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Random Young towers and quenched limit laws

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Abstract. We obtain quenched almost sure invariance principles (with convergence rates) for random Young towers if the average measure of the tail of return times to the base of random towers decays sufficiently fast. We apply our results to some independent and identically distributed perturbations of some non-uniformly expanding maps. These imply that the random systems under study tend to a Brownian motion under various scalings.

Key words: Young towers, stochastic stability, quenched invariance principles
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Contents

1. *Introduction*

A collection $(\Omega, \mathbb{P}, \sigma, (\Delta_\omega)_{\omega\in\Omega}, (\mu_\omega)_{\omega\in\Omega}, (F_\omega)_{\omega\in\Omega})$ is called a *random dynamical system* (RDS) if the following statements hold.

(1) $\sigma : \Omega \to \Omega$ is a $\mathbb{P}$-preserving transformation on a probability space $(\Omega, \mathbb{P})$.
(2) $(\Delta_\omega, \mu_\omega)$ is a probability space, called a fiber, at $\omega \in \Omega$.
(3) $F_\omega : \Delta_\omega \to \Delta_{\sigma\omega}$, is called a fiber map from Δ_ω to $\Delta_{\sigma\omega}$.
(4) $(\mu_\omega)_{\omega\in\Omega}$ are *equivariant probability measures*, namely, for almost every $\omega \in \Omega$,

$$(F_\omega)_*\mu_\omega = \mu_{\sigma\omega}. \tag{1.1}$$

In this paper we only consider an invertible $\sigma : \Omega \to \Omega$.

A decreasing series $\theta_n \searrow 0$ is called an almost sure mixing rate for the RDS if for almost every (a.e.) $\omega \in \Omega$, there is a constant $C_\omega > 0$ and a Banach space $B_\omega \subset L^1(\Delta_\omega, \mu_\omega)$ such that for any $n \in \mathbb{N}$, $\phi_\omega \in B_\omega$, $\Psi_{\sigma^n\omega} \in L^\infty(\Delta_{\sigma^n\omega}, \mu_{\sigma^n\omega})$, there is a constant $C_{\phi,\Psi} > 0$ and

$$\left| \int \phi_\omega \cdot \Psi_{\sigma^n\omega} \circ F^n_\omega \, d\mu_\omega - \int \phi_\omega \, d\mu_\omega \int \Psi_{\sigma^n\omega} \, d\mu_{\sigma^n\omega} \right| \le C_{\phi,\Psi} C_\omega \theta_n \to 0, \tag{1.2}$$

where $\phi(\omega, \cdot) := \phi_\omega(\cdot)$, $\Psi(\omega, \cdot) := \Psi_\omega(\cdot)$ are functions on $\bigcup_{\omega\in\Omega}\{\omega\} \times \Delta_\omega$. We say that an RDS has a uniform almost sure mixing rate if $\operatorname{ess\,sup}_{\omega\in\Omega} C_\omega < \infty$, and a non-uniform almost sure mixing rate if $\operatorname{ess\,sup}_{\omega\in\Omega} C_\omega = \infty$.

A random Young tower (RYT), a random extension of Young towers [**You99**], is a powerful tool to study the almost sure mixing rate for the RDS with a weak hyperbolicity. The original one was constructed by Baladi, Benedicks, and Maume-Deschamps [**BBMD02**] to obtain an almost sure mixing rate for independent and identically distributed (i.i.d.) translations of unimodal maps. In recent years, the RYT has been extended and used intensively. Du [**Du15**] extended [**BBMD02**] to a more general RYT and applied it to i.i.d. perturbations of a wider class of unimodal maps. Li and Vilarinho [**LV18**] applied the RYT [**BBMD02**] to obtain almost sure mixing rates for i.i.d. translations of some non-uniformly expanding maps. Bahsoun, Bose, and Ruziboev [**BBR19**] extended the RYT [**BBMD02**, **Du15**] under additional assumptions (see (P6) and (P7) in [**BBR19**]) and obtained an almost sure mixing rate for i.i.d. perturbations of Liverani–Saussol–Vaienti (LSV) maps.

For an RDS with a uniform almost sure mixing rate, quenched limit laws have been studied by numerous authors; see [**ALS09**, **DFGTV18a**, **DFGTV18b**, **HL20**, **HS20**, **Su19a**, **Su19b**]. However, it is quite natural and more likely to have a RDS (e.g. the RYT) that has a non-uniform almost sure mixing rate. To the best of our knowledge, three papers [**AA16**, **ANV15**, **Kif98**] do make progress for such RDSs: Abdelkader, Aimino, Nicol and Vaienti [**AA16**, **ANV15**] study an RDS with expanding average. Their approach was inspired by the paper [**ALS09**], fixing a reference measure rather than finding equivariant probability measures. However, their applications are limited; see [**ANV15**, §7]. A different approach (assuming equivariant probability measures) is given by Kifer

[**Kif98**]. He proved quenched central limit theorems (QCLTs) under various technical assumptions. However, he remarks in Proposition 2.2 and Remark 6.5 of [**Kif98**] that this method has to work on specific cases with explicit representations, even under the assumption that (Ω, σ) is Bernoulli (that is, i.i.d. perturbations).

In this paper, we will give a different method to prove a quenched limit law, called a *quenched almost sure invariance principle* (QASIP; see Definition 2.1), for the RYT provided that the average measure of the tail of return times to the base of the RYT decays sufficiently fast (see the precise statement in Definition 2.2 and Theorem 2.3). QASIP convergence rates (see (2.1)) are also obtained. The QASIP implies various quenched limit laws including the QCLT.

Finally, we mention the papers [**CDKM20**, **Kor18**] which obtained a sharp QASIP convergence rate for deterministic Young towers (that is, Ω is a singleton).

The structure of the paper is as follows. In §2 we introduce conventions, which are used throughout the paper, give the necessary definitions and formulate the main results. In §3 we revisit the RYT and improve some inequalities in [**Du15**]. In §4 we give several technical lemmas. In §5 we present a proof for the QASIP for the RYT. In §6 we obtain the QASIP for the RDS which can be modeled by the RYT. In §7 we apply our results to some i.i.d. perturbations of some non-uniformly expanding maps. We end our paper with a technical lemma in Appendix A.

2. *Conventions, definitions and main theorems*

Convention 2.1. We start with some conventions.

(1) C_a means a constant depending on a.

(2) $\mathbb{E}_{\mu_\omega}$ means the expectation with respect to (w.r.t.) μ_ω; $\mathbb{E}$ means the expectation of $\mathbb{P}$.

(3) We do not specify the σ-algebra of a measure space if it is clear from context.

(4) $a_n = O_a(b_n)$ and $a_n \precsim_a b_n$ mean that $a_n \le C_a b_n$ for a constant $C_a > 0$ and all $n \in \mathbb{N}$.

(5) $a_n = C_a^{\pm 1} b_n$ means that $C_a^{-1} b_n \le a_n \le C_a b_n$ for a constant $C_a \ge 1$ and all $n \in \mathbb{N}$.

(6) We define $\phi_\omega(\cdot) := \phi(\omega, \cdot)$ for any function $\phi : \bigcup_{\omega\in\Omega}\{\omega\} \times \Delta_\omega \to \mathbb{R}$.

Definition 2.1. (QASIP and coboundary) Consider an RDS $(\Omega, \mathbb{P}, \sigma, (\Delta_\omega)_{\omega\in\Omega}, (\mu_\omega)_{\omega\in\Omega}, (F_\omega)_{\omega\in\Omega})$ and let $\Delta := \bigcup_{\omega\in\Omega}\{\omega\} \times \Delta_\omega$. We say that a fiberwise mean zero function $\phi : \Delta \to \mathbb{R}$ (that is, $\int \phi_\omega \, d\mu_\omega = 0$) has a QASIP if there is a constant $e \in (0, 1/2)$ such that for a.e. $\omega \in \Omega$, there is a Brownian motion B^ω defined on some extension of the probability space $(\Delta_\omega, \mu_\omega)$, say $\mathbf{\Delta}_\omega$, such that

$$\sum_{k\le n} \phi_{\sigma^k\omega} \circ F_\omega^k - B^\omega_{\sigma_n^2(\omega)} = O(n^e) \quad \text{almost surely (a.s.) on } \mathbf{\Delta}_\omega, \tag{2.1}$$

where $\sigma_n^2(\omega) := \int (\sum_{k\le n} \phi_{\sigma^k\omega} \circ F_\omega^k)^2 \, d\mu_\omega < \infty$ for all $n \ge 1$ and the constant in $O(\cdot)$ depends on $\omega \in \Omega$ and $x \in \mathbf{\Delta}_\omega$. e is called a *convergence rate*.

We say that ϕ is a coboundary if there is a function $g : \Delta \to \mathbb{R}$ such that for a.e. $\omega \in \Omega$,

$$\phi_{\sigma\omega} \circ F_\omega = g_{\sigma\omega} \circ F_\omega - g_\omega \quad \mu_\omega\text{-a.s.} \tag{2.2}$$

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Definition 2.2. (Random Young towers; see [Du15]) A random Young tower (Δ, F) is constructed by the following 11 steps.

(1) Fix a probability space $(\Lambda, \mathcal{B}, m)$ and a Bernoulli scheme $(\Omega, \mathbb{P}, \sigma) := (S^{\mathbb{Z}}, \nu^{\mathbb{Z}}, \sigma)$ where ν is a probability on a measurable space S, σ is an invertible left shift on $S^{\mathbb{Z}}$ and $\mathcal{B}$ is a σ-algebra of (Λ, m).

(2) Assume that for a.e. $\omega \in \Omega$, there is a countable partition $\mathcal{P}_\omega$ of a full measure subset $\mathcal{D}_\omega$ of Λ and a function $R_\omega : \Lambda \to \mathbb{N}$ such that R_ω is constant on each $U_\omega \in \mathcal{P}_\omega$.

(3) Assume that $R_\omega(x)$ is a stopping time: if $R_\omega(x) = n$ and $\omega_i = \omega'_i$ for all $0 \le i < n$, then $R_{\omega'}(x) = n$.

(4) For a.e. $\omega \in \Omega$, $l \in \mathbb{N}$, define $\Delta_{\omega,0} := \Lambda \times \{0\}$ and the lth level by

$$\Delta_{\omega,l} := \{(x, l) : x \in \Lambda, R_{\sigma^{-l}\omega}(x) > l\}.$$

Define a tower at ω by

$$\Delta_\omega := \bigcup_{l \ge 0} \Delta_{\omega,l}.$$

Δ_ω is endowed with a measure m_ω, a σ-algebra $\mathcal{B}_\omega$ and a partition $\mathcal{Z}_\omega$ naturally from the probability space $(\Lambda, \mathcal{B}, m)$ and the partitions $(\mathcal{P}_{\sigma^{-l}\omega})_{l \ge 0}$.

(5) For a.e. $\omega \in \Omega$, a dynamics $F_\omega : \Delta_\omega \to \Delta_{\sigma\omega}$ is defined in the following way: if $R_{\sigma^{-l}\omega}(x) > l + 1$, then $F_\omega(x, l) = (x, l + 1)$; if $R_{\sigma^{-l}\omega}(x) = l + 1$, $x \in U_{\sigma^{-l}\omega} \in \mathcal{P}_{\sigma^{-l}\omega}$ and $(x, l) \in U_{\sigma^{-l}\omega} \times \{l\} \subseteq \Delta_\omega$, then F_ω maps $U_{\sigma^{-l}\omega} \times \{l\}$ bijectively onto $\Delta_{\sigma\omega,0}$.

(6) Define $F^n_\omega := F_{\sigma^{n-1}\omega} \circ F_{\sigma^{n-2}\omega} \circ \cdots \circ F_{\sigma\omega} \circ F_\omega$, assume that the partition $\mathcal{Z}_\omega$ is generating for F_ω in the sense that $\bigvee_{j=0}^{\infty} (F^j_\omega)^{-1} \mathcal{Z}_{\sigma^j\omega}$ is a trivial partition into points.

(7) Assume that for a.e. $\omega \in \Omega$, $m_\omega(\Delta_\omega) < \infty$.

(8) Assume that there is an integer $M \in \mathbb{N}$, $\{\epsilon_i > 0, i = 1, \ldots, M\}$ and $\{t_i \in \mathbb{N}, i = 1, \ldots, M\}$ with $\gcd(t_i) = 1$ such that for a.e. $\omega \in \Omega$, all $1 \le i \le M$,

$$m(x \in \Lambda : R_\omega(x) = t_i) > \epsilon_i.$$

(9) Extend R_ω from $\Delta_{\omega,0}$ to Δ_ω (still denoted by R_ω). For any $(x, l) \in \Delta_\omega$,

$$R_\omega(x, l) := R_{\sigma^{-l}\omega}(x) - l,$$

define an nth return time to Δ_ω inductively: for any $x \in \Delta_\omega$,

$$R^0_\omega(x) := 0, \quad R^1_\omega(x) := R_\omega(x), \quad \ldots$$
$$R^n_\omega(x) := R^{n-1}_\omega(x) + R_{\sigma^{R^{n-1}_\omega(x)}\omega}(F^{R^{n-1}_\omega}_\omega(x)),$$

and define a separation time $s_\omega : \Delta_\omega \times \Delta_\omega \to \mathbb{N} \cup \{\infty\}$ by

$$s_\omega(x, y) = \inf\{n : F^{R^n_\omega(x)}_\omega(x), F^{R^n_\omega(y)}_\omega(y) \text{ lie in different elements of } \mathcal{Z}_{\sigma^{R^n_\omega(x)}\omega}\}.$$

Assume that there are constants $C_F > 0$ and $\beta \in (0, 1)$ such that for a.e. $\omega \in \Omega$ and each element $J_\omega \in \mathcal{Z}_\omega$, the map $F^{R_\omega}_\omega|_{J_\omega}$ and its inverse are non-singular w.r.t.

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m, and for each $x, y \in J_\omega$,

$$\left|\frac{JF_\omega^{R_\omega}(x)}{JF_\omega^{R_\omega}(y)} - 1\right| \le C_F \beta^{s_{\sigma^{R_\omega(x)}\omega}(F_\omega^{R_\omega}(x), F_\omega^{R_\omega}(y))}. \tag{2.3}$$

(10) Assume that there is a constant $C > 0$ such that

$$\int m(x \in \Lambda : R_\omega(x) > n)\, d\mathbb{P} \le C\rho_n,$$

where $\rho_n := e^{-an^b}$ or n^{-D} for some constants $a > 0$, $b \in (0, 1]$, $D > 4$.

(11) Define a random Young tower (Δ, F):

$$\Delta := \bigcup_{\omega\in\Omega} \{\omega\} \times \Delta_\omega, \quad F(\omega, x) := (\sigma\omega, F_\omega x).$$

Remark 2.2. For a visualization of the dynamics of the RYT, see Figure 1 in [**BBR19**].

Definition 2.3. (Dynamical Lipschitz cones)

$$\mathcal{F}_\beta^+ := \Big\{\phi : \Delta \to [0, \infty) \mid \text{there is a constant } C_\phi > 0, \text{ such that for any } J_\omega \in \mathcal{Z}_\omega,$$
$$\text{either } \phi_\omega|_{J_\omega} = 0 \text{ or } \phi_\omega|_{J_\omega} > 0, \left|\log \frac{\phi_\omega(x)}{\phi_\omega(y)}\right| \le C_\phi \beta^{s_\omega(x,y)} \text{ for any } x, y \in J_\omega\Big\},$$

where C_ϕ is called a Lipschitz constant for ϕ.

Definition 2.4. (Bounded random Lipschitz functions) For any $p \in (1, \infty]$, define

$$\mathcal{F}_{\beta,p}^{\mathcal{K}} := \{\phi : \Delta \to \mathbb{R} \mid \text{there are constants } C_\phi > 0, \mathcal{K}_\omega \ge 1 \text{ such that for a.e. } \omega \in \Omega,$$
$$\mathcal{K} \in L^p(\Omega), |\phi_\omega(x)| \le C_\phi \text{ and } |\phi_\omega(x) - \phi_\omega(y)| \le C_\phi \mathcal{K}_\omega \beta^{s_\omega(x,y)}\},$$

where C_ϕ is also called a Lipschitz constant for ϕ.

THEOREM 2.3. (Quenched limit laws for the RYT) *The following results hold for* $(\Omega, \mathbb{P}, \sigma, (\Delta_\omega)_{\omega\in\Omega}, (F_\omega)_{\omega\in\Omega})$ *in Definition 2.2.*

(1) *Equivariant probability measures (1.1)* $(\mu_\omega)_{\omega\in\Omega}$ *exist. Define a probability* μ *on* Δ *by*

$$\mu(A) := \int \mu_\omega(A_\omega)\, d\mathbb{P} \tag{2.4}$$

for any measurable subset $A \subseteq \Delta$ *and* $A_\omega := \{x \in \Delta_\omega : (\omega, x) \in A\}$*. Moreover, for any fiberwise mean zero function* $\phi \in \mathcal{F}_{\beta,p}^{\mathcal{K}}$*, suppose that* ρ_n *(see Definition 2.2) is* e^{-an^b} *or* n^{-D} *for some constants* $a > 0$*,* $b \in (0, 1]$ *and* $D > 2 + (4p/(p-1))$*. Then the following statements hold.*

(2) *There is a constant* $\Sigma^2 \ge 0$ *such that* $\lim_{n\to\infty}(\sigma_n^2(\omega)/n) = \Sigma^2$ *for a.e.* $\omega \in \Omega$.

