{-1} \quad (a, b \in R).$$

Given (right) $R$-modules $M$, $N$ let $\mathrm{Hom}_R(M, N)$ be the abelian group of $R$-module morphisms $f: M \to N$.

The *$\alpha$-dual* of an $R$-module $M$ is the $R$-module

$$M^\alpha = \mathrm{Hom}_R(M, R),$$

with $R$ acting by

$$M^\alpha \times R \to M^\alpha; (f, a) \mapsto (x \mapsto \alpha(a)f(x)).$$

For f.g. projective $M$ the $R$-module morphism

$$\iota_u : M \to (M^\alpha)^\alpha;\ x \mapsto (f \mapsto \alpha(f(x))u)$$

is an isomorphism.

The *α-dual* of an $R$-module morphism $f \in \mathrm{Hom}_R(M, N)$ is the $R$-module morphism

$$f^\alpha : N^\alpha \to M^\alpha;\ g \mapsto (x \mapsto g(f(x))).$$

Given a f.g. projective $R$-module chain complex

$$C : \ldots \to C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \to \ldots \quad (n \in \mathbf{Z},\ d^2 = 0)$$

define a $\mathbf{Z}[\mathbf{Z}_2]$-module chain complex $\mathrm{Hom}_{R,\alpha}(C^*, C)$ by

$$\begin{aligned} d : &\mathrm{Hom}_{R,\alpha}(C^*, C)_n \\ &= \sum_{p+q=n} \mathrm{Hom}_R(C_p^\alpha, C_q) \to \mathrm{Hom}_{R,\alpha}(C^*, C)_{n-1}; \\ &\phi \mapsto d\phi + (-)^q \phi d^\alpha \end{aligned}$$

with $T \in \mathbf{Z}_2$ acting by the $(\alpha, u)$-*duality* involution

$$\begin{aligned} T_u : &\mathrm{Hom}_{R,\alpha}(C^*, C)_n \to \mathrm{Hom}_{R,\alpha}(C^*, C)_n; \\ &\phi \mapsto (-)^{pq} \iota_u^{-1} \phi^\alpha. \end{aligned}$$

Define the $(\alpha, u)$-*quadratic Q-groups of C* to be the abelian groups

$$Q_n(C, \alpha, u) = H_n(W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \mathrm{Hom}_{R,\alpha}(C^*, C)) \quad (n \in \mathbf{Z})$$

with $W$ the standard free $\mathbf{Z}[\mathbf{Z}_2]$-module resolution of $\mathbf{Z}$

$$W : \ldots \to \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1-T} \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1+T} \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1-T} \mathbf{Z}[\mathbf{Z}_2].$$

An element $\psi \in Q_n(C, \alpha, u)$ is represented by a collection of chains

$$\{\psi_s \in \mathrm{Hom}_{R,\alpha}(C^*, C)_{n-s} \mid s \geqq 0\}$$

such that

$$\begin{aligned} &d\psi_s + (-)^r \psi_s d^\alpha + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T_u \psi_{s+1}) = 0 \\ &: C^\alpha_{n-r-s-1} \to C_r \quad (r \in \mathbf{Z},\ s \geqq 0). \end{aligned}$$

An *n-dimensional* $(\alpha, u)$-*quadratic Poincaré complex* $(C, \psi)$ *over R* is an $n$-dimensional f.g. projective $R$-module chain complex

$$\begin{aligned} C : \ldots \to 0 \to C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \to \ldots \\ \to C_1 \xrightarrow{d} C_0 \to 0 \to \ldots \quad (n \geqq 0) \end{aligned}$$

together with an element $\psi \in Q_n(C, \alpha, u)$ such that (as in [**11**])

$$(1 + T_u)\psi_0 \in H_n(\mathrm{Hom}_{R,\alpha}(C^*, C))$$

determines $R$-module isomorphisms

$$(1 + T_u)\psi_0 : H^*(C) \xrightarrow{\sim} H_{n-*}(C),$$

where the homology and cohomology $R$-modules of $C$ are defined by

$$\left.\begin{aligned} H_r(C) &= \ker(d:C_r \to C_{r-1})/\mathrm{im}(d:C_{r+1} \to C_r) \\ H^r(C) &= \ker(d^\alpha:C^\alpha_r \to C^\alpha_{r+1})/\mathrm{im}(d^\alpha:C^\alpha_{r-1} \to C^\alpha_r). \end{aligned}\right\} \quad (r \in \mathbf{Z})$$

For example, a 0-dimensional $(\alpha, u)$-quadratic Poincaré complex over $R$ $(C, \psi \in Q_0(C, \alpha, u))$ is the same as a non-singular $(\alpha, u)$-quadratic form $(M, \psi)$ over $R$ in the sense of Wall [**14**], that is a f.g. projective $R$-module $M = C^\alpha_0$ together with an element

$$\begin{aligned} \psi &\in Q_0(C, \alpha, u) \\ &= \mathrm{coker}(1 - T_u:\mathrm{Hom}_R(M, M^\alpha) \to \mathrm{Hom}_R(M, M^\alpha)) \end{aligned}$$

such that $(1 + T_u)\psi \in \mathrm{Hom}_R(M, M^\alpha)$ is an isomorphism, where

$$T_u:\mathrm{Hom}_R(M, M^\alpha) \to \mathrm{Hom}_R(M, M^\alpha);$$
$$\phi \mapsto (\phi^\alpha \iota_u : x \mapsto (y \mapsto \alpha(\phi(y)(x))u)$$

is the $(\alpha, u)$-duality involution on $\mathrm{Hom}_R(M, M^\alpha)$.

Given a chain map of $R$-module chain complexes

$$f:C \to D$$

let $C(f)$ denote the algebraic mapping cone of $f$, the $R$-module chain complex with

$$d_{C(f)} = \begin{pmatrix} d_D & (-)^{n-1}f \\ 0 & d_C \end{pmatrix}$$
$$:C(f)_n = D_n \oplus C_{n-1} \to C(f)_{n-1} = D_{n-1} \oplus C_{n-2}.$$

The relative homology $R$-modules $H_*(f) = H_*(C(f))$ fit into the exact sequence

$$\ldots \to H_{n+1}(D) \to H_{n+1}(f) \to H_n(C) \xrightarrow{f_*} H_n(D) \to \ldots.$$

A chain map of f.g. projective $R$-module chain complexes $f:C \to D$ induces a $\mathbf{Z}[\mathbf{Z}_2]$-module chain map

$$\mathrm{Hom}_{R,\alpha}(f^*, f):\mathrm{Hom}_{R,\alpha}(C^*, C) \to \mathrm{Hom}_{R,\alpha}(D^*, D);\ \phi \mapsto f\phi f^\alpha,$$

so that there are induced $Q$-group morphisms

$$f_\%:Q_n(C, \alpha, u) \to Q_n(D, \alpha, u) \quad (n \in \mathbf{Z}).$$

Define the *relative* $(\alpha, u)$*-quadratic* $Q$*-groups of* $f$

$$Q_{n+1}(f, \alpha, u) = H_{n+1}(W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} C(\mathrm{Hom}_{R,\alpha}(f^*, f))) \quad (n \in \mathbf{Z})$$

to fit into the exact sequence

$$\ldots \to Q_{n+1}(D, \alpha, u) \to Q_{n+1}(f, \alpha, u)$$
$$\to Q_n(C, \alpha, u) \xrightarrow{f_\%} Q_n(D, \alpha, u) \to \ldots .$$

An element $(\delta\psi, \psi) \in Q_{n+1}(f, \alpha, u)$ is represented by a collection of chains

$$\{ (\delta\psi_s, \psi_s) \in \mathrm{Hom}_{R,\alpha}(D^*, D)_{n+1-s} \oplus \mathrm{Hom}_{R,\alpha}(C^*, C)_{n-s} | s \geqq 0\}$$

such that

$$d\delta\psi_s + (-)^r\delta\psi_s d^\alpha + (-)^{n-s}(\delta\psi_{s+1} + (-)^{s+1}T_u\delta\psi_{s+1}) + (-)^n f\psi_s f^\alpha = 0$$
$$: D^\alpha_{n-r-s} \to D_r,$$
$$d\psi_s + (-)^r\psi_s d^\alpha + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1}T_u\psi_{s+1}) = 0$$
$$: C^\alpha_{n-r-s-1} \to C_r \quad (r \in \mathbf{Z}, s \geqq 0).$$

An $(n + 1)$-*dimensional* $(\alpha, u)$-*quadratic Poincaré pair* $(f, (\delta\psi, \psi))$ *over* $R$ consists of a chain map $f:C \to D$ from an $n$-dimensional $R$-module chain complex $C$ to an $(n + 1)$-dimensional $R$-module chain complex $D$ together with an element

$$(\delta\psi, \psi) \in Q_{n+1}(f, \alpha, u)$$

such that

$$(1 + T_u)(\delta\psi_0, \psi_0) \in H_{n+1}(\mathrm{Hom}_{R,\alpha}(f^*, f))$$

determines $R$-module isomorphisms

$$(1 + T_u)(\delta\psi_0, \psi_0): H^*(D) \xrightarrow{\sim} H_{n+1-*}(f).$$

The *boundary* of $(f, (\delta\psi, \psi))$ is the $n$-dimensional $(\alpha, u)$-quadratic Poincaré complex over $R(C, \psi \in Q_n(C, \alpha, u))$.

A *cobordism* of $n$-dimensional $(\alpha, u)$-quadratic Poincaré complexes over $R(C, \psi)$, $(C', \psi')$ is an $(n + 1)$-dimensional $(\alpha, u)$-quadratic Poincaré pair over $R$

$$((f f'): C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$$

with boundary $(C, \psi) \oplus (C', -\psi')$.

A *homotopy equivalence* of $n$-dimensional $(\alpha, u)$-quadratic Poincaré complexes over $R$

$$f:(C, \psi) \xrightarrow{\sim} (C', \psi')$$

is a chain equivalence $f:C \xrightarrow{\simeq} C'$ such that

$$f_{\%}(\psi) = \psi' \in Q_n(C', \alpha, u).$$

Cobordism is an equivalence relation on the set of $n$-dimensional $(\alpha, u)$-quadratic Poincaré complexes over $R$, such that homotopy equivalent complexes are cobordant. The cobordism classes define an abelian group, the *$n$-dimensional $(\alpha, u)$-quadratic $L$-group of $R$* $L_n(R, \alpha, u)$ $(n \geqq 0)$, with addition and inverses by

$$(C, \psi) + (C', \psi') = (C \oplus C', \psi \oplus \psi'),$$
$$-(C, \psi) = (C, -\psi) \in L_n(R, \alpha, u).$$

Given an $R$-module chain complex $C$ define the *suspension* $SC$ to be the $R$-module chain complex with

$$d_{SC} = d_C : SC_r = C_{r-1} \to SC_{r-1} = C_{r-2} \quad (r \in \mathbf{Z}).$$

If $C$ is a f.g. projective $R$-module chain complex there is defined an isomorphism of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes

$$\bar{S}: S^2\mathrm{Hom}_{R,\alpha}(C^*, C) \xrightarrow{\simeq} \mathrm{Hom}_{R,\alpha}(SC^*, SC);$$
$$f \mapsto (-)^p f \quad (f \in \mathrm{Hom}_R(C^\alpha_p, C_q))$$

with $T \in \mathbf{Z}_2$ acting by $T_u$ on $\mathrm{Hom}_{R,\alpha}(C^*, C)$ and by $T_{-u}$ on $\mathrm{Hom}_{R,\alpha}$ $(SC^*, SC)$, so that there are induced isomorphisms in the $Q$-groups

$$\bar{S}: Q_*(C, \alpha, u) \xrightarrow{\simeq} Q_{*+2}(SC, \alpha, -u).$$

The *skew-suspension* maps in the $L$-groups

$$\bar{S}: L_n(R, \alpha, u) \to L_{n+2}(R, \alpha, -u); \ (C, \psi) \mapsto (SC, \bar{S}\psi) \quad (n \geqq 0)$$

are isomorphisms, by Proposition 4.3 of [**11**]. In particular, it follows that the $(\alpha, u)$-quadratic $L$-groups are 4-periodic

$$L_n(R, \alpha, u) = L_{n+4}(R, \alpha, u) \quad (n \geqq 0).$$

Furthermore, working as in Section 5 of [**11**] it is possible to identify

$$\begin{cases} L_{2i}(R, \alpha, u) \\ L_{2i+1}(R, \alpha, u) \end{cases} \quad (i (\mathrm{mod}\ 2))$$

with the Witt group of non-singular

$$(\alpha, (-)^i u)\text{-quadratic} \begin{cases} \text{forms} \\ \text{formations} \end{cases} \text{over } R.$$

**2. Scaling.** Scaling is a classical device for generating isomorphisms between categories of quadratic forms (cf. [**14**]), and hence also of $L$-groups.