(3) *If* $\Sigma^2 > 0$*, then* ϕ *has a QASIP (see Definition 2.1). The convergence rate is* $e = \epsilon_0 + 1/4$*, where* $\epsilon_0 \in (0, 1/4)$ *satisfies the following: if* $\rho_n = e^{-an^b}$*, then* $\epsilon_0 > 0$

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can be arbitrarily small; if $\rho_n = n^{-D}$, then ϵ_0 can be any number in $(\epsilon_D, 1/4)$, where

$$\epsilon_D = \max\left\{\frac{1}{4} + \frac{3\epsilon_1 - 2\epsilon_1^3 - \epsilon_1^2}{4}, \epsilon_1, \frac{1+\epsilon_1}{4}\right\} - \frac{1}{4} \quad \text{and} \quad \epsilon_1 = \frac{2p}{(p-1)(D-2)}. \tag{2.5}$$

(4) *If $\Sigma^2 = 0$, then ϕ is a coboundary (see Definition 2.1), and the function g in (2.2) satisfies the following: if $\rho_n = n^{-D}$, then $g \in L^{(D-2-\delta)(p-1)/(1+\delta)p}(\Delta, \mu)$ for any sufficiently small $\delta > 0$ such that $(D-2-\delta)(p-1)/(1+\delta)p > 4$. In particular, if $\rho_n = e^{-an^b}$ (which implies $\rho_n \le n^{-D}$ for $n, D \gg 1$), then $g \in L^k(\Delta, \mu)$ for all $k \ge 1$.*

Remark 2.4. In §7, p is chosen to be ∞.

Remark 2.5. For any $n \ge 1$, define $S_n^\omega := \sum_{k\le n} \phi_{\sigma^k\omega} \circ F_\omega^k$, and $S^{n,\omega}$ on $[0, 1]$:

$$S_t^{n,\omega} := \frac{S_{i-1}^\omega}{\sqrt{n}} + n\left(t - \frac{i-1}{n}\right)\frac{S_i^\omega - S_{i-1}^\omega}{\sqrt{n}} \quad \text{for any } t \in \left[\frac{i-1}{n}, \frac{i}{n}\right] \text{ and } 1 \le i \le n.$$

Then the QASIP implies the following limit laws for the RYT: for a.e. $\omega \in \Omega$, we have the convergence (w.r.t. the probability μ_ω)

$$\frac{S_n^\omega}{\sqrt{n}} \to_d N(0, \Sigma^2), \quad S^{n,\omega} \to_d \Sigma B, \quad \limsup_{n\to\infty} \frac{S_n^\omega}{\sqrt{n \log\log n}} = \Sigma \text{ a.s.}$$

where B is a standard one-dimensional Brownian motion.

Remark 2.6. For the RYT with $\rho_n = n^{-D}$, Du [**Du15**] obtained a mixing rate (1.2) $\theta_n = n^{-(D-3-\epsilon)}$ for any small $\epsilon \in (0, 1)$ when $D > 4$ (see [**Du15**, Theorem 1.2.6]), while Bahsoun, Bose and Ruziboev [**BBR19**] obtained a better $\theta_n = n^{-(D-1-\epsilon)}$ for any small $\epsilon \in (0, D-1)$ when $D > 1$, under two more restrictive assumptions (P6) and (P7) in [**BBR19**]. In this paper, we only consider the general RYT in [**Du15**] for the following reasons: first, the restrictive RYT in [**BBR19**] is a special case of the general RYT in [**Du15**] when $D > 4$; second, the assumptions (P6) and (P7) for the restrictive RYT in [**BBR19**] are not satisfied by all RDSs in our applications. We believe that the conditions $D > 4$ in [**Du15**] and $D > 2 + (4p/(p-1))$ in our Theorem 2.3 are technical only, and the QASIP for the restrictive RYT in [**BBR19**] should hold for a smaller D.

3. *Random Young towers revisited*

LEMMA 3.1. (See [**Du15**]) *We have the following results for the RYT in Definition 2.2.*

(1) *There is a function $h \in \mathcal{F}_\beta^+$ and a family of absolutely continuous equivariant probability measures $d\mu_\omega := h_\omega dm_\omega$ on Δ_ω such that for a.e. $\omega \in \Omega$,*

$$(F_\omega)_*\mu_\omega = \mu_{\sigma\omega}, \quad \operatorname*{ess\,sup}_{\omega\in\Omega, x\in\Delta_\omega} h_\omega(x) < \infty, \quad h_\omega > 0, \tag{3.1}$$

$$(\Delta, F, \mu) \text{ is exact, mixing and ergodic,}$$

where μ is the probability defined in (2.4) and $h_\omega > 0$ for a.e. $\omega \in \Omega$ (that is, $m_\omega\{h_\omega = 0\} = 0$ for a.e. $\omega \in \Omega$).

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(2) *There is an integer* $l_0 > 0$ *such that for any* $l \geq l_0$, *there is a constant* $\epsilon_l \in (0, 1)$ *such that for a.e.* $\omega \in \Omega$, $m_\omega(\Delta_{\omega,0} \cap F_\omega^{-l}\Delta_{\sigma^l\omega,0}) > \epsilon_l$.

(3) *Define return times on* $\bigcup_{\omega\in\Omega}\{\omega\} \times \Delta_\omega \times \Delta_\omega$ *alternatively and recursively:*

$$\begin{aligned}
&\bar{\tau}_0^\omega(x, x') := 0, \quad \bar{\tau}_1^\omega(x, x') := R_\omega^{l_0}(x),\\
&\bar{\tau}_2^\omega(x, x') := \bar{\tau}_1^\omega(x, x') + R^{l_0}_{\sigma^{\bar{\tau}_1^\omega(x,x')}\omega}(F_\omega^{\bar{\tau}_1^\omega(x,x')}x'),\\
&\bar{\tau}_3^\omega(x, x') := \bar{\tau}_2^\omega(x, x') + R^{l_0}_{\sigma^{\bar{\tau}_2^\omega(x,x')}\omega}(F_\omega^{\bar{\tau}_2^\omega(x,x')}x),\\
&\bar{\tau}_4^\omega(x, x') := \bar{\tau}_3^\omega(x, x') + R^{l_0}_{\sigma^{\bar{\tau}_3^\omega(x,x')}\omega}(F_\omega^{\bar{\tau}_3^\omega(x,x')}x'), \ldots\\
&T^\omega(x, x') := \min\{\bar{\tau}_i^\omega(x, x'), i \geq 1 :\\
&\quad (F_\omega \times F_\omega)^{\bar{\tau}_i^\omega(x,x')}(x, x') \in \Delta_{\sigma^{\bar{\tau}_i^\omega(x,x')}\omega,0} \times \Delta_{\sigma^{\bar{\tau}_i^\omega(x,x')}\omega,0}\},\\
&T_0^\omega := 0, \quad T_1^\omega := T^\omega, \ldots\\
&T_n^\omega(x, x') := T_{n-1}^\omega(x, x') + T^{\sigma^{T_{n-1}^\omega(x,x')}\omega}((F_\omega \times F_\omega)^{T_{n-1}^\omega(x,x')}(x, x')),
\end{aligned}$$

where $R_\omega^{l_0}(x)$ *is the* l_0*th return time of* $x \in \Delta_\omega$ *to the* 0*th level.*

Let $\lambda_\omega, \lambda'_\omega$ *be absolutely continuous probability measures on* Δ_ω *whose density functions are* $d\lambda/dm, d\lambda'/dm \in \mathcal{F}_\beta^+$ *where* $d\lambda/dm(\omega, \cdot) := d\lambda_\omega/dm_\omega(\cdot)$, $d\lambda'/dm(\omega, \cdot) := d\lambda_\omega/dm_\omega(\cdot)$.

Then we have the following matching: there are constants $C = C_{\beta,F,h} > 0$, $r \in (0, 1)$ *(independent of* λ, λ'*) such that for a.e.* $\omega \in \Omega$,

$$\begin{aligned}
|(F_\omega^n)_*\lambda_\omega - (F_\omega^n)_*\lambda'_\omega| :&= \int \left|\frac{d(F_\omega^n)_*\lambda_\omega}{dm_{\sigma^n\omega}} - \frac{d(F_\omega^n)_*\lambda'_\omega}{dm_{\sigma^n\omega}}\right| dm_{\sigma^n\omega}\\
&\leq C \sum_{i\geq 0} r^i(\lambda_\omega \otimes \lambda'_\omega)(T_i^\omega \leq n < T_{i+1}^\omega).
\end{aligned} \tag{3.2}$$

(4) *If* $\rho_n = e^{-an^b}$ *or* n^{-D} *where* $a > 0$, $b \in (0, 1]$, $D > 4$, *then for any small* $\delta \in (0, 1)$, *there is a constant* $C = C_{\beta,F,\delta} > 0$ *and a small* $\alpha = \alpha_\delta > 0$ *such that*

$$\int (m_\omega \otimes m_\omega)(T^\omega_{\lfloor n^\alpha \rfloor} > n)\, d\mathbb{P} \leq Cn^{-(D-2-\delta)}. \tag{3.3}$$

Proof. See Theorem 2.2.1 and Propositions 2.3.1, 2.3.3 and 2.3.4 of [**Du15**] for the proof of (3.1). See Theorem 3.1.1 of [**Du15**] for the proof of (3.2). The proof of Proposition 2.3.4 of [**Du15**] showed that $h_\omega > 0$ for a.e. $\omega \in \Omega$ only. It does not imply a uniform lower bound $\inf_{\omega\in\Omega,x\in\Delta_\omega} h_\omega(x) > 0$. Actually, our proofs do not require such a lower bound.

Since $\rho_n \leq e^{-an^b}$ implies $\rho_n \leq n^{-D}$ for $D, n \gg 1$, we consider $\rho_n \leq n^{-D}$ only and refer to Corollary 7.1.2 of [**Du15**] for the proof of (3.3). □

LEMMA 3.2. *Consider the RYT in Definition 2.2. Suppose that* $\phi \in \mathcal{F}_{\beta,p}^{\mathcal{K}}$ *and* $\rho_n = e^{-an^b}$ *or* n^{-D} *where* $a > 0$, $b \in (0, 1]$, $D > 4$. *Define a probability*

$$d\lambda_\omega := \frac{\phi_\omega + \mathcal{K}_\omega C_\phi + 2C_\phi}{\int(\phi_\omega + \mathcal{K}_\omega C_\phi + 2C_\phi)\, d\mu_\omega}\, d\mu_\omega,$$

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where C_ϕ is a Lipschitz constant for ϕ. Then for any small $\delta \in (0, 1)$, there is a constant $C = C_{h,F,\beta,\delta}$ such that

$$\int |(F_\omega^n)_*\lambda_\omega - (F_\omega^n)_*\mu_\omega| \, d\mathbb{P} \le Cn^{-(D-2-\delta)}. \tag{3.4}$$

Proof. By (3.1) and $\mathcal{K}_\omega \ge 1$,

$$\frac{d\lambda_\omega}{dm_\omega} \le \frac{C_\phi(3+\mathcal{K}_\omega)}{C_\phi(1+\mathcal{K}_\omega)} \operatorname*{ess\,sup}_{\omega\in\Omega} h_\omega \le \frac{3+\mathcal{K}_\omega}{1+\mathcal{K}_\omega} C_h \le 2C_h, \tag{3.5}$$

where C_h is a Lipschitz constant of h. For any $x, y \in \Delta_\omega$, using the inequality $\log z \le z - 1$ when $z \ge 1$, we have

$$\begin{aligned}\left|\log\frac{d\lambda_\omega/dm_\omega(x)}{d\lambda_\omega/dm_\omega(y)}\right| &\le \left|\log\frac{h_\omega(x)}{h_\omega(y)}\right| + \left|\log\frac{\phi_\omega(x)+\mathcal{K}_\omega C_\phi + 2C_\phi}{\phi_\omega(y)+\mathcal{K}_\omega C_\phi + 2C_\phi}\right| \\ &\le C_h\beta^{s_\omega(x,y)} + \frac{|\phi_\omega(x)-\phi_\omega(y)|}{|\phi_\omega(y)+\mathcal{K}_\omega C_\phi + 2\,C_\phi|} \\ &\le C_h\beta^{s_\omega(x,y)} + \frac{C_\phi\mathcal{K}_\omega\beta^{s_\omega(x,y)}}{|\phi_\omega(y)+\mathcal{K}_\omega C_\phi + 2C_\phi|} \le (C_h+1)\beta^{s_\omega(x,y)}.\end{aligned}$$

Therefore, $d\lambda/dm \in \mathcal{F}_\beta^+$ with a Lipschitz constant $2C_h + 1$ where $d\lambda/dm(\omega, \cdot) := d\lambda_\omega/dm_\omega(\cdot)$. By (3.2) and (3.3), there are constants $C' = C_{\beta,F,\delta} > 0$, $\bar{C} = C_{\beta,F,h} > 0$, $\alpha = \alpha_\delta$ such that

$$\begin{aligned}\int |(F_\omega^n)_*\lambda_\omega - (F_\omega^n)_*\mu_\omega| \, d\mathbb{P} &\le \bar{C}\int \sum_{i=0}^{\infty} r^i(\lambda_\omega\otimes\mu_\omega)(T_i^\omega \le n < T_{i+1}^\omega)\, d\mathbb{P} \\ &= \bar{C}\int \sum_{i=\lfloor n^\alpha\rfloor}^{\infty} r^i(\lambda_\omega\otimes\mu_\omega)(T_i^\omega \le n < T_{i+1}^\omega)\, d\mathbb{P} \\ &\quad + \bar{C}\int \sum_{i=0}^{\lfloor n^\alpha\rfloor - 1} r^i(\lambda_\omega\otimes\mu_\omega)(T_i^\omega \le n < T_{i+1}^\omega)\, d\mathbb{P}.\end{aligned}$$

By (3.5) and $r \in (0, 1)$, we can continue the estimate above as

$$\le \frac{\bar{C}r^{\lfloor n^\alpha\rfloor}}{1-r} + 2C_h^2\bar{C}\int (m_\omega\otimes m_\omega)(T_{\lfloor n^\alpha\rfloor}^\omega > n)\, d\mathbb{P} \le \frac{\bar{C}r^{\lfloor n^\alpha\rfloor}}{1-r} + \frac{2C_h^2\bar{C}C'}{n^{D-2-\delta}} \le Cn^{-(D-2-\delta)}$$

where the constant C depends on $\alpha_\delta, \delta, \beta, F, h$. □

Definition 3.1. (Random transfer operators) $P_\omega : L^1(\Delta_\omega, \mu_\omega) \to L^1(\Delta_{\sigma\omega}, \mu_{\sigma\omega})$ is called a random transfer operator for $F_\omega : \Delta_\omega \to \Delta_{\sigma\omega}$ if for any $\Psi_\omega \in L^1(\Delta_\omega, \mu_\omega)$, $\Upsilon_{\sigma\omega} \in L^\infty(\Delta_{\sigma\omega}, \mu_{\sigma\omega})$,

$$\int \Psi_\omega \Upsilon_{\sigma\omega}\circ F_\omega \, d\mu_\omega = \int P_\omega(\Psi_\omega)\Upsilon_{\sigma\omega}\, d\mu_{\sigma\omega}.$$

LEMMA 3.3. (Properties of random transfer operators) *Consider the RYT in Definition 2.2. The random transfer operator P_ω for F_ω has the following expression: for a.e. $\omega \in \Omega$,*

$$(P_\omega \Psi_\omega)(x) = h_{\sigma\omega}^{-1}(x) \sum_{F_\omega(y)=x} \frac{\Psi_\omega(y) h_\omega(y)}{J F_\omega(y)} \quad \text{in } L^1(\mu_{\sigma\omega}), \tag{3.6}$$

where JF_ω is the Jacobian of F_ω w.r.t. m, $\Psi_\omega \in L^1(\Delta_\omega, \mu_\omega)$. Moreover, for any $i, k \geq 0$, any measurable functions Ψ, Υ on Δ, the following results hold for a.e. $\omega \in \Omega$.

If $\Psi \in L^\infty(\Delta, \mu)$, then

$$\| P_\omega \Psi_\omega \|_{L^\infty(\mu_{\sigma\omega})} \leq \| \Psi_\omega \|_{L^\infty(\mu_\omega)}. \tag{3.7}$$

If $\Psi \in L^1(\Delta, \mu)$, then

$$\mathbb{E}_{\mu_\omega}[\Psi_{\sigma^i\omega} \circ F_\omega^i | (F_\omega^{i+1})^{-1} \mathcal{B}_{\sigma^{i+1}\omega}] = [P_{\sigma^i\omega}(\Psi_{\sigma^i\omega})] \circ F_\omega^{i+1} \quad \text{in } L^1(\mu_\omega), \tag{3.8}$$

$$\frac{(F_\omega^i)_*(\Psi_\omega \, d\mu_\omega)}{d\mu_{\sigma^i\omega}} = P_\omega^i(\Psi_\omega) \quad \text{in } L^1(\mu_{\sigma^i\omega}). \tag{3.9}$$

If $\Psi, \Upsilon \in L^2(\Delta, \mu)$, then

$$P_\omega^{i+k}(\Psi_{\sigma^i\omega} \circ F_\omega^i \cdot \Upsilon_\omega) = P_{\sigma^i\omega}^k(\Psi_{\sigma^i\omega} \cdot P_\omega^i(\Upsilon_\omega)) \quad \text{in } L^1(\mu_{\sigma^{i+k}\omega}) \tag{3.10}$$

where $P_\omega^i := P_{\sigma^{i-1}\omega} \circ \cdots \circ P_{\sigma\omega} \circ P_\omega$.

Proof. By (3.1), $h_\omega > 0$ for a.e. $\omega \in \Omega$. Similarly to Ruelle–Perron–Frobenius operators, it is straightforward to verify (3.7)–(3.10) from Definition 3.1. To verify (3.6), let $\Psi_\omega \in L^1(\Delta_\omega, \mu_\omega)$, $\Upsilon_{\sigma\omega} \in L^\infty(\Delta_{\sigma\omega}, \mu_{\sigma\omega})$. By Definition 3.1,

$$\begin{aligned} \int P_\omega(\Psi_\omega) \cdot \Upsilon_{\sigma\omega} \, d\mu_{\sigma\omega} &= \int \Psi_\omega \cdot \Upsilon_{\sigma\omega} \circ F_\omega \, d\mu_\omega = \sum_{I_k \in \mathcal{Z}_\omega} \int_{I_k} \Psi_\omega \cdot \Upsilon_{\sigma\omega} \circ F_\omega \, d\mu_\omega \\ &= \sum_{I_k \in \mathcal{Z}_\omega} \int_{I_k} \Psi_\omega h_\omega \cdot \Upsilon_{\sigma\omega} \circ F_\omega|_{I_k} dm_\omega. \end{aligned}$$

Since F_ω is injective on $I_k \in \mathcal{Z}_\omega$, we can continue the calculation above as

$$= \sum_{I_k \in \mathcal{Z}_\omega} \int_{F_\omega(I_k)} (\Psi_\omega h_\omega) \circ F_\omega|_{I_k}^{-1} \cdot \Upsilon_{\sigma\omega} \frac{dm_\omega \circ F_\omega|_{I_k}^{-1}}{dm_{\sigma\omega}} dm_{\sigma\omega}.$$

Since $h_{\sigma\omega} > 0$ for a.e. $\omega \in \Omega$, we can continue the calculation above as

$$\begin{aligned} &= \sum_{I_k \in \mathcal{Z}_\omega} \int_{F_\omega(I_k)} (\Psi_\omega h_\omega) \circ F_\omega|_{I_k}^{-1} \cdot \Upsilon_{\sigma\omega} \frac{dm_\omega \circ F_\omega|_{I_k}^{-1}}{dm_{\sigma\omega}} h_{\sigma\omega}^{-1} \, d\mu_{\sigma\omega} \\ &= \int \sum_{I_k \in \mathcal{Z}_\omega} 1_{F_\omega(I_k)} \cdot \frac{(\Psi_\omega h_\omega)}{J F_\omega} \circ F_\omega|_{I_k}^{-1} \cdot h_{\sigma\omega}^{-1} \Upsilon_{\sigma\omega} \, d\mu_{\sigma\omega}. \end{aligned}$$

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Therefore $\sum_{I_k\in\mathcal{Z}_\omega} 1_{F_\omega(I_k)}(\Psi_\omega h_\omega)/JF_\omega\circ F_\omega|_{I_k}^{-1}h_{\sigma\omega}^{-1}$ is in $L^1(\mu_{\sigma\omega})$, finite almost everywhere on $\Delta_{\sigma\omega}$ and equal to $P_\omega(\Psi_\omega)$ in $L^1(\mu_{\sigma\omega})$. Observe that

$$h_{\sigma\omega}^{-1}(x)\sum_{F_\omega(y)=x}\frac{\Psi_\omega(y)h_\omega(y)}{JF_\omega(y)}=\sum_{I_k\in\mathcal{Z}_\omega}1_{F_\omega(I_k)}(x)\frac{(\Psi_\omega h_\omega)}{JF_\omega}\circ F_\omega|_{I_k}^{-1}(x)h_{\sigma\omega}^{-1}(x).$$

Thus our lemma holds. □

LEMMA 3.4. *Consider the RYT in Definition 2.2. Suppose that $\rho_n=e^{-an^b}$ or n^{-D} where $a>0$, $b\in(0,1]$, $D>4$ and $\phi\in\mathcal{F}_{\beta,p}^{\mathcal{K}}$. Then for any small $\delta\in(0,1)$, there is a constant $C:=C_\phi C_{h,F,\beta,\delta,p}\|\mathcal{K}\|_p$ such that*

$$\mathbb{E}\int\bigg|P_\omega^n\bigg(\phi_\omega-\int\phi_\omega\,d\mu_\omega\bigg)\bigg|\,d\mu_{\sigma^n\omega}\le Cn^{-((D-2-\delta)(p-1)/p)}.$$

Proof. Let $d\lambda_\omega:=(\phi_\omega+\mathcal{K}_\omega C_\phi+2C_\phi)/\int(\phi_\omega+\mathcal{K}_\omega C_\phi+2C_\phi)\,d\mu_\omega\,d\mu_\omega$. By (3.9) and the Hölder inequality,

$$\begin{aligned}
&\mathbb{E}\int\bigg|P_\omega^n\bigg(\phi_\omega-\int\phi_\omega\,d\mu_\omega\bigg)\bigg|\,d\mu_{\sigma^n\omega}\\
&\quad=\int\bigg|\int(\phi_\omega+2C_\phi+C_\phi\mathcal{K}_\omega)\,d\mu_\omega\bigg|\cdot|(F_\omega^n)_*\lambda_\omega-(F_\omega^n)_*\mu_\omega|\,d\mathbb{P}\\
&\quad\le 3C_\phi\mathbb{E}|(F_\omega^n)_*\lambda_\omega-(F_\omega^n)_*\mu_\omega|+C_\phi\mathbb{E}\mathcal{K}_\omega|(F_\omega^n)_*\lambda_\omega-(F_\omega^n)_*\mu_\omega|\\
&\quad\le 3C_\phi\mathbb{E}|(F_\omega^n)_*\lambda_\omega-(F_\omega^n)_*\mu_\omega|+C_\phi\|\mathcal{K}\|_p[\mathbb{E}|(F_\omega^n)_*\lambda_\omega-(F_\omega^n)_*\mu_\omega|^{p'}]^{1/p'},
\end{aligned}$$

where $1/p'=1-1/p$. Using $|(F_\omega^n)_*\lambda_\omega-(F_\omega^n)_*\mu_\omega|\le 2$ and (3.4), we continue the estimate

$$\begin{aligned}
&\le 3C_\phi C_{h,F,\beta,\delta}n^{-(D-2-\delta)}+2^{(p'-1)/p'}C_\phi\|\mathcal{K}\|_p[\mathbb{E}|(F_\omega^n)_*\lambda_\omega-(F_\omega^n)_*\mu_\omega|]^{1/p'}\\
&\le 3C_\phi C_{h,F,\beta,\delta}n^{-(D-2-\delta)}+2^{(p'-1)/p'}C_\phi\|\mathcal{K}\|_pC_{h,F,\beta,\delta}^{1/p'}n^{-(D-2-\delta)/p'}\\
&\le C_\phi C_{h,F,\beta,\delta,p}\|\mathcal{K}\|_pn^{-((D-2-\delta)(p-1)/p)},
\end{aligned}$$

where the last inequality is due to $1/p'=(p-1)/p$. □

4. *Several lemmas*

LEMMA 4.1. *Suppose that $\Psi\in L^q(\Delta,\mu)$, $q\ge 2$. Then for any $\delta>0$, for a.e. $\omega\in\Omega$,*

$$\int|\Psi_{\sigma^n\omega}\circ F_\omega^n|^q\,d\mu_\omega=O_{\omega,q}(n),\qquad\int|\Psi_{\sigma^n\omega}\circ F_\omega^n|^2\,d\mu_\omega=O_{\omega,q}(n^{2/q}),$$
$$\Psi_{\sigma^n\omega}\circ F_\omega^n(x)=O_{\omega,x,\delta}(n^{(2+\delta)/q})\quad a.s.\ x\in\Delta_\omega.$$

Proof. By Birkhoff's ergodic theorem, $\lim_{n\to 0}(\sum_{i\le n}\int|\Psi_{\sigma^i\omega}\circ F_\omega^i|^q\,d\mu_\omega/n)=\mathbb{E}\int|\Psi_\omega|^q\,d\mu_\omega<\infty$ for a.e. $\omega\in\Omega$. Thus $\int|\Psi_{\sigma^n\omega}\circ F_\omega^n|^q\,d\mu_\omega=O_{\omega,q}(n)$ and

$$\int|\Psi_{\sigma^n\omega}\circ F_\omega^n|^2\,d\mu_\omega\le\bigg(\int|\Psi_{\sigma^n\omega}\circ F_\omega^n|^q\,d\mu_\omega\bigg)^{2/q}=O_{\omega,q}(n^{2/q}).$$

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Since $\int |\Psi_{\sigma^n\omega} \circ F_\omega^n / n^{(2+\delta)/q}|^q \, d\mu_\omega = O_{\omega,q}(n/n^{2+\delta}) = O_{\omega,q}(n^{-(1+\delta)})$, by the Borel–Cantelli lemma we have $\Psi_{\sigma^n\omega} \circ F_\omega^n(x) = O_{\omega,x,\delta}(n^{(2+\delta)/q})$ a.s. $x \in \Delta_\omega$. □

LEMMA 4.2. (Martingale convergence rates) *Suppose that $\Psi \in L^q(\Delta, \mu)$, $q \geq 2$, and $(\Psi_{\sigma^n\omega} \circ F_\omega^n)_{n\geq 0}$ is a sequence of reverse martingale differences for a.e. $\omega \in \Omega$. Then for any $\delta > 0$, for a.e. $\omega \in \Omega$,*

$$\bigg\| \sum_{i\leq n} \Psi_{\sigma^i\omega} \circ F_\omega^i \bigg\|_{L^q(\mu_\omega)} = O_\omega(\sqrt{n}), \quad \sum_{i\leq n} \Psi_{\sigma^i\omega} \circ F_\omega^i(x)$$
$$= O_{x,\omega,q,\delta}(n^{(1/2)+(1+\delta/q)}) \quad \text{a.s. } x \in \Delta_\omega.$$

Proof. By the Burkholder–Davis–Gundy inequality and Minkowski inequality, there is a constant C_q such that for a.e. $\omega \in \Omega$,

$$\begin{aligned}
\bigg\| \sum_{i\leq n} \Psi_{\sigma^i\omega} \circ F_\omega^i \bigg\|_{L^q(\mu_\omega)} &\leq C_q \bigg\| \bigg(\sum_{i\leq n} \Psi^2_{\sigma^i\omega} \circ F_\omega^i \bigg)^{1/2} \bigg\|_{L^q(\mu_\omega)} \\
&\leq C_q \bigg(\sum_{i\leq n} \|\Psi^2_{\sigma^i\omega} \circ F_\omega^i\|_{L^{q/2}(\mu_\omega)} \bigg)^{1/2} \\
&= C_q \bigg(\sum_{i\leq n} \|\Psi^2_{\sigma^i\omega}\|_{L^{q/2}(\mu_{\sigma^i\omega})} \bigg)^{1/2} = O_{\omega,q}(n^{1/2}),
\end{aligned}$$

where the last equality is due to $\mathbb{E}\|\Psi^2_\omega\|_{L^{q/2}(\mu_\omega)} \leq (\mathbb{E} \int |\Psi_\omega|^q d\mu_\omega)^{2/q} < \infty$ and Birkhoff's ergodic theorem. Then for any $\delta > 0$,

$$\int \bigg| \frac{\sum_{i\leq n} \Psi_{\sigma^i\omega} \circ F_\omega^i}{n^{(1/2)+((1+\delta)/q)}} \bigg|^q d\mu_\omega = O_{\omega,q}\bigg(\frac{n^{q/2}}{n^{q/2+1+\delta}} \bigg) = O_{\omega,q}(n^{-(1+\delta)}),$$
$$\implies \sum_{i\leq n} \Psi_{\sigma^i\omega} \circ F_\omega^i(x) = O_{x,\omega,q,\delta}(n^{(1/2)+(1+\delta/q)}) \quad \text{a.s. } x \in \Delta_\omega,$$

where the last equality is due to the Borel–Cantelli lemma. □

LEMMA 4.3. *Suppose that $\psi \in L^q(\Delta, \mu)$, $q \geq 2$, satisfies*

$$\|P_\omega^n \psi_\omega\|_{L^q(\Delta,\mu)} = O_{q,\psi}(n^{-d}) \quad \text{with } d > 1.$$

Then for any $\delta > 0$, for a.e. $\omega \in \Omega$,

$$\bigg\| \sum_{i\leq n} \psi_{\sigma^i\omega} \circ F_\omega^i \bigg\|_{L^q(\mu_\omega)} = O_{\omega,\psi,q}(n^{1/2}),$$
$$\sum_{i\leq n} \psi_{\sigma^i\omega} \circ F_\omega^i(x) = O_{x,\omega,q,\delta,\psi}(n^{(1/2)+(1+\delta/q)}) \quad \text{a.s. } x \in \Delta_\omega.$$

Proof. Define $g_\omega := \sum_{i\geq 0} P^i_{\sigma^{-i}\omega}(\psi_{\sigma^{-i}\omega})$. This is well defined because

$$\bigg\| \sum_{i\geq 0} P^i_{\sigma^{-i}\omega}(\psi_{\sigma^{-i}\omega}) \bigg\|_{L^q(\Delta,\mu)} \leq \sum_{i\geq 0} \|P^i_{\sigma^{-i}\omega}(\psi_{\sigma^{-i}\omega})\|_{L^q(\Delta,\mu)} = O_{q,\psi}\bigg(\sum_{i\geq 1} i^{-d} \bigg) < \infty.$$

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By Lemma 4.1, $\|g_{\sigma^n\omega} \circ F_\omega^n\|_{L^q(\mu_\omega)} = O_{\omega,q}(n^{1/q})$. Let $\Psi_\omega := \psi_{\sigma\omega} \circ F_\omega - g_{\sigma\omega} \circ F_\omega + g_\omega$. By (3.10) we have for a.e. $\omega \in \Omega$,

$$\begin{aligned} P_\omega \Psi_\omega &= P_\omega(\psi_{\sigma\omega} \circ F_\omega) - P_\omega(g_{\sigma\omega} \circ F_\omega) + P_\omega g_\omega = \psi_{\sigma\omega} - g_{\sigma\omega} + P_\omega g_\omega \\ &= \psi_{\sigma\omega} - \sum_{i \ge 0} P^i_{\sigma^{-i}\sigma\omega}(\psi_{\sigma^{-i}\sigma\omega}) + \sum_{i \ge 0} P^{i+1}_{\sigma^{-i}\omega}(\psi_{\sigma^{-i}\omega}) = 0. \end{aligned}$$

Then by (3.8), $\mathbb{E}_{\mu_\omega}[\Psi_{\sigma^i\omega} \circ F_\omega^i | (F_\omega^{i+1})^{-1}\mathcal{B}_{\sigma^{i+1}\omega}] = [P_{\sigma^i\omega}(\Psi_{\sigma^i\omega})] \circ F_\omega^{i+1} = 0$, that is, $(\Psi_{\sigma^i\omega} \circ F_\omega^i)_{i \ge 0}$ is a sequence of reverse martingale differences w.r.t. $((F_\omega^i)^{-1}\mathcal{B}_{\sigma^i\omega})_{i \ge 0}$. Then by Lemma 4.2, we have $\| \sum_{i \le n} \Psi_{\sigma^i\omega} \circ F_\omega^i\|_{L^q(\mu_\omega)} = O_\omega(n^{1/2})$. Therefore,

$$\begin{aligned} \bigg\| \sum_{1 \le i \le n} \psi_{\sigma^i\omega} \circ F_\omega^i \bigg\|_{L^q(\mu_\omega)} &= \bigg\| \sum_{1 \le i \le n} \Psi_{\sigma^{i-1}\omega} \circ F_\omega^{i-1} + g_{\sigma^n\omega} \circ F_\omega^n - g_\omega \bigg\|_{L^q(\mu_\omega)} \\ &\le \bigg\| \sum_{1 \le i \le n} \Psi_{\sigma^{i-1}\omega} \circ F_\omega^{i-1} \bigg\|_{L^q(\mu_\omega)} + \|g_{\sigma^n\omega} \circ F_\omega^n\|_{L^q(\mu_\omega)} \\ &\quad + \|g_\omega\|_{L^q(\mu_\omega)} \\ &= O_\omega(n^{1/2}) + O_\omega(n^{1/q}) + O_\omega(1) = O_\omega(n^{1/2}). \end{aligned}$$