The *scaling* of the antistructure $(\alpha, u)$ on $R$ by the unit $v \in R$ is the antistructure on $R$

$$(\alpha, u)^v = (\beta, w)$$

defined by

$$\beta: R \to R;\ a \mapsto v\alpha(a)v^{-1},\ w = v\alpha(v^{-1})u \in R.$$

For any $R$-module $M$ there is defined a scaling isomorphism of the $\alpha$-dual and $\beta$-dual $R$-modules

$$\sigma^v: M^\alpha \xrightarrow{\sim} M^\beta;\ f \mapsto (f^v: x \mapsto vf(x)).$$

If $C$ is a f.g. projective $R$-module chain complex there is defined a scaling isomorphism of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes

$$\sigma^v: \mathrm{Hom}_{R,\alpha}(C^*, C) \xrightarrow{\sim} \mathrm{Hom}_{R,\beta}(C^*, C);\ \phi \mapsto \phi^v$$

sending $\phi \in \mathrm{Hom}_R(C_p^\alpha, C_q)$ to the composite

$$\phi^v: C_p^\beta \xrightarrow{(\sigma^v)^{-1}} C_p^\alpha \xrightarrow{\phi} C_q.$$

There are induced scaling isomorphisms of $Q$-groups

$$\sigma^v: Q_n(C, \alpha, u) \xrightarrow{\sim} Q_n(C, \beta, w);\ \psi \mapsto \psi^v$$

and hence also of $L$-groups

$$\sigma^v: L_n(R, \alpha, u) \xrightarrow{\sim} L_n(R, \beta, w);\ (C, \psi) \mapsto (C, \psi^v).$$

**3. Twisted quadratic extensions.** A *structure* $(\rho, a)$ on a ring $R$ is a pair consisting of a ring automorphism $\rho: R \to R$ and a unit $a \in R$ such that

$$\rho^2(x) = axa^{-1} \in R \quad (x \in R)$$

and $\rho(a) = a \in R$. The $(\rho, a)$-*twisted quadratic extension of* $R$ is the ring

$$S = R_\rho[\sqrt{a}] = R_\rho[t]/(t^2 - a)$$

with $t$ an indeterminate over $R$ such that

$$tx = \rho(x)t \quad (x \in R).$$

The extension of $\rho$ to an automorphism of $S$ is denoted by

$$\rho: S \to S;\ x + yt \mapsto t(x + yt)t^{-1} \quad (x, y \in R).$$

Let now $R$ be a ring with antistructure $(\alpha_0, u)$ and structure $(\rho, a)$ such that $\alpha_0$ extends to an antiautomorphism of $S$

$$\alpha: R_\rho[\sqrt{a}] = S \to R_\rho[\sqrt{a}]$$

with $\alpha(\sqrt{a}).\sqrt{a} \in R \subset S$ and $\alpha^2(\sqrt{a}) = u\sqrt{a}u^{-1} \in S$. Thus $(\alpha, u)$ is an antistructure on $S$, and the inclusion $i:R \to S$ defines a morphism of rings with antistructure

$$i:(R, \alpha_0, u) \to (S, \alpha, u).$$

Use scaling by the unit $\sqrt{a} \in S$ and the Galois automorphism of $S$ over $R$

$$\gamma:S \to S;\ x + yt \mapsto x - yt \quad (x, y \in R)$$

to define an antistructure on $S$

$$\widetilde{(\alpha, u)} = (\widetilde{\alpha}, \widetilde{u})$$

by

$$\begin{aligned}(\widetilde{\alpha}, \widetilde{u}) &= (\gamma\alpha, u)^{\sqrt{a}} = (z \mapsto \sqrt{a}\gamma\alpha(z)(\sqrt{a})^{-1}, \sqrt{a}\gamma\alpha((\sqrt{a})^{-1})u) \\ &= (\rho\gamma\alpha, -\sqrt{a}\alpha((\sqrt{a})^{-1})u).\end{aligned}$$

Then $(\widetilde{\alpha}, \widetilde{u})$ restricts to another antistructure $(\widetilde{\alpha}_0, \widetilde{u})$ on $R$, with a morphism of rings with antistructure

$$i:(R, \widetilde{\alpha}_0, \widetilde{u}) \to (S, \widetilde{\alpha}, \widetilde{u}).$$

Given an $R$-module $M$ denote the induced $S$-module by

$$i_!M = M \otimes_R S.$$

If $M$ is a f.g. projective $R$-module then $i_!M$ is a f.g. projective $S$-module, and there is defined a natural $S$-module isomorphism

$$i_!(M^{\alpha_0}) \xrightarrow{\sim} (i_!M)^{\alpha};\ f \otimes x \mapsto (u \otimes y \mapsto \alpha(x)f(u)y) \qquad (f \in M^{\alpha_0},\ u \in M,\ x, y \in S).$$

If $C$ is a f.g. projective $R$-module chain complex then $i_!C$ is a f.g. projective $S$-module chain complex, and there is defined a $\mathbf{Z}[\mathbf{Z}_2]$-module chain map

$$\begin{aligned}&i_!:\mathrm{Hom}_{R,\alpha_0}(C^*, C) \to \mathrm{Hom}_{S,\alpha}(i_!C^*, i_!C);\\ &\phi \mapsto (i_!\phi:f \otimes x \mapsto \phi(f) \otimes x) \qquad (\phi \in \mathrm{Hom}_R(C_p^{\alpha}, C_q), f \in C_p^{\alpha}, x \in S)\end{aligned}$$

inducing $Q$-group morphisms

$$i_!:Q_*(C, \alpha_0, u) \to Q_*(i_!C, \alpha, u);\ \psi \mapsto i_!\psi.$$

The induced $L$-group morphisms

$$i_!:L_*(R, \alpha_0, u) \to L_*(S, \alpha, u);\ (C, \psi) \mapsto (i_!C, i_!\psi)$$

fit into an exact sequence

$$\ldots \to L_n(R, \alpha_0, u) \xrightarrow{i_!} L_n(S, \alpha, u) \to L_n(i_!, \alpha, u) \to L_{n-1}(R, \alpha_0, u) \to \ldots$$

in which the *relative L-groups* $L_n(i_!, \alpha, u)$ $(n \geqq 1)$ are defined as in Section 2 of [**12**] to be the cobordism groups of pairs

$$((C, \psi), (f: i_! C \to D, (\delta\psi, i_!\psi)))$$

consisting of an $(n - 1)$-dimensional $(\alpha_0, u)$-quadratic Poincaré complex over $R$

$$(C, \psi \in Q_{n-1}(C, \alpha_0, u))$$

and an $n$-dimensional $(\alpha, u)$-quadratic Poincaré pair over $S$

$$(f: i_! C \to D, (\delta\psi, i_!\psi) \in Q_n(f, \alpha, u))$$

with boundary $i_!(C, \psi)$.

Given an $S$-module $N$ denote by $i^! N$ the $R$-module with the same additive group and $R$ acting by the restriction of the $S$-action to $R \subset S$. If $N$ is a f.g. projective $S$-module then $i^! N$ is a f.g. projective $R$-module, and there is defined a natural $R$-module isomorphism

$$i^!(N^\alpha) \xrightarrow{\sim} (i^! N)^{\alpha_0}; f \mapsto (u \mapsto x)$$
$$f \in N^\alpha, u \in N, f(u) = x + y\sqrt{a} \in S, x, y \in R).$$