Then for any $\delta > 0$,

$$\begin{aligned} &\int \bigg| \frac{\sum_{i \le n} \psi_{\sigma^i\omega} \circ F_\omega^i}{n^{(1/2)+((1+\delta)/q)}} \bigg|^q \, d\mu_\omega = O_{\omega,q}\bigg(\frac{n^{q/2}}{n^{q/2+1+\delta}} \bigg) = O_{\omega,q}(n^{-(1+\delta)}), \\ &\implies \sum_{i \le n} \psi_{\sigma^i\omega} \circ F_\omega^i(x) = O_{x,\omega,q,\delta,\psi}(n^{(1/2)+(1+\delta/q)}) \quad \text{a.s. } x \in \Delta_\omega. \end{aligned}$$

where the last equality is due to the Borel–Cantelli lemma. □

LEMMA 4.4. (Regularities) *Suppose that $\phi \in \mathcal{F}^{\mathcal{K}}_{\beta,p}$ with a Lipschitz constant C_ϕ. Define $\Phi_n(\omega, \cdot) = (P_\omega^n \phi_\omega)(\cdot)$ for any $n \in \mathbb{N}$ and $C_{h,F} := C_h + e^{C_h} C_F + e^{C_h + C_F} C_h$. Then for any $n \in \mathbb{N}$,*

$$\Phi_n \in \mathcal{F}^{\mathcal{K} \circ \sigma^{-n} + C_{h,F}}_{\beta,p} \quad \textit{with a Lipschitz constant } C_\phi.$$

Proof. By Lemma 3.3, $\|P_\omega^n \phi_\omega\|_{L^\infty(\mu_{\sigma^n\omega})} < \infty$. Suppose that $x, y \in \Delta_{\sigma^n\omega,l}$ for some $l \in \mathbb{N}$ and $s_{\sigma^n\omega}(x, y) > 0$. Then by (3.6), we have

$$\begin{aligned} P_\omega^n \phi_\omega(x) &= h^{-1}_{\sigma^n\omega}(x) \sum_{F_\omega^n(z_x) = x} \phi_\omega(z_x) h_\omega(z_x) / J F_\omega^n(z_x), \\ P_\omega^n \phi_\omega(y) &= h^{-1}_{\sigma^n\omega}(y) \sum_{F_\omega^n(z_y) = y} \phi_\omega(z_y) h_\omega(z_y) / J F_\omega^n(z_y), \end{aligned}$$

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where z_x, z_y are in the same element of $\bigvee_{j=0}^{n}(F_\omega^j)^{-1}\mathcal{Z}_{\sigma^j\omega}$. Therefore,

$$\begin{aligned}
|P_\omega^n\phi_\omega(x) - P_\omega^n\phi_\omega(y)| &= \bigg|\frac{1}{h_{\sigma^n\omega}(x)}\sum_{F_\omega^n(z_x)=x}\frac{\phi_\omega(z_x)h_\omega(z_x)}{JF_\omega^n(z_x)} \\
&\quad - \frac{1}{h_{\sigma^n\omega}(y)}\sum_{F_\omega^n(z_y)=y}\frac{\phi_\omega(z_y)h_\omega(z_y)}{JF_\omega^n(z_y)}\bigg| \\
&= \bigg|\frac{1}{h_{\sigma^n\omega}(x)}\sum_{F_\omega^n(z_x)=x}\frac{(\phi_\omega(z_x)-\phi_\omega(z_y))h_\omega(z_x)}{JF_\omega^n(z_x)} \\
&\quad + \sum_{F_\omega^n(z_y)=y}\phi_\omega(z_y)\bigg(\frac{h_\omega(z_x)}{JF_\omega^n(z_x)h_{\sigma^n\omega}(x)} - \frac{h_\omega(z_y)}{JF_\omega^n(z_y)h_{\sigma^n\omega}(y)}\bigg)\bigg| \\
&\le C_\phi\mathcal{K}_\omega\beta^{s_{\sigma^n\omega}(x,y)} + C_\phi\sum_{F_\omega^n(z_y)=y}\frac{h_\omega(z_y)}{JF_\omega^n(z_y)h_{\sigma^n\omega}(y)} \\
&\quad \times \bigg|1 - \frac{h_\omega(z_x)}{JF_\omega^n(z_x)h_{\sigma^n\omega}(x)}\bigg(\frac{h_\omega(z_y)}{JF_\omega^n(z_y)h_{\sigma^n\omega}(y)}\bigg)^{-1}\bigg| \\
&\le C_\phi(\mathcal{K}\circ\sigma^{-n} + C_{h,F})_{\sigma^n\omega}\beta^{s_{\sigma^n\omega}(x,y)}
\end{aligned}$$

where the last inequality is due to $|1-z_1z_2z_3| \le |1-z_1| + |z_1||1-z_2| + |z_1||z_2||1-z_3| = |1-z_1| + e^{\ln|z_1|}|1-z_2| + e^{\ln|z_1|+\ln|z_2|}|1-z_3|$, $h \in \mathcal{F}_\beta^+$ and (2.3). □

5. *Proof of Theorem 2.3*

The equivariant probability measures $(\mu_\omega)_{\omega\in\Omega}$ have been obtained in Lemma 3.1. Thus it remains to prove the coboundary or the QASIP and its convergence rate. In Theorem 2.3 we suppose that $\phi \in \mathcal{F}_{\beta,p}^{\mathcal{K}}$, $\int \phi_\omega\, d\mu_\omega = 0$, $\rho_n = n^{-D}$ for some $D > 2 + 4p/(p-1)$. In particular, $\rho_n = e^{-an^b}$ is a special case of $\rho_n = n^{-D}$ when $a > 0$, $b \in (0, 1]$ and $D, n \gg 1$.

5.1. *Martingale decompositions.*

LEMMA 5.1. (Decompositions) *Let $\delta > 0$ be small such that $(D-2-\delta)(p-1)/(\delta+1)p > 4$, and*

$$g_\omega := \sum_{i\ge 0} P_{\sigma^{-i}\omega}^i(\phi_{\sigma^{-i}\omega}), \quad g(\omega, \cdot) := g_\omega(\cdot),$$

$$\psi_\omega := \phi_{\sigma\omega}\circ F_\omega - g_{\sigma\omega}\circ F_\omega + g_\omega, \quad \psi(\omega, \cdot) := \psi_\omega(\cdot).$$

Then $\psi, g \in L^{(D-2-\delta)(p-1)/(\delta+1)p}(\Delta, \mu) \subseteq L^4(\Delta, \mu)$ and for a.e. $\omega \in \Omega$,

$$\psi_\omega \in \ker P_\omega, \quad \sum_{1\le i\le n}\phi_{\sigma^i\omega}\circ F_\omega^i = \sum_{1\le i\le n}\psi_{\sigma^{i-1}\omega}\circ F_\omega^{i-1} + g_{\sigma^n\omega}\circ F_\omega^n - g_\omega, \tag{5.1}$$

namely, this is a martingale decomposition where $(\psi_{\sigma^i\omega} \circ F_\omega^i)_{i\geq 0}$ *are reverse martingale differences w.r.t.* $((F_\omega^i)^{-1}\mathcal{B}_{\sigma^i\omega})_{i\geq 0}$ *for a.e.* $\omega \in \Omega$*. Moreover,*

$$\int |g_{\sigma^n\omega} \circ F_\omega^n|^2 \, d\mu_\omega = O_{\omega,\delta}(n^{2(1+\delta)p/(D-2-\delta)(p-1)}) \quad \text{a.e. } \omega \in \Omega,$$
$$g_{\sigma^n\omega} \circ F_\omega^n(x) = O_{\omega,x,\delta}(n^{(2+\delta)(1+\delta)p/(D-2-\delta)(p-1)}) \quad \text{a.s. } x \in \Delta_\omega.$$

Proof. Let $q := (D-2-\delta)(p-1)/(1+\delta)p > 4$ for a small $\delta > 0$. By Lemma 3.4, (3.7) and $\phi \in L^\infty(\Delta, \mu)$,

$$\begin{aligned}\|g\|_{L^q(\Delta,\mu)} &\leq \sum_{i\geq 0} \|P^i_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})\|_{L^q(\Delta,\mu)} \\ &\leq C_\phi + \sum_{i\geq 1}\bigg[\mathbb{E}\int |P^i_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})| \, d\mu_\omega\bigg]^{1/q} C_\phi^{(q-1)/q} \\ &\leq C_\phi + C_\phi^{(q-1)/q}(C_\phi C_{h,F,\beta,\delta,p}\|\mathcal{K}\|_{L^p})^{1/q}\sum_{i\geq 1} i^{-(D-2-\delta)(p-1)/(pq)} < \infty.\end{aligned}$$

Then by Lemma 4.1,

$$\int |g_{\sigma^n\omega} \circ F_\omega^n|^2 \, d\mu_\omega = O_{\omega,\delta}(n^{2(1+\delta)p/(D-2-\delta)(p-1)}) \quad \text{a.e. } \omega \in \Omega,$$
$$g_{\sigma^n\omega} \circ F_\omega^n(x) = O_{\omega,x,\delta}(n^{(2+\delta)(1+\delta)p/(D-2-\delta)(p-1)}) \quad \text{a.s. } x \in \Delta_\omega.$$

$\psi \in L^q(\Delta, \mu)$ follows from $g \in L^q(\Delta, \mu)$ and $\phi \in \mathcal{F}^{\mathcal{K}}_{\beta,p}$. By (3.10),

$$\begin{aligned}P_\omega\psi_\omega &= P_\omega(\phi_{\sigma\omega} \circ F_\omega) - P_\omega(g_{\sigma\omega} \circ F_\omega) + P_\omega g_\omega = \phi_{\sigma\omega} - g_{\sigma\omega} + P_\omega g_\omega \\ &= \phi_{\sigma\omega} - \sum_{i\geq 0} P^i_{\sigma^{-i}\sigma\omega}(\phi_{\sigma^{-i}\sigma\omega}) + \sum_{i\geq 0} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) = 0\end{aligned}$$

for a.e. $\omega \in \Omega$. By (3.8), $\mathbb{E}_{\mu_\omega}(\psi_{\sigma^i\omega} \circ F_\omega^i | (F_\omega^{i+1})^{-1}\mathcal{B}_{\sigma^{i+1}\omega}) = [P_{\sigma^i\omega}(\psi_{\sigma^i\omega})] \circ F_\omega^{i+1} = 0$, that is, $(\psi_{\sigma^i\omega} \circ F_\omega^i)_{i\geq 0}$ are reverse martingale differences w.r.t. $((F_\omega^i)^{-1}\mathcal{B}_{\sigma^i\omega})_{i\geq 0}$. □

5.2. *A coboundary.*

LEMMA 5.2. *Let* $\eta_n^2(\omega) := \int (\sum_{i\leq n} \psi_{\sigma^i\omega} \circ F_\omega^i)^2 \, d\mu_\omega$, $\Sigma^2 := \mathbb{E}\int \psi_\omega^2 \, d\mu_\omega$ *and consider any small* $\delta > 0$ *such that* $(D-2-\delta)(p-1)/(\delta+1)p > 4$*. Then for a.e.* $\omega \in \Omega$*,*

$$\begin{aligned}\sigma_n^2(\omega) - \eta_{n-1}^2(\omega) &= O_{\omega,\delta}(n^{(1/2)+((1+\delta)p/(D-2-\delta)(p-1))}), \\ \lim_{n\to\infty}\frac{\sigma_n^2(\omega)}{n} &= \lim_{n\to\infty}\frac{\eta_n^2(\omega)}{n} = \Sigma^2.\end{aligned} \tag{5.2}$$

If $\Sigma^2 > 0$*, then there is a constant* $C'_\omega \in [1, \infty)$ *such that* $\eta_n^2(\omega) = C'^{\pm 1}_\omega n$*; if* $\Sigma^2 = 0$*, then* ϕ *is a coboundary (see (2.2)), and the function g in (2.2) is in* $L^{(D-2-\delta)(p-1)/(1+\delta)p}(\Delta, \mu)$*.*

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Proof. Let $q := (D-2-\delta)(p-1)/(1+\delta)p > 4$ for a small $\delta > 0$. By (5.1) and the Hölder inequality,

$$\sigma_n^2(\omega) - \eta_{n-1}^2(\omega) = \int \bigg(\sum_{i\le n} \phi_{\sigma^i\omega} \circ F_\omega^i\bigg)^2 \, d\mu_\omega - \int \bigg(\sum_{i\le n} \psi_{\sigma^{i-1}\omega} \circ F_\omega^{i-1}\bigg)^2 \, d\mu_\omega$$

$$= \int (g_{\sigma^n\omega} \circ F_\omega^n - g_\omega) \bigg(g_{\sigma^n\omega} \circ F_\omega^n - g_\omega + 2\sum_{i\le n} \psi_{\sigma^{i-1}\omega} \circ F_\omega^{i-1}\bigg) \, d\mu_\omega$$

$$= \int (g_{\sigma^n\omega} \circ F_\omega^n - g_\omega)^2 \, d\mu_\omega + 2\int (g_{\sigma^n\omega} \circ F_\omega^n - g_\omega) \bigg(\sum_{i\le n} \psi_{\sigma^{i-1}\omega} \circ F_\omega^{i-1}\bigg) \, d\mu_\omega$$

$$\le \int (g_{\sigma^n\omega} \circ F_\omega^n - g_\omega)^2 \, d\mu_\omega + 2\|g_{\sigma^n\omega} \circ F_\omega^n - g_\omega\|_{L^2(\mu_\omega)} \bigg\| \sum_{i\le n} \psi_{\sigma^{i-1}\omega} \circ F_\omega^{i-1} \bigg\|_{L^2(\mu_\omega)}.$$

Using Lemmas 5.1 and 4.2, we can continue the estimate: for a.e. $\omega \in \Omega$,

$$= O_{\omega,\delta}(n^{2/q}) + O_{\omega,\delta}(n^{1/q}) \bigg\| \sum_{i\le n} \psi_{\sigma^{i-1}\omega} \circ F_\omega^{i-1} \bigg\|_{L^2(\mu_\omega)}$$

$$= O_{\omega,\delta}(n^{2/q}) + O_{\omega,\delta}(n^{1/q}) O_\omega(n^{1/2}) = O_{\omega,\delta}(n^{1/q+1/2}).$$

Using results above and applying Birkhoff's ergodic theorem to $\eta_n^2(\omega)/n$, we have

$$\lim_{n\to\infty} \frac{\sigma_n^2(\omega)}{n} = \lim_{n\to\infty} \frac{\eta_n^2(\omega)}{n} = \mathbb{E} \int \psi_\omega^2 \, d\mu_\omega = \Sigma^2 \quad \text{for a.e. } \omega \in \Omega.$$

If $\Sigma^2 = \mathbb{E} \int \psi_\omega^2 \, d\mu_\omega > 0$, then there is a constant $C'_\omega \ge 1$ such that

$$\eta_n^2(\omega) = n C_\omega'^{\pm 1}. \tag{5.3}$$

If $\Sigma^2 = \mathbb{E} \int \psi_\omega^2 \, d\mu_\omega = \mathbb{E} \int (\phi_{\sigma\omega} \circ F_\omega - g_{\sigma\omega} \circ F_\omega + g_\omega)^2 \, d\mu_\omega = 0$, then for a.e. $\omega \in \Omega$,

$$\phi_{\sigma\omega} \circ F_\omega - g_{\sigma\omega} \circ F_\omega + g_\omega = 0 \quad \mu_\omega\text{-a.s.}$$

which means that ϕ is a coboundary. By Lemma 5.1, $g \in L^q(\Delta, \mu)$. □

5.3. *Approximations by Brownian motions.* From now on, we assume (5.3), that is, $\Sigma^2 = \mathbb{E} \int \psi_\omega^2 \, d\mu_\omega > 0$.

LEMMA 5.3. (Approximations for martingale differences) *Let $\epsilon \in (0, 1/2)$, $\gamma := 1/(4\epsilon)$. Define*

$$R_n(\omega) := \sum_{i\ge n} \frac{\psi_{\sigma^i\omega} \circ F_\omega^i}{\eta_i^{2\gamma}(\omega)}, \quad \delta_n^2(\omega) := \int R_n^2(\omega) \, d\mu_\omega,$$

$$\eta_n^2(\omega) := \int \bigg(\sum_{i\le n} \psi_{\sigma^i\omega} \circ F_\omega^i\bigg)^2 \, d\mu_\omega.$$

Then for a.e. $\omega \in \Omega$, there is a constant $C_{\omega,\gamma} \geq 1$, a probability space $(\mathbf{\Delta}_\omega, \mathbf{Q}_\omega)$ (which is an extension of $(\Delta_\omega, \mu_\omega)$), a Brownian motion B^ω and decreasing stopping times $\tau_i^\omega \searrow 0$ defined on $\mathbf{\Delta}_\omega$ such that

$$\delta_n^2(\omega) = C_{\omega,\gamma}^{\pm 1} \sigma_n^{2-4\gamma}(\omega) = C_{\omega,\gamma}^{\pm 1} \eta_n^{2-4\gamma}(\omega) \to 0, \tag{5.4}$$
$$R_n(\omega) = B^\omega_{\tau_n^\omega}.$$

Moreover, if

$$\tau_n^\omega - \delta_n^2(\omega) = O(\delta_n^{2+2\epsilon}(\omega)) \quad a.s. \tag{5.5}$$

then we have

$$\Big| \sum_{i \leq n} \psi_{\sigma^i\omega} \circ F_\omega^i - B^\omega_{\eta_n^2(\omega)} \Big| = O(n^{(1/4)+(3\epsilon-2\epsilon^3-\epsilon^2/4)}) \quad a.s. \tag{5.6}$$

where the constants in $O(\cdot)$ of (5.5) and (5.6) depend on ω, ϵ and $x \in \mathbf{\Delta}_\omega$.

Proof. Equation (5.3) implies that there is a constant $C_\omega \geq 1$ such that for all $n \geq 1$,

$$\eta_n^2(\omega) = C_\omega^{\pm 1} n, \quad \eta_{n+1}^2(\omega) = C_\omega^{\pm 1} \eta_n^2(\omega). \tag{5.7}$$

Since $(\psi_{\sigma^i\omega} \circ F_\omega^i)_{i\geq 1}$ are reverse martingale differences, it follows that by (5.7),

$$\delta_n^2(\omega) = \sum_{i\geq n} \frac{\int \psi^2_{\sigma^i\omega} \circ F_\omega^i \, d\mu_\omega}{\eta_i^{4\gamma}(\omega)} = \sum_{i \geq n} \frac{\eta_i^2(\omega) - \eta_{i-1}^2(\omega)}{\eta_i^{4\gamma}(\omega)} \leq \int_{\eta^2_{n-1}(\omega)}^\infty x^{-2\gamma}\, dx = \frac{\eta_{n-1}^{2-4\gamma}(\omega)}{1-2\gamma},$$

$$\delta_n^2(\omega) \geq C_\omega^{-2\gamma} \sum_{i \geq n} \frac{\eta_i^2(\omega) - \eta_{i-1}^2(\omega)}{\eta_{i-1}^{4\gamma}(\omega)} \geq C_\omega^{-2\gamma} \int_{\eta^2_{n-1}(\omega)}^\infty x^{-2\gamma}\, dx = C_\omega^{-2\gamma} \frac{\eta_{n-1}^{2-4\gamma}(\omega)}{1-2\gamma},$$

which implies (5.4) using (5.2) and (5.3). Since $(\psi_{\sigma^i\omega} \circ F_\omega^i)_{i\geq 1}$ is a sequence of reverse martingale differences and $R_n(\omega)$ is $(F_\omega^n)^{-1}\mathcal{B}_{\sigma^n\omega}$-measurable, we have that $(R_n(\omega))_{n\geq 0}$ is a reverse martingale w.r.t. $((F_\omega^n)^{-1}\mathcal{B}_{\sigma^n\omega})_{n\geq 0}$. Therefore, by Theorem 2 of [SH83], there is a probability space $(\mathbf{\Delta}_\omega, \mathbf{Q}_\omega)$ (which is an extension of $(\Delta_\omega, \mu_\omega)$), a Brownian motion B^ω, a decreasing family of stopping times $\tau_i^\omega \searrow 0$ defined on $\mathbf{\Delta}_\omega$, a decreasing family of σ-algebras $\mathcal{G}_n^\omega \supseteq \sigma\{\tau_i^\omega, (F_\omega^i)^{-1}\mathcal{B}_{\sigma^i\omega}, i \geq n\}$ and a constant $C_\omega \geq 1$ such that for any $q \geq 1$,

$$R_n(\omega) = B^\omega_{\tau_n^\omega}, \tag{5.8}$$

$$\mathbb{E}_{\mathbf{Q}_\omega}[\tau_n^\omega - \tau_{n+1}^\omega | \mathcal{G}_{n+1}^\omega] = \mathbb{E}_{\mu_\omega}\bigg[\frac{\psi_n^2 \circ F_\omega^n}{\eta_n^{4\gamma}} \bigg| (F_\omega^{n+1})^{-1}\mathcal{B}_{\sigma^{n+1}\omega}\bigg], \tag{5.9}$$

$$\mathbb{E}_{\mu_\omega}\bigg[\frac{\psi_n^{2q} \circ F_\omega^n}{\eta_n^{4q\gamma}} \bigg| (F_\omega^{n+1})^{-1}\mathcal{B}_{\sigma^{n+1}\omega}\bigg] = C_\omega^{\pm 1}\mathbb{E}_{\mathbf{Q}_\omega}[(\tau_n^\omega - \tau_{n+1}^\omega)^q | \mathcal{G}_{n+1}^\omega]. \tag{5.10}$$

Note that in Theorem 2 of [SH83], $\sigma\{\tau_i^\omega, i \geq n\} \subseteq \mathcal{G}_n^\omega$ only, but (5.9) and (5.10) still hold after conditioning on $\sigma\{\mathcal{G}_i^\omega, \tau_i^\omega, (F_\omega^i)^{-1}\mathcal{B}_{\sigma^i\omega}, i \geq n\}$. Therefore we assume without loss of generality that $\mathcal{G}_n^\omega \supseteq \sigma\{\tau_i^\omega, (F_\omega^i)^{-1}\mathcal{B}_{\sigma^i\omega}, i \geq n\}$.

By (5.8), $(B^\omega_{\tau^\omega_i} - B^\omega_{\tau^\omega_{i+1}})\eta_i^{2\gamma}(\omega) = \psi_{\sigma^i\omega} \circ F^i_\omega$. Then

$$\begin{aligned}\left|\sum_{i\le n}\psi_{\sigma^i\omega}\circ F^i_\omega - B^\omega_{\eta_n^2(\omega)}\right| &= \left|\sum_{i\le n}\psi_{\sigma^i\omega}\circ F^i_\omega - \sum_{i\le n}(B^\omega_{\delta_i^2}(\omega) - B^\omega_{\delta^2_{i+1}(\omega)})\eta_i^{2\gamma}(\omega)\right| \\ &= \left|\sum_{i\le n}[(B^\omega_{\tau^\omega_i} - B^\omega_{\tau^\omega_{i+1}}) - (B^\omega_{\delta_i^2(\omega)} - B^\omega_{\delta^2_{i+1}(\omega)})]\eta_i^{2\gamma}(\omega)\right| \\ &= \left|\sum_{i\le n}(B^\omega_{\tau^\omega_i} - B^\omega_{\delta_i^2(\omega)})(\eta_i^{2\gamma}(\omega) - \eta_{i-1}^{2\gamma}(\omega)) - (B^\omega_{\tau^\omega_{n+1}} - B^\omega_{\delta^2_{n+1}(\omega)})\eta_n^{2\gamma}(\omega)\right|.\end{aligned}$$

Since Brownian motions are locally Hölder continuous with exponent $(1-\epsilon^2)/2$, we can continue the estimate above: for a.s. $x \in \mathbf{\Delta}_\omega$,

$$\begin{aligned}\precsim_{\omega,\epsilon,x}\Bigg[\sum_{i\le n}|\tau^\omega_i - \delta_i^2(\omega)|^{(1-\epsilon^2)/2}(\eta_i^{2\gamma}(\omega) - \eta_{i-1}^{2\gamma}(\omega)) \\ + |\tau^\omega_{n+1} - \delta^2_{n+1}(\omega)|^{(1-\epsilon^2)/2}\eta_n^{2\gamma}(\omega)\Bigg].\end{aligned}$$

Using (5.5) and $\delta_n^2(\omega) \precsim_\gamma \eta_{n-1}^{2-4\gamma}(\omega)$, we can continue the estimate above: for a.s. $x \in \mathbf{\Delta}_\omega$,

$$\begin{aligned}&\precsim_{\omega,\epsilon,x}\left[\sum_{i\le n}|\delta_i^{2(1+\epsilon)}(\omega)|^{(1-\epsilon^2)/2}(\eta_i^{2\gamma}(\omega) - \eta_{i-1}^{2\gamma}(\omega)) + |\delta_{n+1}^{2(1+\epsilon)}(\omega)|^{(1-\epsilon^2)/2}\eta_n^{2\gamma}(\omega)\right] \\ &\precsim_{\omega,\epsilon,x}\Bigg[\sum_{i\le n}\eta_{i-1}^{(2-4\gamma)(1+\epsilon)(1-\epsilon^2)/2}(\omega)(\eta_i^{2\gamma}(\omega) \\ &\quad - \eta_{i-1}^{2\gamma}(\omega)) + \eta_n^{(2-4\gamma)(1+\epsilon)(1-\epsilon^2)/2}(\omega)\eta_n^{2\gamma}(\omega)\Bigg] \\ &\precsim_{\omega,\epsilon,x}\left[\int_0^{\eta_n^{2\gamma}(\omega)} x^{((2-4\gamma)(1+\epsilon)/2\gamma)(1-\epsilon^2/2)}\,dx + \eta_n^{2\gamma+(2-4\gamma)(1+\epsilon)(1-\epsilon^2)/2}(\omega)\right] \\ &= O_{\omega,\epsilon,x}(n^{\gamma-(2\gamma-1)(1+\epsilon)(1-\epsilon^2)/2}) = O_{\omega,\epsilon,x}(n^{(1/4)+(3\epsilon-2\epsilon^3-\epsilon^2)/4})\end{aligned}$$

where the last two equalities are due to (5.7) and $\gamma = 1/(4\epsilon)$. □

5.4. *Proof of the QASIP for martingale differences.* To prove the QASIP (5.6), we will verify (5.5) in Lemma 5.3: for some $\epsilon \in (0, 1/2)$,

$$\tau^\omega_n - \delta_n^2(\omega) = O(\delta_n^{2(1+\epsilon)}(\omega)) \quad \text{a.s. for a.e. } \omega \in \Omega.$$

LEMMA 5.4. (Stopping times decompositions) *The following decompositions hold:*

$$\tau^\omega_n - \delta_n^2(\omega) := R'_n(\omega) + R''_n(\omega) + S'_n(\omega), \tag{5.11}$$

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where

$$S_n'(\omega) := \sum_{i\ge n}\bigg(\frac{\psi^2_{\sigma^i\omega}\circ F^i_\omega}{\eta_i^{4\gamma}(\omega)} - \mathbb{E}_{\mu_\omega}\frac{\psi^2_{\sigma^i\omega}\circ F^i_\omega}{\eta_i^{4\gamma}(\omega)}\bigg),$$

$$R_n'(\omega) := \sum_{i\ge n}\tau_i^\omega - \tau_{i+1}^\omega - \mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^i\omega}\circ F^i_\omega}{\eta_i^{4\gamma}(\omega)}\bigg|(F_\omega^{i+1})^{-1}\mathcal{B}_{\sigma^{i+1}\omega}\bigg],$$

$$R_n''(\omega) := \sum_{i\ge n}\mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^i\omega}\circ F^i_\omega}{\eta_i^{4\gamma}(\omega)}\bigg|(F_\omega^{i+1})^{-1}\mathcal{B}_{\sigma^{i+1}\omega}\bigg] - \frac{\psi^2_{\sigma^i\omega}\circ F^i_\omega}{\eta_i^{4\gamma}(\omega)}.$$

$R'(\omega)$, $R''(\omega)$ are reverse martingales w.r.t. $(\mathcal{G}_i^\omega)_{i\ge0}$ and $((F^i_\omega)^{-1}\mathcal{B}_{\sigma^i\omega})_{i\ge0}$ respectively. And for a.e. $\omega\in\Omega$,

$$R_n'(\omega) = O(\delta_n^{2+2\epsilon}(\omega)),\quad R_n''(\omega) = O(\delta_n^{2+2\epsilon}(\omega))\quad a.s.$$

where the constants in $O(\cdot)$ depend on ω,ϵ and $x\in\mathbf{\Delta}_\omega$ and $\epsilon := 2p(1+\delta)^2/(p-1)(D-2-\delta)$ for a sufficiently small $\delta>0$ such that $\epsilon\in(0,1/2)$.

Proof. It is straightforward to verify the decompositions. We will now prove that $(R_n'(\omega))_{n\ge0}$, $(R_n''(\omega))_{n\ge0}$ are reverse martingales w.r.t. $(\mathcal{G}_i^\omega)_{i\ge0}$ and $((F^i_\omega)^{-1}\mathcal{B}_{\sigma^i\omega})_{i\ge0}$, respectively. It is obvious that $R_n^{'}(\omega)$ and $R_n^{''}(\omega)$ are measurable w.r.t. $(\mathcal{G}_i^\omega)_{i\ge0}$ and $(F^n_\omega)^{-1}\mathcal{B}_{\sigma^n\omega}$, respectively.

We now study $R_n'(\omega)$. By (5.9),

$$\begin{aligned}
&\mathbb{E}_{\mathbf{Q}_\omega}[R_n'(\omega)|\mathcal{G}^\omega_{n+1}]\\
&\quad= \mathbb{E}_{\mathbf{Q}_\omega}(\tau_n^\omega - \tau_{n+1}^\omega|\mathcal{G}^\omega_{n+1}) - \mathbb{E}_{\mathbf{Q}_\omega}\bigg\{\mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^n\omega}\circ F^n_\omega}{\eta_n^{4\gamma}(\omega)}\bigg|(F_\omega^{n+1})^{-1}\mathcal{B}_{\sigma^{n+1}\omega}\bigg]\bigg|\mathcal{G}^\omega_{n+1}\bigg\}\\
&\qquad+ \sum_{i\ge n+1}\mathbb{E}_{\mathbf{Q}_\omega}(\tau_i^\omega - \tau_{i+1}^\omega|\mathcal{G}^\omega_{n+1}) - \mathbb{E}_{\mathbf{Q}_\omega}\bigg\{\mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^i\omega}\circ F^i_\omega}{\eta_i^{4\gamma}(\omega)}\bigg|(F_\omega^{i+1})^{-1}\mathcal{B}_{\sigma^{i+1}\omega}\bigg]\bigg|\mathcal{G}^\omega_{n+1}\bigg\}\\
&\quad= \mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^n\omega}\circ F^n_\omega}{\eta_n^{4\gamma}(\omega)}\bigg|(F_\omega^{n+1})^{-1}\mathcal{B}_{\sigma^{n+1}\omega}\bigg]\\
&\qquad- \mathbb{E}_{\mathbf{Q}_\omega}\bigg\{\mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^n\omega}\circ F^n_\omega}{\eta_n^{4\gamma}(\omega)}\bigg|(F_\omega^{n+1})^{-1}\mathcal{B}_{\sigma^{n+1}\omega}\bigg]\bigg|\mathcal{G}^\omega_{n+1}\bigg\}\\
&\qquad+ \sum_{i\ge n+1}\mathbb{E}_{\mathbf{Q}_\omega}[\tau_i^\omega - \tau_{i+1}^\omega|\mathcal{G}^\omega_{n+1}] - \mathbb{E}_{\mathbf{Q}_\omega}\bigg\{\mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^i\omega}\circ F^i_\omega}{\eta_i^{4\gamma}(\omega)}\bigg|(F_\omega^{i+1})^{-1}\mathcal{B}_{\sigma^{i+1}\omega}\bigg]\bigg|\mathcal{G}^\omega_{n+1}\bigg\}.
\end{aligned}$$

Since $(F_\omega^{n+1+i})^{-1}\mathcal{B}_{\sigma^{n+1+i}\omega}\subseteq\mathcal{G}^\omega_{n+1}$ and τ^ω_{n+1+i} is $\mathcal{G}^\omega_{n+1}$-measurable for any $i\ge0$, we can continue the calculation above as

$$\begin{aligned}
&= \mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^n\omega}\circ F^n_\omega}{\eta_n^{4\gamma}(\omega)}\bigg|(F_\omega^{n+1})^{-1}\mathcal{B}_{\sigma^{n+1}\omega}\bigg] - \mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^n\omega}\circ F^n_\omega}{\eta_n^{4\gamma}(\omega)}\bigg|(F_\omega^{n+1})^{-1}\mathcal{B}_{\sigma^{n+1}\omega}\bigg]\\
&\quad+ \sum_{i\ge n+1}\tau_i^\omega - \tau_{i+1}^\omega - \mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^i\omega}\circ F^i_\omega}{\eta_i^{4\gamma}(\omega)}\bigg|(F_\omega^{i+1})^{-1}\mathcal{B}_{\sigma^{i+1}\omega}\bigg] = R_{n+1}'(\omega).
\end{aligned}$$

Therefore $(R_n'(\omega))_{n\ge0}$ is a reverse martingale w.r.t. $(\mathcal{G}_i^\omega)_{i\ge0}$.

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We now turn to $R_n''(\omega)$. Since $((F_\omega^n)^{-1}\mathcal{B}_{\sigma^n\omega})_{n\geq 0}$ is a decreasing filtration,

$$\begin{aligned}
&\mathbb{E}_{\mathbf{Q}_\omega}[R_n''(\omega)|(F_\omega^{n+1})^{-1}\mathcal{B}_{\sigma^{n+1}\omega}]\\
&= \mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^n\omega}\circ F_\omega^n}{\eta_n^{4\gamma}(\omega)}\bigg|(F_\omega^{n+1})^{-1}\mathcal{B}_{\sigma^{n+1}\omega}\bigg] - \mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^n\omega}\circ F_\omega^n}{\eta_n^{4\gamma}(\omega)}\bigg|(F_\omega^{n+1})^{-1}\mathcal{B}_{\sigma^{n+1}\omega}\bigg]\\
&\quad + \mathbb{E}_{\mathbf{Q}_\omega}\bigg\{\sum_{i\geq n+1}\bigg[\mathbb{E}_{\mathbf{Q}_\omega}\frac{\psi^2_{\sigma^i\omega}\circ F_\omega^i}{\eta_i^{4\gamma}(\omega)}\bigg|(F_\omega^{i+1})^{-1}\mathcal{B}_{\sigma^{i+1}\omega}\bigg] - \frac{\psi^2_{\sigma^i\omega}\circ F_\omega^i}{\eta_i^{4\gamma}(\omega)}\bigg|(F_\omega^{n+1})^{-1}\mathcal{B}_{\sigma^{n+1}\omega}\bigg\}\\
&= R''_{n+1}(\omega).
\end{aligned}$$

Therefore $(R_n''(\omega))_{n\geq 0}$ is a reverse martingale w.r.t. $((F_\omega^i)^{-1}\mathcal{B}_{\sigma^i\omega})_{i\geq 0}$.