If $D$ is a f.g. projective $S$-module chain complex then $i^! D$ is a f.g. projective $R$-module chain complex, and there is defined a $\mathbf{Z}[\mathbf{Z}_2]$-module chain map

$$i^!: \mathrm{Hom}_{S,\alpha}(D^*, D) \to \mathrm{Hom}_{R,\alpha_0}(i^! D^*, i^! D); \phi \mapsto (i^!\phi: f \mapsto \phi(f))$$
$$(\phi \in \mathrm{Hom}_S(D_p^\alpha, D_q), f \in (i^! D_p)^{\alpha_0} = i^!(D_p^\alpha))$$

inducing $Q$-groups morphisms

$$i^!: Q_*(D, \alpha, u) \to Q_*(i^! D, \alpha_0, u); \psi \mapsto i^!\psi.$$

The induced $L$-group morphisms

$$i^!: L_*(S, \alpha, u) \to L_*(R, \alpha_0, u); (D, \psi) \mapsto (i^! D, i^!\psi)$$

fit into an exact sequence

$$\ldots \to L_n(S, \alpha, u) \xrightarrow{i^!} L_n(R, \alpha_0, u) \to L_n(i^!, \alpha, u) \to L_{n-1}(S, \alpha, u) \to \ldots$$

in which the *relative L-groups* $L_n(i^!, \alpha, u)$ $(n \geqq 1)$ are defined as in Section 2 of [**12**] to be the cobordism groups of pairs

$$((D, \psi), (f: i^! D \to C, (\delta\psi, i^!\psi)))$$

consisting of an $(n-1)$-dimensional $(\alpha, u)$-quadratic Poincaré complex over $S$

$$(D, \psi \in Q_{n-1}(D, \alpha, u))$$

and an $n$-dimensional $(\alpha_0, u)$-quadratic Poincaré pair over $R$

$$(f:i^!D \to C, (\delta\psi, i^!\psi) \in Q_n(f, \alpha_0, u))$$

with boundary $i^!(D, \psi)$.

If $M$ is an $R$-module and $N$ is an $S$-module there are defined natural abelian group isomorphisms

$$\mathrm{Hom}_R(M, i^!N) \xrightarrow{\sim} \mathrm{Hom}_S(i_!M, N); f \mapsto (x \otimes s \mapsto f(x)s)$$

$$\mathrm{Hom}_R(i^!N, M) \xrightarrow{\sim} \mathrm{Hom}_S(N, i_!M);$$

$$g \mapsto (y \mapsto g(y) \otimes 1 + g(y\sqrt{a}) \otimes (\sqrt{a})^{-1}) \; (x \in M, y \in N, s \in S)$$

which we shall use as identifications.

Given a f.g. projective $R$-module $M$ let $\rho M$ denote the f.g. projective $R$-module with the same additive group and $R$ acting by

$$\rho M \times R \to \rho M; (x, r) \mapsto x\rho(r).$$

The isomorphism of abelian groups

$$\rho:\mathrm{Hom}_R(M, M^{\alpha_0}) \xrightarrow{\sim} \mathrm{Hom}_R(\rho M, (\rho M)^{\alpha_0});$$

$$\phi \mapsto (\rho\phi:x \mapsto (y \mapsto \alpha(\sqrt{a})(\phi(x)(y))\sqrt{a}))$$

is such that $T_u(\rho\phi) = \rho(T_u\phi)$, so that it is an isomorphism of $\mathbf{Z}[\mathbf{Z}_2]$-modules. Thus if $C$ is a f.g. projective $R$-module chain complex there is defined an isomorphism of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes

$$\rho:\mathrm{Hom}_{R,\alpha_0}(C^*, C) \xrightarrow{\sim} \mathrm{Hom}_{R,\alpha_0}(\rho C^*, \rho C)$$

inducing $Q$-group isomorphisms

$$\rho:Q_*(C, \alpha_0, u) \xrightarrow{\sim} Q_*(\rho C, \alpha_0, u).$$

Furthermore, there is defined an isomorphism of $R$-module chain complexes

$$i^!i_!C \xrightarrow{\sim} C \oplus \rho C; x \otimes (r + s\sqrt{a}) \mapsto (xr, xs) \quad (x \in C, r, s \in R),$$

allowing the identifications

$$\begin{aligned}\mathrm{Hom}_{S,\alpha}(i_!C^*, i_!C) &= \mathrm{Hom}_{R,\alpha_0}(C^*, i^!i_!C)\\ &= \mathrm{Hom}_{R,\alpha_0}(C^*, C) \oplus \mathrm{Hom}_{R,\alpha_0}(C^*, \rho C)\\ &= \mathrm{Hom}_{R,\alpha_0}(C^*, C) \oplus \mathrm{Hom}_{R,\tilde{\alpha}_0}(C^*, C),\end{aligned}$$

https://doi.org/10.4153/CJM-1987-017-x Published online by Cambridge University Press

$$Q_*(i_!C, \alpha, u) = Q_*(C, \alpha_0, u) \oplus Q_*(C, \widetilde{\alpha}_0, -\widetilde{u}).$$

The identity $i^!i_!C = C \oplus \rho C$ has the following geometric interpretation. Let $X$ be a connected topological space with fundamental group $\pi$, and let $\pi \subset \pi'$ be the inclusion of $\pi$ as an index 2 subgroup in a group $\pi'$. Then $S = \mathbf{Z}[\pi']$ is a $(\rho, a)$-quadratic extension of $R = \mathbf{Z}[\pi]$ with $\sqrt{a} \in \pi' - \pi$, and the chain complex of the universal cover $\widetilde{X}$ of $X$ is an $R$-module chain complex $C = C(\widetilde{X})$. The composite

$$X \to K(\pi, 1) \to K(\pi', 1)$$

classifies a covering $\widetilde{X}'$ of $X$ with group of covering translations $\pi'$, such that $C(\widetilde{X}') = i_!C$. As a $\pi$-space $\widetilde{X}' = \widetilde{X} \cup \rho\widetilde{X}$, and the chain level decomposition

$$C(\widetilde{X}') = C(\widetilde{X}) \oplus \rho C(\widetilde{X})$$

is precisely $i^!i_!C = C \oplus \rho C$.

Given a f.g. projective $S$-module $N$ let $\gamma N$ denote the f.g. projective $S$-module with the same additive group and $S$ acting by

$$\gamma N \times S \to \gamma N; (x, s) \mapsto x\gamma(s).$$

The isomorphism of abelian groups

$$\gamma:\mathrm{Hom}_S (N, N^\alpha) \xrightarrow{\sim} \mathrm{Hom}_S (\gamma N, (\gamma N)^\alpha);$$
$$\phi \mapsto (\gamma\phi:x \mapsto (y \mapsto \gamma(\phi(x)(y))))$$

is such that $T_u(\gamma\phi) = \gamma(T_u\phi)$, so that it is an isomorphism of $\mathbf{Z}[\mathbf{Z}_2]$-modules. Thus if $D$ is a f.g. projective $S$-module chain complex there is defined an isomorphism of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes

$$\gamma:\mathrm{Hom}_{S,\alpha} (D^*, C) \xrightarrow{\sim} \mathrm{Hom}_{S,\alpha} (\gamma D^*, \gamma D)$$

inducing $Q$-group isomorphisms

$$\gamma:Q_*(D, \alpha, u) \xrightarrow{\sim} Q_*(\gamma D, \alpha, u).$$

Furthermore, there is defined a short exact sequence of $S$-module chain complexes

$$0 \to \gamma D \to i_!i^!D \to D \to 0$$

with

$$\gamma D \to i_!i^!D; x \mapsto x \otimes 1 - x(\sqrt{a})^{-1} \otimes \sqrt{a}$$
$$i_!i^!D \to D; x \otimes s \mapsto xs \quad (x \in D, s \in S),$$

giving rise to a short exact sequence of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes

$$0 \to \mathrm{Hom}_{S,\gamma\alpha}(D^*, D) \xrightarrow{i^!} \mathrm{Hom}_{R,\alpha_0}(i^!D^*, i^!D) \to \mathrm{Hom}_{S,\alpha}(D^*, D) \to 0$$

and a long exact sequence of $Q$-groups

$$\ldots \to Q_n(D, \gamma\alpha, u) \xrightarrow{i^!} Q_n(i^!D, \alpha_0, u) \to Q_n(D, \alpha, u) \to Q_{n-1}(D, \gamma\alpha, u) \to \ldots .$$