We now estimate $R_n'(\omega)$. Let $q := (p-1)(D-2-\delta)/2p(1+\delta) > 2$, $\epsilon := 2p(1+\delta)^2/(p-1)(D-2-\delta) \in (0, 1/2)$ for a sufficiently small $\delta > 0$. By the Burkholder–Davis–Gundy inequality and Minkowski inequality, there is a constant C_q such that

$$\begin{aligned}
&\|R_n'(\omega)\|_{L^q(\mathbf{Q}_\omega)}\\
&\quad\leq C_q\bigg\|\sum_{i\geq n}\bigg|\tau_i^\omega - \tau_{i+1}^\omega - \mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^i\omega}\circ F_\omega^i}{\eta_i^{4\gamma}(\omega)}\bigg|(F_\omega^{i+1})^{-1}\mathcal{B}_{\sigma^{i+1}\omega}\bigg]\bigg|^2\bigg\|^{1/2}_{L^{q/2}(\mathbf{Q}_\omega)}\\
&\quad\leq C_q\bigg(\sum_{i\geq n}\bigg\|\bigg|\tau_i^\omega - \tau_{i+1}^\omega - \mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^2_{\sigma^i\omega}\circ F_\omega^i}{\eta_i^{4\gamma}(\omega)}\bigg|(F_\omega^{i+1})^{-1}\mathcal{B}_{\sigma^{i+1}\omega}\bigg]\bigg|^2\bigg\|_{L^{q/2}(\mathbf{Q}_\omega)}\bigg)^{1/2}\\
&\quad\leq \sqrt{2}C_q\bigg(\sum_{i\geq n}\bigg\||\tau_i^\omega - \tau_{i+1}^\omega|^2 + \mathbb{E}_{\mathbf{Q}_\omega}\bigg[\frac{\psi^4_{\sigma^i\omega}\circ F_\omega^i}{\eta_i^{8\gamma}(\omega)}\bigg|(F_\omega^{i+1})^{-1}\mathcal{B}_{\sigma^{i+1}\omega}\bigg]\bigg\|_{L^{q/2}(\mathbf{Q}_\omega)}\bigg)^{1/2}.
\end{aligned}$$

Using (5.10) and the fact that $(\mathbf{\Delta}_\omega, \mathbf{Q}_\omega)$ is an extension of $(\Delta_\omega, \mu_\omega)$, we can continue the estimate above as

$$\leq C_{\omega,q}\bigg(\sum_{i\geq n}\frac{\|\psi^4_{\sigma^i\omega}\circ F_\omega^i\|_{L^{q/2}(\mathbf{Q}_\omega)}}{\eta_i^{8\gamma}(\omega)}\bigg)^{1/2} = C_{\omega,q}\bigg(\sum_{i\geq n}\frac{\|\psi^4_{\sigma^i\omega}\circ F_\omega^i\|_{L^{q/2}(\mu_\omega)}}{\eta_i^{8\gamma}(\omega)}\bigg)^{1/2} \tag{5.12}$$

for some constant $C_{\omega,q} > 0$.

Let $K_n(\omega) := \sum_{i\leq n}\|\psi^4_{\sigma^i\omega}\circ F_\omega^i\|_{L^{q/2}(\mu_\omega)}$. Then $\mathbb{E}\|\psi_\omega^4\|_{L^{q/2}(\mu_\omega)} \leq (\mathbb{E}\int\psi_\omega^{2q}\,d\mu_\omega)^{2/q} < \infty$ due to Lemma 5.1 and $2/q < 1$. By Birkhoff's ergodic theorem, for a.e. $\omega\in\Omega$, there is a constant $C_\omega \geq 1$ such that $K_n(\omega) = C_\omega^{\pm 1}n$ for all $n\in\mathbb{N}$. Using (5.3) and (5.12), we have

$$\begin{aligned}
\|R_n'(\omega)\|_{L^q(\mathbf{Q}_\omega)} &\leq C_{\omega,q}\bigg(\sum_{i\geq n}\frac{K_i(\omega) - K_{i-1}(\omega)}{\eta_i^{8\gamma}(\omega)}\bigg)^{1/2}\\
&\leq C_{\omega,q}\bigg(\frac{K_{n-1}(\omega)}{\eta_n^{8\gamma}(\omega)} + \sum_{i\geq n}K_i(\omega)\frac{(\eta_{i+1}^{8\gamma}(\omega) - \eta_i^{8\gamma}(\omega))}{\eta_i^{16\gamma}(\omega)}\bigg)^{1/2}\\
&\leq C_{\omega,q}\sqrt{C_{\omega,\gamma}}\bigg(\frac{n}{n^{4\gamma}} + \int_{\eta_n^{8\gamma}(\omega)}^\infty x^{-(16\gamma-2/8\gamma)}\,dx\bigg)^{1/2} \leq C'_{\omega,q,\gamma}n^{-(4\gamma-1)/2}
\end{aligned}$$

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for some constants $C_{\omega,\gamma}, C'_{\omega,q,\gamma} > 0$. Recall that $q := (p-1)(D-2-\delta)/2p(1+\delta) > 2$, $\epsilon := 2p(1+\delta)^2/(p-1)(D-2-\delta)$ and $\gamma = (4\epsilon)^{-1}$. Using (5.4), there are constants $C_{\omega,q,\gamma}, C''_{\omega,q,\gamma} > 0$ such that

$$\left\| \frac{R'_n(\omega)}{\delta_n^{2+2\epsilon}(\omega)} \right\|^q_{L^q(\mathbf{Q}_\omega)} \le C''_{\omega,q,\gamma} \frac{n^{-(4\gamma-1/2)q}}{\sigma_n^{(1+\epsilon)(2-4\gamma)q}} \le C_{\omega,q,\gamma} \frac{n^{(2\gamma-1)(1+\epsilon)q}}{n^{(4\gamma-1)/2q}} = C_{\omega,q,\gamma} n^{-(1+\delta)}.$$

By the Borel–Cantelli lemma, we have $R'_n(\omega) = O(\delta_n^{2+2\epsilon}(\omega))$ a.s. The estimate for $R''_n(\omega)$ is similar. □

LEMMA 5.5. (Estimates for $S'_n(\omega)$ in (5.11)) *Define* $S_n(\omega) := \sum_{i\le n}(\psi^2_{\sigma^i\omega} \circ F^i_\omega - \int \psi^2_{\sigma^i\omega} \circ F^i_\omega d\mu_\omega)$, $\gamma = (4\epsilon)^{-1}$. *Suppose that*

$$S_n(\omega) = O(\eta_n^{2-(4\gamma-2)\epsilon}(\omega)) = O(n^{(\epsilon+1)/2}) \quad \mu_\omega\text{-a.s.}$$

Then $S'_n(\omega) = O(\delta_n^{2(1+\epsilon)}(\omega))\mu_\omega$*-a.s. All constants in* $O(\cdot)$ *here depend on* ω, ϵ *and* $x \in \Delta_\omega$.

Proof. Since $S'_n(\omega) := \sum_{i\ge n}(\psi^2_{\sigma^i\omega} \circ F^i_\omega - \mathbb{E}_{\mu_\omega}\psi^2_{\sigma^i\omega} \circ F^i_\omega)\eta_i^{-4\gamma}(\omega)$, we have that

$$S'_n(\omega) = \sum_{i\ge n} \frac{S_i(\omega) - S_{i-1}(\omega)}{\eta_i^{4\gamma}(\omega)} = -\frac{S_{n-1}(\omega)}{\eta_n^{4\gamma}(\omega)} + \sum_{i\ge n} S_i(\omega)(\eta_i^{-4\gamma}(\omega) - \eta_{i+1}^{-4\gamma}(\omega)).$$

Using (5.7), we can continue the calculation above as

$$= \frac{O(\eta_n^{2-(4\gamma-2)\epsilon}(\omega))}{\eta_n^{4\gamma}(\omega)} + \sum_{i\ge n} O(\eta_{i+1}^{2-(4\gamma-2)\epsilon}(\omega)) \frac{\eta_{i+1}^{4\gamma}(\omega) - \eta_i^{4\gamma}(\omega)}{\eta_{i+1}^{8\gamma}(\omega)}$$
$$= O(\eta_n^{-(4\gamma-2)(1+\epsilon)}(\omega)) + O\bigg(\int_{\eta_n^{4\gamma}(\omega)}^\infty x^{-((8\gamma-2+(4\gamma-2)\epsilon)/4\gamma)}\, dx\bigg) = O(\delta_n^{2(1+\epsilon)}(\omega))$$

where the last equality is due to (5.4). All constants in $O(\cdot)$ depend on ω, ϵ and $x \in \Delta_\omega$. □

We will use Lemma 5.5 to estimate $S_n(\omega)$ to control $S'_n(\omega)$. Using $\psi_\omega = \phi_{\sigma\omega} \circ F_\omega - g_{\sigma\omega} \circ F_\omega + g_\omega$, we have

$$\sum_{i\le n} \psi^2_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega = \sum_{i\le n} (\phi_{\sigma^i\omega} \circ F^i_\omega - g_{\sigma^i\omega} \circ F^i_\omega + g_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega)^2$$
$$= \sum_{i\le n} \phi^2_{\sigma^i\omega} \circ F^i_\omega + g^2_{\sigma^i\omega} \circ F^i_\omega + g^2_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega + 2\phi_{\sigma^i\omega} \circ F^i g_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega$$
$$- 2\phi_{\sigma^i\omega} \circ F^i_\omega g_{\sigma^i\omega} \circ F^i_\omega - 2g_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega g_{\sigma^i\omega} \circ F^i_\omega$$
$$= \sum_{i\le n} \phi^2_{\sigma^i\omega} \circ F^i_\omega - g^2_{\sigma^i\omega} \circ F^i_\omega + g^2_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega + 2\phi_{\sigma^i\omega} \circ F^i_\omega g_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega$$
$$+ 2g^2_{\sigma^i\omega} \circ F^i_\omega - 2\phi_{\sigma^i\omega} \circ F^i_\omega g_{\sigma^i\omega} \circ F^i_\omega - 2g_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega g_{\sigma^i\omega} \circ F^i_\omega.$$

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Using $\psi_\omega = \phi_{\sigma\omega} \circ F_\omega - g_{\sigma\omega} \circ F_\omega + g_\omega$ again, we can continue the calculation above as

$$= \sum_{i \le n} \phi^2_{\sigma^i\omega} \circ F^i_\omega - g^2_{\sigma^i\omega} \circ F^i_\omega + g^2_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega + 2\phi_{\sigma^i\omega} \circ F^i_\omega g_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega$$
$$- 2\psi_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega g_{\sigma^i\omega} \circ F^i_\omega$$
$$= \sum_{i \le n} \phi^2_{\sigma^i\omega} \circ F^i_\omega - g^2_{\sigma^n\omega} \circ F^n_\omega + g^2_\omega + 2\sum_{i \le n} \phi_{\sigma^i\omega} \circ F^i_\omega g_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega$$
$$- 2\sum_{i \le n} \psi_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega g_{\sigma^i\omega} \circ F^i_\omega.$$

Then we have

$$S_{n-1}(\omega) = \sum_{i \le n} (\phi^2_{\sigma^i\omega} \circ F^i_\omega - \mathbb{E}_{\mu_\omega} \phi^2_{\sigma^i\omega} \circ F^i_\omega)$$
$$+ 2\sum_{i \le n} \bigg(\phi_{\sigma^i\omega} \circ F^i_\omega g_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega - \int \phi_{\sigma^i\omega} \circ F^i_\omega g_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega \, d\mu_\omega\bigg)$$
$$- 2\sum_{i \le n} \bigg(\psi_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega g_{\sigma^i\omega} \circ F^i_\omega - \int \psi_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega g_{\sigma^i\omega} \circ F^i_\omega \, d\mu_\omega\bigg)$$
$$- g^2_{\sigma^n\omega} \circ F^n_\omega + g^2_\omega + \mathbb{E}_{\mu_\omega} g^2_{\sigma^n\omega} \circ F^n_\omega - \mathbb{E}_{\mu_\omega} g^2_\omega$$
$$= \sum_{i \le n} (\phi^2_{\sigma^i\omega} \circ F^i_\omega - \mathbb{E}_{\mu_\omega} \phi^2_{\sigma^i\omega} \circ F^i_\omega) \tag{5.13}$$
$$+ 2\sum_{i \le n} \bigg(\phi_{\sigma^i\omega} \circ F^i_\omega g_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega - \int \phi_{\sigma^i\omega} \circ F^i_\omega g_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega \, d\mu_\omega\bigg) \tag{5.14}$$
$$- 2\sum_{i \le n} \psi_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega g_{\sigma^i\omega} \circ F^i_\omega \tag{5.15}$$
$$- g^2_{\sigma^n\omega} \circ F^n_\omega + \mathbb{E}_{\mu_\omega} g^2_{\sigma^n\omega} \circ F^n_\omega + g^2_\omega - \mathbb{E}_{\mu_\omega} g^2_\omega \tag{5.16}$$

where (5.15) is due to

$$\mathbb{E}_{\mu_\omega} \psi_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega g_{\sigma^i\omega} \circ F^i_\omega = \int g_{\sigma^i\omega} \circ F^i_\omega \mathbb{E}_{\mu_\omega}[\psi_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega | (F^i_\omega)^{-1}\mathcal{B}_{\sigma^i\omega}] \, d\mu_\omega = 0.$$

To estimate $S_{n-1}(\omega)$, we will estimate (5.13), (5.14), (5.15) and (5.16).

LEMMA 5.6. (Estimates for (5.16)) *For a small* $\delta > 0$ *such that* $\epsilon := \max\{2(2+\delta)(1+\delta)p/(D-2-\delta)(p-1) - \frac{1}{2}, 0\} \in [0, 1/2)$, *we have for a.e.* $\omega \in \Omega$,

$$(5.16) = O(n^{(\epsilon+1)/2}) \quad \mu_\omega\text{-a.s.}$$

where the constant in $O(\cdot)$ *depends on* ω, ϵ *and* $x \in \Delta_\omega$.

Proof. By Lemma 5.1,

$$(5.16) = O_{\omega,x,\delta}(n^{2(2+\delta)(1+\delta)p/(D-2-\delta)(p-1)}) + O_{\omega,\delta}(n^{2(1+\delta)p/(D-2-\delta)(p-1)})$$
$$+ O_{\omega,x}(1) = O_{\omega,x,\delta}(n^{(\epsilon+1)/2})$$

where $2(2+\delta)(1+\delta)p/(D-2-\delta)(p-1) \le \epsilon + 1/2$ for a sufficiently small $\delta > 0$. □

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LEMMA 5.7. (Estimates for (5.13)) *For a small $\delta > 0$ such that $\epsilon := p(1+\delta)^2/(p-1)(D-2-\delta) \in (0, 1/4)$, we have for a.e. $\omega \in \Omega$,*

$$(5.13) = O(n^{(\epsilon+1)/2}) \quad \mu_\omega\text{-a.s.}$$

where the constant in $O(\cdot)$ depends on ω, ϵ and $x \in \Delta_\omega$.

Proof. Let $q := (D-2-\delta)(p-1)/(1+\delta)p$. Since $\phi^2_{(\cdot)} - \mathbb{E}_{\mu_{(\cdot)}}\phi^2_{(\cdot)} \in L^\infty(\Delta, \mu) \bigcap \mathcal{F}^{\mathcal{K}}_{\beta,p} \subseteq L^q(\Delta, \mu)$ with a Lipschitz constant $2C^2_\phi$, there is a $C_{\phi,q,p,h,F} > 0$ by Lemma 3.4 and (3.7) such that

$$\begin{aligned}
&\bigg(\mathbb{E}\int |P^n_\omega(\phi^2_\omega - \mathbb{E}_{\mu_\omega}\phi^2_\omega)|^q \, d\mu_\omega\bigg)^{1/q} \\
&\quad\leq (2C^2_\phi)^{(q-1)/q}\bigg(\mathbb{E}\int |P^n_\omega(\phi^2_\omega - \mathbb{E}_{\mu_\omega}\phi^2_\omega)| \, d\mu_\omega\bigg)^{1/q} \\
&\quad\leq C_{\phi,q,p,h,F}\, n^{-((D-2-\delta)(p-1)/qp)} = C_{\phi,q,p,h,F}\, n^{-(1+\delta)}.
\end{aligned}$$

By Lemma 4.3, for a.e. $\omega \in \Omega$, $(5.13) = O_{\omega,x,\delta}(n^{(1/2)+((1+\delta)/q)}) = O_{\omega,x,\delta}(n^{(\epsilon+1)/2})$ μ_ω-a.s. □

LEMMA 5.8. (Estimates for (5.15)) *For a small $\delta > 0$ such that $\epsilon := 2p(1+\delta)^2/(p-1)(D-2-\delta) \in (0, 1/2)$, we have for a.e. $\omega \in \Omega$,*

$$(5.15) = O(n^{(\epsilon+1)/2}) \quad \mu_\omega\text{-a.s.}$$

where the constant in $O(\cdot)$ depends on ω, ϵ and $x \in \Delta_\omega$.