If $D = i_!C$ for some f.g. projective $R$-module chain complex $C$ the long exact sequence of $Q$-groups is naturally isomorphic to the direct sum of the exact sequence

$$\ldots \to Q_n(C, \alpha_0, u) \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} Q_n(C, \alpha_0, u) \oplus Q_n(C, \alpha_0, u) \xrightarrow{(1\ \ -1)} Q_n(C, \alpha_0, u) \xrightarrow{0} Q_{n-1}(C, \alpha_0, u) \to \ldots$$

and the exact sequence

$$\ldots \to Q_n(C, \widetilde{\alpha}_0, \widetilde{u}) \to H_n(\mathrm{Hom}_{R,\widetilde{\alpha}_0}(C^*, C)) \to Q_n(C, \widetilde{\alpha}_0, -\widetilde{u}) \xrightarrow{S} Q_{n-1}(C, \widetilde{\alpha}_0, \widetilde{u}) \to \ldots$$

of Proposition 1.3 of [**11**], with $S$ the suspension map. The exact sequence

$$0 \to \gamma D \to i_!i^!D \to D \to 0$$

has the following geometric interpretation.

Let $Y$ be a connected topological space with fundamental group $\pi_1(Y) = \pi'$, and let $\pi \subset \pi'$ be a subgroup of index 2 classifying a non-trivial $(D^1, S^0)$-bundle $\xi$ over $Y$

$$(D^1, S^0) \to (E(\xi), S(\xi)) \to Y$$

with $\pi_1(S(\xi)) = \pi$. As before, $S = \mathbf{Z}[\pi']$ is a $(\rho, a)$-quadratic extension of $R = \mathbf{Z}[\pi]$. The chain complex of the universal cover $\widetilde{Y}$ of $Y$ is an $S$-module chain complex $D = C(\widetilde{Y})$. Let $\widetilde{\xi}$ be the $(D^1, S^0)$-bundle over $\widetilde{Y}$ obtained from $\xi$ by pullback along the covering projection $\widetilde{Y} \to Y$

$$(D^1, S^0) \to (E(\widetilde{\xi}), S(\widetilde{\xi})) \to \widetilde{Y}.$$

Then

$$C(S(\widetilde{\xi})) = i_!i^!D, \quad C(E(\widetilde{\xi})) = D$$

(up to chain equivalence) and

$$C(E(\widetilde{\xi}), S(\widetilde{\xi})) = S\gamma D$$

by the chain level Thom isomorphism.

**4. The main result.** As in Section 3 let $(\rho, a)$ be a structure on a ring $R$, and let $(\alpha, u)$ be an antistructure on the $(\rho, a)$-twisted quadratic extension ring $S = R_\rho[\sqrt{a}]$ such that there are defined morphisms of rings with antistructure

$$i:(R, \alpha_0, u) \to (S, \alpha, u), \widetilde{i}:(R, \widetilde{\alpha}_0, \widetilde{u}) \to (S, \widetilde{\alpha}, \widetilde{u}).$$

MAIN RESULT. *The relative L-groups of* $i_!, \widetilde{i}_!, i^!, \widetilde{i}^!$ *are related by natural isomorphisms*

$$\Gamma_!:L_n(\widetilde{i}_!) \to L_{n+1}(i_!),$$
$$\Gamma^!:L_n(i^!) \to L_{n+1}(\widetilde{i}^!).$$

The isomorphisms $\Gamma_!$, $\Gamma^!$ are defined using the following constructions.

Given an $n$-dimensional $(\widetilde{\alpha}_0, \widetilde{u})$-quadratic Poincaré complex over $R(C, \psi \in Q_n(C, \widetilde{\alpha}_0, \widetilde{u}))$ there is defined an $(n + 1)$-dimensional $(\alpha_0, u)$-quadratic Poincaré pair over $R$

$$(g_C:i^!i_!C \to C, (0, i^!\sigma^{\sqrt{a}}i_!\psi) \in Q_{n+1}(g_C, \alpha_0, u))$$

with

$$g_C = (1, \quad 0):i^!i_!C = C \oplus \rho C \to C,$$

and

$$\begin{aligned} i^!\sigma^{\sqrt{a}}i_!\psi &= (0, 0, (1 + T_u)\psi_0) \\ &\in Q_n(i^!i_!C, \alpha_0, u) \\ &= Q_n(C, \alpha_0, u) \oplus Q_n(C, \alpha_0, u) \oplus H_n(\mathrm{Hom}_{R,\widetilde{\alpha}_0}(C^*, C)) \end{aligned}$$

the image of $\psi \in Q_n(C, \widetilde{\alpha}_0, \widetilde{u})$ under the composite

$$Q_n(C, \widetilde{\alpha}_0, \widetilde{u}) \xrightarrow{i_!} Q_n(i_!C, \widetilde{\alpha}, \widetilde{u}) \overset{\sigma\sqrt{a}}{\rightrightarrows} Q_n(i_!C, \gamma\alpha, u) \xrightarrow{i^!} Q_n(i^!i_!C, \widetilde{\alpha}_0, \widetilde{u}).$$

Given an $n$-dimensional $(\alpha, u)$-quadratic Poincaré complex over $S$

$$(D, \psi \in Q_n(D, \alpha, u))$$

there is defined an $(n + 1)$-dimensional $(\widetilde{\alpha}, \widetilde{u})$-quadratic Poincaré pair over $S$

$$(e_D:i_!i^!D \to D, (0, i_!i^!\psi) \in Q_{n+1}(e_D, \widetilde{\alpha}, \widetilde{u}))$$

with

$$e_D:i_!i^!D \to D; \ x \otimes s \mapsto xs \quad (x \in D, s \in S),$$

and

$$(0, i_!i^!\psi) \in Q_{n+1}(e_D, \widetilde{\alpha}, \widetilde{u})$$

https://doi.org/10.4153/CJM-1987-017-x Published online by Cambridge University Press

the image of $\psi \in Q_n(D, \alpha, u)$ under the map

$$Q_n(D, \alpha, u) \to Q_{n+1}(e_D, \tilde{\alpha}, \tilde{u})$$

appearing in the morphism of exact sequences

$$\begin{array}{ccccccccc}
\dots \to & Q_n(D, \alpha, u) & \xrightarrow{i^!} & Q_n(i^! D, \alpha_0, u) & \longrightarrow & Q_n(D, \gamma\alpha, u) & \longrightarrow & Q_{n-1}(D, \alpha, u) & \to \dots \\
& \downarrow & & \downarrow{\scriptstyle \sigma^{\sqrt{a_{i_!}}}} & & \downarrow{\scriptstyle 2\sigma^{\sqrt{a}}} & & \downarrow & \\
\dots \to & Q_{n+1}(e_D, \tilde{\alpha}, \tilde{u}) & \longrightarrow & Q_n(i_! i^! D, \tilde{\alpha}, \tilde{u}) & \xrightarrow{e_{D\%}} & Q_n(D, \tilde{\alpha}, \tilde{u}) & \longrightarrow & Q_n(e_D, \tilde{\alpha}, \tilde{u}) & \to \dots
\end{array}$$