Proof. From Lemma 5.1, for a.e. $\omega \in \Omega$, $(\psi_{\sigma^i\omega} \circ F^i_\omega)_{i\geq 0}$ and $(\psi_{\sigma^{i-1}\omega} \circ F^{i-1}_\omega \cdot g_{\sigma^i\omega} \circ F^i_\omega)_{i\geq 1}$ are reverse martingale differences w.r.t. $((F^i_\omega)^{-1}\mathcal{B}_{\sigma^i\omega})_{i\geq 0}$. Let $q := (D-2-\delta)(p-1)/2(1+\delta)p$. By Lemma 5.1 again and the Hölder inequality,

$$\begin{aligned}
&\bigg(\mathbb{E}\int |\psi_\omega \cdot g_{\sigma\omega} \circ F_\omega|^q \, d\mu_\omega\bigg)^{1/q} \\
&\quad\leq \bigg(\mathbb{E}\int |g_{\sigma\omega} \circ F_\omega|^{2q} \, d\mu_\omega\bigg)^{1/(2q)}\bigg(\mathbb{E}\int |\psi_\omega|^{2q} \, d\mu_\omega\bigg)^{1/(2q)}
\end{aligned}$$

is finite. Then by Lemma 4.2, $(5.15) = O_{x,\omega,q,\delta}(n^{(1/2)+((1+\delta)/q)}) = O_{x,\omega,q,\delta}(n^{(\epsilon+1)/2})$ μ_ω-a.s. □

LEMMA 5.9. (Estimates for (5.14)) *For any small $\delta > 0$ such that $\epsilon := 2p(1+\delta)^2/(p-1)(D-2-\delta) \in (0, 1/2)$, we have for a.e. $\omega \in \Omega$,*

$$(5.14) = O(n^{(\epsilon+1)/2}) \quad \mu_\omega\text{-a.s.}$$

where the constant in $O(\cdot)$ depends on ω, ϵ and $x \in \Delta_\omega$.

Proof. Let $q := (D-2-\delta)(p-1)/2(1+\delta)p > 2$ for a small $\delta > 0$. Denote $\Phi_\omega := \phi_{\sigma\omega} \circ F_\omega g_\omega - \int \phi_{\sigma\omega} \circ F_\omega g_\omega \, d\mu_\omega$, $\Phi(\omega, \cdot) := \Phi_\omega(\cdot)$. By Lemma 5.1, $\|\Phi\|_{L^q(\Delta,\mu)} \leq 2C_\phi\|g\|_{L^q(\Delta,\mu)} < \infty$.

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Therefore, by the Minkowski inequality and (3.10),

$$\bigg(\mathbb{E}\int |P^k_\omega(\Phi_\omega)|^q \, d\mu_{\sigma^k\omega}\bigg)^{1/q}$$

$$= \bigg(\mathbb{E}\int \bigg|P^k_\omega\bigg[\phi_{\sigma\omega}\circ F_\omega g_\omega - \int \phi_{\sigma\omega}\circ F_\omega g_\omega \, d\mu_\omega\bigg]\bigg|^q \, d\mu_{\sigma^k\omega}\bigg)^{1/q}$$

$$\leq \sum_{i\geq 0}\bigg(\mathbb{E}\int \bigg|P^k_\omega\bigg[\phi_{\sigma\omega}\circ F_\omega P^i_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) - \int \phi_{\sigma\omega}\circ F_\omega P^i_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) \, d\mu_\omega\bigg]\bigg|^q \, d\mu_{\sigma^k\omega}\bigg)^{1/q}$$

$$\leq \sum_{i\geq 0}\bigg(\mathbb{E}\int \bigg|P^{k-1}_{\sigma\omega}\bigg[\phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) - \int \phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) \, d\mu_{\sigma\omega}\bigg]\bigg|^q \, d\mu_{\sigma^k\omega}\bigg)^{1/q}$$

$$= \sum_{i< k}\bigg(\mathbb{E}\int \bigg|P^{k-1}_{\sigma\omega}\bigg[\phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) - \int \phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) \, d\mu_{\sigma\omega}\bigg]\bigg|^q \, d\mu_{\sigma^k\omega}\bigg)^{1/q}$$
$$+ \sum_{i\geq k}\bigg(\mathbb{E}\int \bigg|P^{k-1}_{\sigma\omega}\bigg[\phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) - \int \phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) \, d\mu_{\sigma\omega}\bigg]\bigg|^q \, d\mu_{\sigma^k\omega}\bigg)^{1/q}.$$

By (3.7), $\|\phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})\|_{L^\infty(\mu_{\sigma\omega})} \leq C^2_\phi$. By Definition 3.1,

$$\int |P^{k-1}_{\sigma\omega}[\phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})]| \, d\mu_{\sigma^k\omega}$$
$$= \sup_{\xi:\|\xi\|_\infty\leq 1}\int \xi P^{k-1}_{\sigma\omega}[\phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})] \, d\mu_{\sigma^k\omega}$$
$$= \sup_{\xi:\|\xi\|_\infty\leq 1}\int \xi\circ F^{k-1}_{\sigma\omega}\cdot \phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) \, d\mu_{\sigma\omega}$$
$$\leq C_\phi \int |P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})| \, d\mu_{\sigma\omega}.$$

Then we can continue the estimate: there are constants $C_{\phi,q}, C'_{\phi,q} > 0$ such that

$$\leq C'_{\phi,q}\sum_{i< k}\bigg(\mathbb{E}\int \bigg|P^{k-1}_{\sigma\omega}\bigg[\phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) - \int \phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) \, d\mu_{\sigma\omega}\bigg]\bigg| \, d\mu_{\sigma^k\omega}\bigg)^{1/q}$$

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$$+ C'_{\phi,q} \sum_{i\geq k} \bigg(\mathbb{E}\int \bigg|P^{k-1}_{\sigma\omega}\bigg[\phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) - \int \phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})\, d\mu_{\sigma\omega}\bigg]\bigg|\, d\mu_{\sigma^k\omega}\bigg)^{1/q}$$

$$\leq C_{\phi,q} \sum_{i<k} \bigg(\mathbb{E}\int \bigg|P^{k-1}_{\sigma\omega}\bigg[\phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega}) - \int \phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})\, d\mu_{\sigma\omega}\bigg]\bigg|\, d\mu_{\sigma^k\omega}\bigg)^{1/q}$$

$$+ C_{\phi,q} \sum_{i\geq k} \bigg(\mathbb{E}\int \big|P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})\big|\, d\mu_{\sigma\omega} + \mathbb{E}\int \big|P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})\big|\, d\mu_{\sigma\omega}\bigg)^{1/q}.$$

To continue the estimate, we need the regularity of $\phi_{\sigma\omega} P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})$: by Lemma 4.4 and (3.7), for any $x, y \in \Delta_{\sigma\omega}$,

$$\begin{aligned}
&|\phi_{\sigma\omega}(x) P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})(x) - \phi_{\sigma\omega}(y) P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})(y)| \\
&\leq |\phi_{\sigma\omega}(x) - \phi_{\sigma\omega}(y)| C_\phi + |P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})(y) - P^{i+1}_{\sigma^{-i}\omega}(\phi_{\sigma^{-i}\omega})(x)| C_\phi \\
&\leq \mathcal{K}_{\sigma\omega}\beta^{s_{\sigma\omega}(x,y)} C_\phi^2 + (\mathcal{K}_{\sigma^{-i}\omega} + C_{h,F})\beta^{s_{\sigma\omega}(x,y)} C_\phi^2.
\end{aligned}$$

Thus $\phi_{(\cdot)} P^{i+1}_{\sigma^{-(i+1)}(\cdot)}(\phi_{\sigma^{-(i+1)}(\cdot)}) \in \mathcal{F}^{\mathcal{K}+\mathcal{K}\circ\sigma^{-(i+1)}+C_{h,F}}_{\beta,p}$ with a Lipschitz constant C_ϕ^2. Now we can continue our estimate: by Lemma 3.4, there is a constant $C = [C_\phi^2 C_{h,F,\beta,\delta,p} \|\mathcal{K} + \mathcal{K}\circ\sigma^{-(i+1)} + C_{h,F}\|_{L^p}]^{1/q} \leq [C_\phi^2 C_{h,F,\beta,\delta,p}(2\|\mathcal{K}\|_{L^p} + C_{h,F})]^{1/q}$ such that

$$\begin{aligned}
&\bigg(\mathbb{E}\int \big|P^k_\omega(\Phi_\omega)\big|^q\, d\mu_{\sigma^k\omega}\bigg)^{1/q} \\
&\leq CC_{\phi,q} \sum_{i<k} (k-1)^{-((D-2-\delta)(p-1)/qp)} + 2CC_{\phi,q} \sum_{i\geq k} i^{-((D-2-\delta)(p-1)/qp)} \\
&\leq C_{\phi,q,\delta,p}(k-1)^{-((D-2-\delta)(p-1)/qp)+1} \precsim_{\phi,q,\delta,p} k^{-(2\delta+1)}
\end{aligned}$$

for a constant $C_{\phi,q,\delta,p} > 0$. Therefore, by Lemma 4.3, $\sum_{i\leq n}(\phi_{\sigma^i\omega}\circ F^i_\omega \cdot g_{\sigma^{i-1}\omega}\circ F^{i-1}_\omega - \int \phi_{\sigma^i\omega}\circ F^i_\omega \cdot g_{\sigma^{i-1}\omega}\circ F^{i-1}_\omega\, d\mu_\omega) = O_{x,\omega,q,\delta}(n^{(1/2)+((1+\delta)/q)}) = O_{x,\omega,q,\delta}(n^{(\epsilon+1)/2})$ μ_ω-a.s. □

We now turn to the QASIP for martingale differences $(\psi_{\sigma^i\omega}\circ F^i_\omega)_{i\geq 0}$.

LEMMA 5.10. (QASIP for $(\psi_{\sigma^i\omega}\circ F^i_\omega)_{i\geq 0}$) *For any sufficiently small $\delta > 0$ such that $\epsilon = 2p(1+\delta)^2/(p-1)(D-2-\delta) \in (0, 1/2)$, we have for a.e. $\omega \in \Omega$,*

$$\bigg|\sum_{i\leq n} \psi_{\sigma^i\omega}\circ F^i_\omega - B^\omega_{\eta_n^2(\omega)}\bigg| = O(n^{(1/4)+(3\epsilon-2\epsilon^3-\epsilon^2/4)}) \quad a.s.$$

where the constant in $O(\cdot)$ depends on ω, ϵ and $x \in \mathbf{\Delta}_\omega$.

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Proof. By Lemmas 5.6–5.9, we have for a small enough $\delta > 0$ and a.e. $\omega \in \Omega$,

$$S_n(\omega) = O_{\omega,\epsilon,x}(n^{(\epsilon+1)/2}) \quad \mu_\omega\text{-a.s.}$$

where $\epsilon = \max\{2p(1+\delta)^2/(p-1)(D-2-\delta), \max\{(2(2+\delta)(1+\delta)p/(D-2)(p-1)) - \frac{1}{2}, 0\}, p(1+\delta)^2/(p-1)(D-2-\delta)\} = 2p(1+\delta)^2/(p-1)(D-2-\delta)$.

By Lemma 5.5, $S'_n(\omega) = O_{\omega,\epsilon,x}(\delta_n^{2+2\epsilon})\mu_\omega$-a.s. By Lemma 5.4,

$$\tau_n^\omega - \delta_n^2(\omega) = O_{\omega,\epsilon,x}(\delta_n^{2+2\epsilon}(\omega)) \quad \text{a.s.}$$

By Lemma 5.3, $|\sum_{i\le n} \psi_{\sigma^i\omega} \circ F_\omega^i - B^\omega_{\eta_n^2(\omega)}| = O_{\omega,\epsilon,x}(n^{(1/4)+(3\epsilon-2\epsilon^3-\epsilon^2)/4})$ a.s. □

5.5. *Proof of the QASIP for Birkhoff sums.*

LEMMA 5.11. (QASIP for $(\phi_{\sigma^i\omega} \circ F_\omega^i)_{i\ge 0}$) *For a.e. $\omega \in \Omega$,*

$$\left| \sum_{i\le n} \phi_{\sigma^i\omega} \circ F_\omega^i - B^\omega_{\sigma_n^2(\omega)} \right| = O(n^{\epsilon_0+1/4}) \quad \textit{a.s.}$$

where the constant in $O(\cdot)$ depends on ω, ϵ and $x \in \mathbf{\Delta}_\omega$.

If $\rho_n = n^{-D}$, then ϵ_0 is any number in $(\epsilon_D, 1/4)$, where ϵ_D is defined in (2.5). In particular, if $\rho_n = e^{-an^b}$ for some $a > 0$, $b \in (0, 1]$ (which implies $\rho_n \le n^{-D}$ for $n, D \gg 1$), then ϵ_D can be arbitrarily small, as is $\epsilon_0 > 0$.

Proof. Since $\epsilon_1 = 2p/(p-1)(D-2) \in (0, 1/2)$, there is a small $\delta > 0$ such that $\epsilon = 2p(1+\delta)^2/(p-1)(D-2-\delta) \in (\epsilon_1, 1/2)$. By Lemmas 5.1 and 5.10,

$$\begin{aligned}
\sum_{1\le i\le n} \phi_{\sigma^i\omega} \circ F_\omega^i &= \sum_{1\le i\le n} \psi_{\sigma^{i-1}\omega} \circ F_\omega^{i-1} + g_{\sigma^n\omega} \circ F_\omega^n - g_\omega \\
&= \sum_{1\le i\le n} \psi_{\sigma^{i-1}\omega} \circ F_\omega^{i-1} + O(n^{(2+\delta)(1+\delta)p/(D-2-\delta)(p-1)}) + O(1) \\
&= B^\omega_{\eta_{n-1}^2(\omega)} + O(n^{(1/4)+((3\epsilon-2\epsilon^3-\epsilon^2)/4)}) + O(n^\epsilon) + O(1) \\
&= B^\omega_{\eta_{n-1}^2(\omega)} + O(n^{\max\{(1/4)+((3\epsilon-2\epsilon^3-\epsilon^2)/4)\epsilon\}}).
\end{aligned}$$

Using (5.2) and the basic property of Brownian motion, we can continue the estimate above as

$$\begin{aligned}
&= B^\omega_{\sigma_n^2(\omega)} + O(n^{(1/4)+((1+\delta)^2 p/2(D-2-\delta)(p-1))}) + O(n^{\max\{(1/4)+((3\epsilon-2\epsilon^3-\epsilon^2)/4),\epsilon\}}) \\
&= B^\omega_{\sigma_n^2(\omega)} + O(n^{(1+\epsilon)/4}) + O(n^{\max\{(1/4)+((3\epsilon-2\epsilon^3-\epsilon^2)/4),\epsilon\}}) \\
&= B^\omega_{\sigma_n^2(\omega)} + O(n^{\max\{(1/4)+((3\epsilon-2\epsilon^3-\epsilon^2)/4),\epsilon,(1+\epsilon)/4\}}) = B^\omega_{\sigma_n^2(\omega)} + O(n^{(1/4)+\epsilon_0}) \quad \text{a.s.}
\end{aligned}$$

where $\epsilon_0 := \max\{\frac{1}{4} + (3\epsilon - 2\epsilon^3 - \epsilon^2/4), \epsilon, (1+\epsilon)/4\} - \frac{1}{4} \in (\epsilon_D, 1/4)$. All constants in $O(\cdot)$ depend on ω, ϵ and $x \in \mathbf{\Delta}_\omega$. □

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6. *Projection from towers*

In this section we consider the RDS which can be described by the RYT.

Definition 6.1. (Induced random Markov maps) We say that $(f_\omega)_{\omega\in\Omega}$ are induced random Markov maps if they satisfy the following conditions.

(1) Let $(\Omega, \mathbb{P}, \sigma) := (I^{\mathbb{Z}}, \nu^{\mathbb{Z}}, \sigma)$ be a Bernoulli scheme where (I, ν) is a probability space and σ is an invertible left shift on $I^{\mathbb{Z}}$. (M, Leb, d) is a compact Riemannian manifold with a Riemannian volume Leb and a Riemannian distance d. $(f_\omega)_{\omega\in\Omega}$ are non-singular random transformations w.r.t. Leb on M. Define $f_\omega^n := f_{\sigma^{n-1}\omega} \circ f_{\sigma^{n-2}\omega} \circ \cdots \circ f_{\sigma\omega} \circ f_\omega$.

(2) Assume that Λ is an open geodesic ball in M, with a normalized probability m inherited from Leb.

(3) Assume that for a.e. $\omega \in \Omega$, there is a countable partition $\mathcal{P}_\omega$ of a full measure subset $\mathcal{D}_\omega$ of Λ and a function $R_\omega : \Lambda \to \mathbb{N}$ such that R_ω is constant on each $U_\omega \in \mathcal{P}_\omega$, $R_\omega(x)$ is a stopping time (see Definition 2.2) and $f_\omega^{R_\omega}|_{U_\omega}$ is a diffeomorphism from U_ω to Λ.