Use the constructions and the algebraic gluing operation of Section 1.7 of [**12**] to define the abelian group morphisms $\Gamma_!$, $\Gamma^!$ by

$$\Gamma_!: L_n(i_!, \tilde{\alpha}, \tilde{u}) \to L_{n+1}(i_!, \alpha, u);$$

$$((C, \psi \in Q_{n-1}(C, \tilde{\alpha}_0, \tilde{u})), (f: i_! C \to D, (\delta\psi, i_!\psi) \in Q_n(f, \tilde{\alpha}, \tilde{u})))$$

$$\mapsto ((C', \psi' \in Q_n(C', \alpha_0, u)),$$

$$(f': i_! C' \to D', (0, i_!\psi') \in Q_{n+1}(f', \alpha, u)))$$

with

$$C' = C \cup_{i^! i_! C^{i^!}} D = C\left(\begin{pmatrix} g_C \\ i^! f \end{pmatrix}: i^! i_! C \to C \oplus i^! D\right),$$

$$D' = C(f),\ \psi' = 0 \cup_{i^! \sigma\sqrt{a}_{i_!\psi^{i^!}}} \sigma^{\sqrt{a}} \delta\psi,$$

$$f' = \begin{pmatrix} 0 & e_{i_! C} & 0 \\ 0 & 0 & e_D \end{pmatrix}: i_! C'_r = i_! C_r \oplus i_! i^! i_! C_{r-1} \oplus i_! i^! D_r$$

$$\to D'_r = i_! C_{r-1} \oplus D_r \quad (r \in \mathbf{Z}),$$

and

$$\Gamma^!: L_n(i^!, \alpha, u) \to L_{n+1}(i^!, \tilde{\alpha}, \tilde{u});$$

$$((D, \psi \in Q_{n-1}(D, \alpha, u)), (f: i^! D \to C, (\delta\psi, i^!\psi) \in Q_n(f, \alpha_0, u)))$$

$$\mapsto ((D', \psi' \in Q_n(D', \tilde{\alpha}, \tilde{u})), (f': i^! D' \to C', (0, i^!\psi')$$

$$\in Q_n(f', \tilde{\alpha}_0, \tilde{u})))$$

with

$$D' = D \cup_{i_! i^! D^{i_!}} C = C\left(\begin{pmatrix} e_D \\ i_! f \end{pmatrix}: i_! i^! \to D \oplus i_! C\right),\ C' = C(f),$$

$$\psi' = 0 \cup_{\sigma^{\sqrt{a_{i_!}}} i^! \psi} \sigma^{\sqrt{a_{i_!}}} \delta\psi,$$

$$f' = \begin{pmatrix} 0 & g_{i^! D} & 0 \\ 0 & 0 & g_C \end{pmatrix}: i^! D'_r = i^! D_r \oplus i^! i_! i^! D_{r-1} \oplus i^! i_! C_r$$

$$\to C'_r = i^! D_{r-1} \oplus C_r \quad (r \in \mathbf{Z}).$$

(The definition of $\Gamma^!$ corrects the expression for the ill-defined isomorphism

$$L_*(i^!, \alpha, u) \xrightarrow{\sim} L_{*+1}(i^!, \tilde{\alpha}, \tilde{u})$$

given on pp. 704-705 of [**12**].)

The maps

$$\Gamma_!: L_*(i_!, \tilde{\alpha}, \tilde{u}) \to L_{*+1}(i_!, \alpha, u)$$

are isomorphisms because there is defined a commutative diagram

$$\begin{array}{ccccc} L_{*-2}(i_!, \tilde{\tilde{\alpha}}, \tilde{\tilde{u}}) & \xrightarrow{\Gamma_!} & L_{*-1}(i_!, \tilde{\alpha}, \tilde{u}) & \xrightarrow{\Gamma_!} & L_*(i_!, \alpha, u) \\ & \searrow \sigma^a \wr & & \wr t \nearrow & \\ & L_{*-2}(i_!, \alpha, -u) & \xrightarrow[\sim]{\bar{S}} & L_*(i_!, \alpha, u) & \end{array}$$

involving the scaling isomorphism $\sigma^a$ for $(\tilde{\tilde{\alpha}}, \tilde{\tilde{u}}) = (\alpha, -u)^a$, the skew-suspension isomorphism $\bar{S}$ and the automorphism

$$t: L_*(i_!, \alpha, u) \xrightarrow{\sim} L_*(i_!, \alpha, u);$$

$$((C, \psi), (f: i_!C \to D, (\delta\psi, i_!\psi)))$$

$$\mapsto ((\rho C, \rho\psi), (tf: i_!\rho C \to \gamma D; x \otimes s$$

$$\mapsto f(x \otimes \sqrt{a}\gamma(s)), (\gamma\delta\psi, i_!\rho\psi))).$$

The diagram actually commutes on the homotopy (rather than cobordism) level: given a representative

$$x = ((C, \psi), (f: i_!C \to D, (\delta\psi, i_!\psi)))$$

of an element of $L_{*-2}(i_!, \tilde{\tilde{\alpha}}, \tilde{\tilde{u}})$ let

$$\Gamma_!(x) = ((C', \psi'), (f': i_!C' \to D', (0, i_!\psi')))$$

$$\Gamma_!\Gamma_!(x) = ((C'', \psi''), (f'': i_!C'' \to D'', (0, i_!\psi''))).$$

Now

$$C'' = C\left(\begin{pmatrix} g_{C'} \\ i^!f' \end{pmatrix}: i^!i_!C' \to C' \oplus i^!D'\right)$$

$$= C\left(\begin{pmatrix} F \\ G \end{pmatrix}: C\left(\begin{pmatrix} g_{i^!i_!C} \\ i^!e_{i_!C} \end{pmatrix}\right) \to C\left(\begin{pmatrix} g_{i^!D} \\ i^!e_D \end{pmatrix}\right) \oplus C(g_C)\right),$$

$$D'' = C(f') = C\left(H: C\begin{pmatrix} e_{i_!C} \\ i_!g_C \end{pmatrix}\right) \to C(e_D)),$$

with

$$F = \begin{pmatrix} i^!f & 0 & 0 \\ 0 & i^!i_!i^!f & 0 \\ 0 & 0 & i^!f \end{pmatrix}:$$

$$C\left(\begin{pmatrix} g_{i^!i_!C} \\ i^!e_{i_!C} \end{pmatrix}\right)_r = i^!i_!C_r \oplus i^!i_!i^!i_!C_{r-1} \oplus i^!i_!C_r$$

$$\to C\left(\begin{pmatrix} g_{i^!D} \\ i^!e_D \end{pmatrix}\right)_r = i^!D_r \oplus i^!i_!i^!D_{r-1} \oplus i^!D_r,$$

$$G = \begin{pmatrix} g_C & 0 & 0 \\ 0 & i^!i_!g_C & 0 \end{pmatrix}:$$

$$C\begin{pmatrix} g_{i^!i_!C} \\ i^!e_{i_!C} \end{pmatrix}_r = i^!i_!C_r \oplus i^!i_!i^!i_!C_{r-1} \oplus i^!i_!C_r$$

$$\to C(g_C)_r = C_r \oplus i^!i_!C_{r-1},$$

$$H = \begin{pmatrix} f & 0 & 0 \\ 0 & i_!i^!f & 0 \end{pmatrix}:$$

$$C\left(\begin{pmatrix} e_{i_!C} \\ i_!g_C \end{pmatrix}\right)_r = i_!C_r \oplus i_!i^!i_!C_{r-1} \oplus i_!C_r$$

$$\to C(e_D)_r = D_r \oplus i_!i^!D_{r-1} \quad (r \in \mathbf{Z}).$$

The chain maps

$$\begin{pmatrix} g_{i^!i_!C} \\ i^!e_{i_!C} \end{pmatrix}: i^!i_!i^!i_!C \to i^!i_!C \oplus i^!i_!C$$

$$\begin{pmatrix} g_{i^!D} \\ i^!e_D \end{pmatrix}: i^!i_!i^!D \to i^!D \oplus i^!D$$

$$\begin{pmatrix} e_{i_!C} \\ i_!g_C \end{pmatrix}: i_!i^!i_!C \to i_!C \oplus i_!C$$

are isomorphisms, so that up to chain equivalence

$$C\left(\begin{pmatrix} g_{i^!i_!C} \\ i^!e_{i_!C} \end{pmatrix}\right) = 0,\ C\left(\begin{pmatrix} g_{i^!D} \\ i^!e_D \end{pmatrix}\right) = 0,\ C\left(\begin{pmatrix} e_{i_!C} \\ i_!g_C \end{pmatrix}\right) = 0,$$

$$C'' = C(g_C) = S\rho C,\ D'' = C(e_D) = S\gamma D.$$

The quadratic structures follow suit, and

$$\Gamma_!\Gamma_!(x) = t\bar{S}\sigma^a(x)$$

up to homotopy equivalence.

https://doi.org/10.4153/CJM-1987-017-x Published online by Cambridge University Press

Similarly, the maps

$$\Gamma^!: L_*(i^!, \alpha, u) \to L_{*+1}(i^!, \tilde{\alpha}, \tilde{u})$$

are isomorphisms because there is defined a commutative diagram

$$\begin{array}{ccccc} L_{*-2}(i^!, \alpha, u) & \xrightarrow{\Gamma^!} & L_{*-1}(i^!, \tilde{\alpha}, \tilde{u}) & \xrightarrow{\Gamma^!} & L_*(i^!, \tilde{\tilde{\alpha}}, \tilde{\tilde{u}}) \\ \bar{S} \downarrow \wr & & & \nearrow \sigma^a \wr & \\ L_*(i^!, \alpha, u) & \overset{t}{\underset{\sim}{\to}} & L_*(i^!, \alpha, -u) & & \end{array}$$

involving the scaling isomorphism $\sigma^a$, the skew-suspension isomorphism $\bar{S}$ and the automorphism

$$t: L_*(i^!, \alpha, -u) \overset{\sim}{\to} L_*(i^!, \alpha, -u);$$

$$((D, \psi), (f: i^! D \to C, (\delta\psi, i^!\psi)))$$

$$\mapsto ((\gamma D, \gamma\psi), (tf: i^!\gamma D \to \rho C;\ x \mapsto f(x(\sqrt{a})^{-1}), (\rho\delta\psi, i^!\gamma\psi))).$$

As before, the diagram actually commutes on the homotopy level: given a representative

$$x = ((D, \psi), (f: i^! D \to C, (\delta\psi, i^!\psi)))$$

of an element of $L_{*-2}(i^!, \alpha, u)$ let

$$\Gamma^!(x) = ((D', \psi'), (f': i^! D' \to C', (0, i^!\psi')))$$

$$\Gamma^!\Gamma^!(x) = ((D'', \psi''), (f'': i^! D'' \to C'', (0, i^!\psi''))).$$

Now

$$D'' = C\left(\begin{pmatrix} e_{D'} \\ i_! f' \end{pmatrix}: i_! i^! D' \to D' \oplus i_! C'\right)$$

$$= C\left(\begin{pmatrix} F \\ G \end{pmatrix}: C\left(\begin{pmatrix} e_{i_! i^! D} \\ i_! g_{i^! D} \end{pmatrix}\right) \to C\left(\begin{pmatrix} e_{i_! C} \\ i_! g_C \end{pmatrix}\right) \oplus C(e_D)\right),$$

$$C'' = C(f') = C\left(H: C\left(\begin{pmatrix} g_{i^! D} \\ i^! e_D \end{pmatrix}\right) \to C(g_C)\right),$$

with

$$F = \begin{pmatrix} i_! f & 0 & 0 \\ 0 & i_! i^! i_! f & 0 \\ 0 & 0 & i_! f \end{pmatrix}$$

$$: C\left(\begin{pmatrix} e_{i_! i^! D} \\ i_! g_{i^! D} \end{pmatrix}\right)_r = i_! i^! D_r \oplus i_! i^! i_! i^! D_{r-1} \oplus i_! i^! D_r$$

$$\to C\left(\begin{pmatrix} e_{i_!C} \\ i_!g_C \end{pmatrix}\right)_r = i_!C_r \oplus i_!i^!i_!C_{r-1} \oplus i_!C_r,$$

$$G = \begin{pmatrix} e_D & 0 & 0 \\ 0 & i_!i^!e_D & 0 \end{pmatrix}$$

$$:C\left(\begin{pmatrix} e_{i_!i^!D} \\ i_!g_{i^!D} \end{pmatrix}\right)_r = i_!i^!D_r \oplus i_!i^!i_!i^!D_{r-1} \oplus i_!i^!D_r$$

$$\to C(e_D)_r = D_r \oplus i_!i^!D_{r-1},$$

$$H = \begin{pmatrix} f & 0 & 0 \\ 0 & i^!i_!f & 0 \end{pmatrix}$$

$$:C\left(\begin{pmatrix} g_{i^!D} \\ i^!e_D \end{pmatrix}\right)_r = i^!D_r \oplus i^!i_!i^!D_{r-1} \oplus i^!D_r$$

$$\to C(g_C)_r = C_r \oplus i^!i_!C_{r-1} \quad (r \in \mathbf{Z}).$$

The chain maps

$$\begin{pmatrix} e_{i_!i^!D} \\ i_!g_{i^!D} \end{pmatrix}: i_!i^!i_!i^!D \to i_!i^!D \oplus i_!i^!D$$

$$\begin{pmatrix} e_{i_!C} \\ i_!g_C \end{pmatrix}: i_!i^!i_!C \to i_!C \oplus i_!C$$

$$\begin{pmatrix} g_{i^!D} \\ i^!e_D \end{pmatrix}: i^!i_!i^!D \to i^!D \oplus i^!D$$

are isomorphisms, so that up to chain equivalence

$$C\left(\begin{pmatrix} e_{i_!i^!D} \\ i_!g_{i^!D} \end{pmatrix}\right) = 0,\ C\left(\begin{pmatrix} e_{i_!C} \\ i_!g_C \end{pmatrix}\right) = 0,\ C\left(\begin{pmatrix} g_{i^!D} \\ i^!e_D \end{pmatrix}\right) = 0,$$

$$D'' = C(e_D) = S\gamma D,\ C'' = C(g_C) = S\rho C.$$

The quadratic structures follow suit, as before, so that

$$\Gamma^!\Gamma^!(x) = \sigma^a t\bar{S}(x)$$

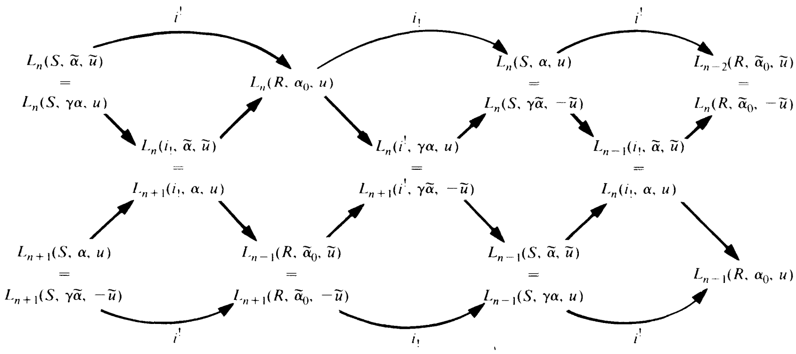
up to homotopy equivalence.

The isomorphisms

$$\Gamma_!: L_*(i_!, \tilde{\alpha}, \tilde{u}) \xrightarrow{\sim} L_{*+1}(i_!, \alpha, u),$$

$$\Gamma^!: L_*(i^!, \gamma\alpha, u) \xrightarrow{\sim} L_*(i^!, \gamma\tilde{\alpha}, -\tilde{u})$$

(using $(\gamma\alpha, u) = (\alpha, u)^{\sqrt{a}} = (\gamma\tilde{\alpha}, -\tilde{u})$) can be combined to define a commutative braid of exact sequences




(This is the Twisting Diagram (0.1) required by Hambleton, Taylor and Williams [**5**].) It follows that the chain complexes of abelian groups

$$\dots \to L_n(S, \gamma\alpha, u) \xrightarrow{i^!} L_n(R, \alpha_0, u) \xrightarrow{i_!} L_n(S, \alpha, u)$$

$$\xrightarrow{i^!\sigma^{\sqrt{a}}} L_n(R, \tilde{\alpha}_0, -\tilde{u}) \xrightarrow{i_!} L_n(S, \tilde{\alpha}, -\tilde{u}) \to \dots$$

$$\dots \to L_{n+1}(S, \gamma\tilde{\alpha}, -\tilde{u}) \xrightarrow{i^!} L_{n+1}(R, \tilde{\alpha}_0, -\tilde{u}) \xrightarrow{i_!} L_{n+1}(S, \tilde{\alpha}, -\tilde{u})$$

$$\xrightarrow{i^!\sigma^{\sqrt{a}}} L_{n+1}(R, \alpha_0, -u) \xrightarrow{i_!} L_{n+1}(S, \alpha, -u) \to \dots$$

have isomorphic homology groups. This homology isomorphism was first obtained by Harsiladze [**6**], [**7**] in the special case when $S = R[\mathbf{Z}_2]$ is the untwisted quadratic extension of $R$ and $u = \pm 1 \in R$. Indeed, it is possible to generalize the methods of [**6**], [**7**] to obtain the isomorphisms $\Gamma_!$, $\Gamma^!$ of relative $L$-groups, replacing the quadratic Poincaré complexes of Ranicki [**11**], [**12**] by the quadratic forms and formations of Ranicki [**10**].

## References


**1.** W. Browder and R. Livesay, *Fixed point free involutions on homotopy spheres*, Tohoku J. Math. *25* (1973), 69-88.

**2.** S. Cappell and J. Shaneson, *Pseudo-free group actions I*, Proc. 1978 Arhus Conference on Algebraic Topology, (Springer Lecture Notes *763*, 1979), 395-447.

**3.** ——— *A counterexample on the oozing problem for closed manifolds*, ibid. (1979), 627-634.

**4.** I. Hambleton, *Projective surgery obstructions on closed manifolds*, Proc. 1980 Oberwolfach Conference on Algebraic $K$-theory (Springer Lecture Notes *967*, 1982), 101-131.

**5.** I. Hambleton, L. Taylor and B. Williams, *An introduction to maps between surgery obstruction groups*, Proc. 1982 Arhus Conference on Algebraic Topology (Springer Lecture Notes 1051, 1984), 49-127.

6. A. F. Harsiladze, *The construction of discriminants in hermitian K-theory*, Uspekhi Mat. Nauk *34* (1979), 204-207. (English translation in Russian Math. Surveys 34:6, 179-183 (1979) ).
7. ——— *Hermitian K-theory and quadratic extensions*, Trudy Moskov. Mat. Obshch. *41* (1980), 3-36.
8. D. W. Lewis, *Exact sequences of Witt groups of equivariant forms*, L'Enseignement Mathematique *29* (1983), 45-51.
9. Lopez de Medrano, *Involutions on manifolds* (Springer, 1971).
10. A. A. Ranicki, *Algebraic L-theory I. Foundations*, Proc. London Math. Soc. *27* (1973), 101-125.
11. ——— *The algebraic theory of surgery I. Foundations*, ibid. *40* (1980), 87-192.
12. ——— *Exact sequences in the algebraic theory of surgery*, Mathematical Notes *26* (Princeton, 1981).
13. C. T. C. Wall, *Surgery on compact manifolds* (Academic Press, 1970).
14. ——— *On the axiomatic foundations of the theory of Hermitian forms*, Proc. Camb. Phil. Soc. *67* (1970), 243-250.
15. M. Warshauer, *The Witt group of degree k maps and asymmetric inner product spaces* (Springer Lecture Notes, 914, 1982).

*University of Edinburgh,*
*Edinburgh, Scotland*