(4) Assume that there is an integer $N \in \mathbb{N}$, $\{\epsilon_i > 0, i = 1, \ldots, N\}$ and $\{t_i \in \mathbb{N}, i = 1, \ldots, N\}$ with $\gcd(t_i) = 1$ such that for a.e. $\omega \in \Omega$, $m(x \in \Lambda : R_\omega(x) = t_i) > \epsilon_i$ for all $1 \le i \le N$.

(5) Assume that there are constants $\beta \in (0, 1)$, $C \ge 1$ and a function $\mathcal{K} \in L^p(\Omega)$ ($\mathcal{K} \ge 1$, $p \in (1, \infty]$) such that for a.e. $\omega \in \Omega$, any $U_\omega \in \mathcal{P}_\omega$, $x, y \in U_\omega$, and $0 \le k \le R_\omega|_{U_\omega}$,

$$d(f_\omega^{R_\omega}(x), f_\omega^{R_\omega}(y)) \ge \beta^{-1} d(x, y), \tag{6.1}$$

$$\left| \log \frac{Jf_\omega^{R_\omega}(x)}{Jf_\omega^{R_\omega}(y)} \right| \le C d(f_\omega^{R_\omega}(x), f_\omega^{R_\omega}(y)), \tag{6.2}$$

$$d(f_\omega^k(x), f_\omega^k(y)) \le C\mathcal{K}_{\sigma^k\omega} d(f_\omega^{R_\omega}(x), f_\omega^{R_\omega}(y)). \tag{6.3}$$

(6) Assume that there is a constant $C > 0$ and a decreasing sequence $(\rho_n)_{n\ge1}$ such that

$$\int m(x \in \Lambda : R_\omega(x) > n)\, d\mathbb{P} \le C\rho_n \searrow 0.$$

THEOREM 6.1. (Quenched limit laws for the RDS) *Let $M_\omega := M$. Then the following statements hold for $(\Omega, \mathbb{P}, \sigma, (M_\omega)_{\omega\in\Omega}, (f_\omega)_{\omega\in\Omega})$ in Definition 6.1.*

(1) *Equivariant probability measures (1.1) $(\upsilon_\omega)_{\omega\in\Omega}$ exist. Define a probability υ by*

$$\upsilon(A) := \int \upsilon_\omega(A_\omega)\, d\mathbb{P}$$

for any measurable subset $A \subseteq \Omega \times M$ and $A_\omega := \{x \in M : (\omega, x) \in A\}$. For any Hölder function φ on M with a Hölder exponent $\gamma \in (0, 1]$, define

$$\varphi_\omega := \varphi - \int \varphi d\upsilon_\omega, \quad \Phi(\omega, \cdot) := \varphi_\omega(\cdot), \quad \sigma_n^2(\omega) := \int \bigg(\sum_{k\le n} \varphi_{\sigma^k\omega} \circ f_\omega^k \bigg)^2 d\upsilon_\omega.$$

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Suppose that $\rho_n := e^{-an^b}$ or n^{-D} for some constants $a > 0$, $b \in (0, 1]$, $D > 2 + (4p/(p-\gamma))$, $\gamma \in (0, 1]$. Then the following statements hold for the RDS $(\Omega, \mathbb{P}, \sigma, (M_\omega)_{\omega\in\Omega}, (\upsilon_\omega)_{\omega\in\Omega}, (f_\omega)_{\omega\in\Omega})$.

(2) *There is a constant $\Sigma^2 \geq 0$ such that $\lim_{n\to\infty}(\sigma_n^2(\omega)/n) = \Sigma^2$ a.e. $\omega \in \Omega$.*

(3) *If $\Sigma^2 > 0$, then Φ has the QASIP (see Definition 2.1). The convergence rate is $e = \epsilon_0 + 1/4$, where $\epsilon_0 \in (0, 1/4)$ satisfies the following: if $\rho_n = e^{-an^b}$, then $\epsilon_0 > 0$ is any small number; if $\rho_n = n^{-D}$, then ϵ_0 is any number in $(\epsilon_D, 1/4)$, where ϵ_D is defined in (2.5) with a different $\epsilon_1 = 2p/(p-\gamma)(D-2)$.*

(4) *If $\Sigma^2 = 0$, then Φ is a coboundary (see Definition 2.1). The function g in (2.2) satisfies the following: if $\rho_n = n^{-D}$, then $g \in L^{(D-2-\delta)(p-\gamma)/(1+\delta)p}(\upsilon)$ for any small $\delta > 0$ such that $(D-2-\delta)(p-\gamma)/(1+\delta)p > 4$. In particular, if $\rho_n = e^{-an^b}$ (which implies $\rho_n \leq n^{-D}$ for $n, D \gg 1$), then $g \in L^k(\upsilon)$ for all $k \geq 1$.*

Proof of Theorem 6.1. We identify the 0th levels of all $(\Delta_\omega)_{\omega\in\Omega}$ with the base Λ and denote $A := \sup_{x,y\in M} d(x, y)$. From Definition 6.1, we can construct Δ and F such that $F_\omega^{R_\omega} = f_\omega^{R_\omega}$. To show that (Δ, F) is an RYT, we just need to verify the distortion (2.3) from (6.2): if the separation time for $x, y \in \Delta_\omega$ is $s_\omega(x, y) = n$, then for any $i < n$, $F_\omega^{R_\omega^i(x)}(x), F_\omega^{R_\omega^i(y)}(y)$ lie in the same element of $\mathcal{P}_{\sigma^{R_\omega^i(x)}\omega}$ and $F_\omega^{R_\omega^n(x)}(x), F_\omega^{R_\omega^n(y)}(y)$ lie in different elements of $\mathcal{P}_{\sigma^{R_\omega^n(x)}\omega}$. By (6.1) and (6.2),

$$d(x, y) \leq \beta d(f_\omega^{R_\omega}(x), f_\omega^{R_\omega}(y)) \leq \cdots \leq \beta^n d(f_\omega^{R_\omega^n}(x), f_\omega^{R_\omega^n}(y)) \leq \beta^n A,$$

$$\left|\log \frac{JF_\omega^{R_\omega}(x)}{JF_\omega^{R_\omega}(y)}\right| \leq Cd(f_\omega^{R_\omega}(x), f_\omega^{R_\omega}(y)) \leq CA\beta^{n-1},$$

that is, there is a constant $C' > 0$ such that

$$\left|\frac{JF_\omega^{R_\omega}(x)}{JF_\omega^{R_\omega}(y)} - 1\right| \leq C'\beta^{s_{\sigma^{R_\omega(x)}\omega}(F_\omega^{R_\omega}(x), F_\omega^{R_\omega}(y))}.$$

Thus, by Definition 2.2 and Lemma 3.1, (Δ, F) is an RYT and there are equivariant probability measures $(\mu_\omega)_{\omega\Omega}$ for (Δ, F). Define a projection $\pi_\omega : \Delta_\omega \to M$ by

$$\pi_\omega(x, l) := f_{\sigma^{-l}\omega}^l(x)$$

which is a random semiconjugacy: $f_\omega \circ \pi_\omega = \pi_{\sigma\omega} \circ F_\omega$. Therefore

$$(\upsilon_\omega)_{\omega\in\Omega} := ((\pi_\omega)_*\mu_\omega)_{\omega\in\Omega}$$

are equivariant probability measures (see §3.1 in [**BBR19**]).

Define functions $\phi_\omega := \varphi_\omega \circ \pi_\omega$ and $\phi(\omega, \cdot) := \phi_\omega(\cdot)$ satisfying

$$\|\phi_\omega\|_{L^\infty(\Delta_\omega)} \leq \max_{x\in M} |\varphi(x)|, \qquad \int \phi_\omega \, d\mu_\omega = 0.$$

Claim. $\phi \in \mathcal{F}_{\beta^\gamma, p/\gamma}^{\mathcal{K}^\gamma}$ with a Lipschitz constant $C_\varphi C^\gamma A^\gamma(\beta^\gamma)^{-1}$.

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This claim holds since for any $(x,l),(y,l)\in\Delta_\omega$ with $s_\omega((x,l),(y,l))=n$, we have, using (6.3),

$$\begin{aligned}|\phi_\omega(x,l)-\phi_\omega(y,l)| &= |\varphi_\omega(f^l_{\sigma^{-l}\omega}x)-\varphi_\omega(f^l_{\sigma^{-l}\omega}y)|\le C_\varphi d(f^l_{\sigma^{-l}\omega}x, f^l_{\sigma^{-l}\omega}y)^\gamma\\ &\le C_\varphi C^\gamma \mathcal{K}^\gamma_\omega d(f^{R_{\sigma^{-l}\omega}}_{\sigma^{-l}\omega}(x), f^{R_{\sigma^{-l}\omega}}_{\sigma^{-l}\omega}(y))^\gamma\\ &\le C_\varphi C^\gamma \mathcal{K}^\gamma_\omega A^\gamma(\beta^\gamma)^{-1}(\beta^\gamma)^{s_\omega((x,l),(y,l))}.\end{aligned}$$

We now apply Theorem 2.3 to ϕ.

(1) There is a constant $\Sigma^2\ge 0$ such that

$$\lim_{n\to\infty}\frac{\int(\sum_{k\le n}\varphi_{\sigma^k\omega}\circ f^k_\omega)^2 d\upsilon_\omega}{n}=\lim_{n\to\infty}\frac{\int(\sum_{k\le n}\phi_{\sigma^k\omega}\circ F^k_\omega)^2\, d\mu_\omega}{n}=\Sigma^2.$$

(2) If $\Sigma^2>0$, then ϕ has the QASIP: there is a constant $e>0$ such that for a.e. $\omega\in\Omega$, we have a Brownian motion $\bar{B}^\omega$ defined on an extended probability space $(\mathbf{\Delta}_\omega,\mathbf{Q}_\omega)$ and

$$\sum_{k\le n}\varphi_{\sigma^k\omega}\circ f^k_\omega\circ\pi_\omega-\bar{B}^\omega_{\sigma_n^2(\omega)}=\sum_{k\le n}\phi_{\sigma^k\omega}\circ F^k_\omega-\bar{B}^\omega_{\sigma_n^2(\omega)}=O(n^e)\quad \mathbf{Q}_\omega\text{-a.s.}$$

Here $\sigma_n^2(\omega)=\int(\sum_{k\le n}\phi_{\sigma^k\omega}\circ F^k_\omega)^2\,d\mu_\omega$, and $e=1/4+\epsilon_0$ satisfies the following: if $\rho_n=e^{-an^b}$, then $\epsilon_0>0$ is any small number; if $\rho_n=n^{-D}$, then ϵ_0 is any number in $(\epsilon_D,1/4)$, where ϵ_D is defined in (2.5) with a different $\epsilon_1=2p/(p-\gamma)(D-2)$.

(3) If $\Sigma^2=0$, then ϕ is a coboundary: there is a function g' on Δ such that for a.e. $\omega\in\Omega$,

$$\phi_{\sigma\omega}\circ F_\omega=g'_{\sigma\omega}\circ F_\omega-g'_\omega\quad \mu_\omega\text{-a.s.};$$

in other words,

$$\phi\circ F=g'\circ F-g'\quad \mu\text{-a.s.}\tag{6.4}$$

If $\rho_n=n^{-D}$, then $g'\in L^{(D-2-\delta)(p-\gamma)/(1+\delta)p}(\Delta,\mu)$ for a sufficiently small $\delta>0$ such that $(D-2-\delta)(p-\gamma)/(1+\delta)p>4$.

Now we show how to project this QASIP and coboundary to the RDS. Let $\chi(\omega,x):=(\sigma\omega,f_\omega(x))$, χ^* be its transfer operator w.r.t. υ, and $\mathcal{M}=:\bigcup_{\omega\in\Omega}(\{\omega\}\times M_\omega)$.

Projection for the coboundary. We will apply Theorem 1.1 of [**Liv96**] to the stationary system $(\mathcal{M},\chi,\upsilon)$ and the fiberwise mean zero function Φ, verifying three conditions of the theorem.

By Lemma 3.4, there is a constant $C=C_{\phi,h,F,\beta^\gamma,\delta,p/\gamma}\|\mathcal{K}^\gamma\|_{L^{p/\gamma}}$ such that

$$\begin{aligned}\sum_{i\ge1}\left|\int\Phi\Phi\circ\chi^i d\upsilon\right| &= \sum_{i\ge1}\left|\int\int\varphi_\omega\varphi_{\sigma^i\omega}\circ f^i_\omega d\upsilon_\omega\, d\mathbb{P}\right|\\ &= \sum_{i\ge1}\left|\int\int\phi_\omega\phi_{\sigma^i\omega}\circ F^i_\omega\, d\mu_\omega\, d\mathbb{P}\right|\end{aligned}$$

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$$\leq C_\phi \sum_{i\geq 1} \int \int |P_\omega^i(\phi_\omega)| \, d\mu_{\sigma^i\omega} \, d\mathbb{P}$$

$$\leq C_\phi C \sum_{i\geq 1} i^{-(D-2-\delta)(p/\gamma-1)/p/\gamma} < \infty,$$

$$\sum_{n\geq 1} \int |(\chi^*)^n \Phi| d\upsilon = \sum_{n\geq 1} \sup_{\|\psi\|_\infty \leq 1} \int \psi \circ \chi^n \Phi d\upsilon$$

$$= \sum_{n\geq 1} \sup_{\|\psi\|_\infty \leq 1} \int \int \psi_{\sigma^n\omega} \circ \pi_\omega \circ F_\omega^n \phi_\omega \, d\mu_\omega \, d\mathbb{P}$$

$$\leq \sum_{n\geq 1} \int \int |P_\omega^n(\phi_\omega)| \, d\mu_{\sigma^n\omega} \, d\mathbb{P}$$

$$\leq C \sum_{i\geq 1} i^{-(D-2-\delta)(p/\gamma-1)/p/\gamma} < \infty.$$

Using $\int \phi\phi \circ F^n \, d\mu \leq Cn^{-(D-2-\delta)(p-\gamma)/p} \leq Cn^{-4}$ and following the same computations as in the proof of Corollary 3.10 of [Su19b] (we skip this here), we have

$$\int \Phi^2 d\upsilon + 2 \sum_{i\geq 1} \int \Phi\Phi \circ \chi^i d\upsilon = \int \phi^2 \, d\mu + 2 \sum_{i\geq 1} \int \phi\phi \circ F^i \, d\mu$$

$$= \lim_{n\to\infty} \frac{\int (\sum_{i\leq n} \phi \circ F^i)^2 \, d\mu}{n} = \lim_{n\to\infty} \frac{\int (\phi + g' \circ F^n - g')^2 \, d\mu}{n} = 0$$

where the last two equalities are due to (6.4) and $g' \in L^4(\Delta, \mu)$. Therefore, by Theorem 1.1 of [Liv96], there is a measurable function g on $(\mathcal{M}, \upsilon)$ such that $\Phi \circ \chi = g \circ \chi - g\upsilon$-a.s. and $g := \sum_{n\geq 0} (\chi^*)^n \Phi$. Let $q := (D-2-\delta)(p-\gamma)/(1+\delta)p$. Using $\|\Phi\|_\infty \leq C_\phi$, we have

$$\|g\|_q \leq \left\| \sum_{n\geq 0} (\chi^*)^n \Phi \right\|_{L^q(\upsilon)} \leq C_\varphi + C_\varphi^{(q-1)/q} \sum_{n\geq 1} \left(\int |(\chi^*)^n \Phi| d\upsilon \right)^{1/q}$$

$$= C_\varphi + C_\varphi^{(q-1)/q} \sum_{n\geq 1} \left(\sup_{\xi:\|\xi\|_\infty \leq 1} \int \xi \circ \chi^n \Phi d\upsilon \right)^{1/q}$$

$$= C_\varphi + C_\varphi^{(q-1)/q} \sum_{n\geq 1} \left(\sup_{\xi:\|\xi\|_\infty \leq 1} \int \int \xi_{\sigma^n\omega} \circ f_\omega^n \varphi_\omega d\upsilon_\omega \, d\mathbb{P} \right)^{1/q}$$

$$= C_\varphi + C_\varphi^{(q-1)/q} \sum_{n\geq 1} \left(\sup_{\xi:\|\xi\|_\infty \leq 1} \int \int \xi_{\sigma^n\omega} \circ \pi_{\sigma^n\omega} \circ F_\omega^n \phi_\omega \, d\mu_\omega \, d\mathbb{P} \right)^{1/q}.$$

By Definition 3.1, $|\xi_{\sigma^n\omega} \circ \pi_{\sigma^n\omega}| \leq 1$ and Lemma 3.4, there is a constant $C = (C_{\phi,h,F,\beta^\gamma,\delta,p/\gamma} \|\mathcal{K}^\gamma\|_{L^{p/\gamma}})^{1/q}$ such that the last expression above can be

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estimated as

$$\leq C_\varphi + C_\varphi^{(q-1)/q} \sum_{n\geq 1} \bigg(\int\int |P_\omega^n(\phi_\omega)|\, d\mu_{\sigma^n\omega}\, d\mathbb{P}\bigg)^{1/q}$$
$$\leq C_\varphi + CC_\varphi^{(q-1)/q} \sum_{n\geq 1} n^{-(D-2-\delta)(p-\gamma)/(qp)} = C_\varphi + CC_\varphi^{(q-1)/q} \sum_{n\geq 1} n^{-(1+\delta)} < \infty.$$

Therefore $g \in L^{(D-2-\delta)(p-\gamma)/(1+\delta)p}(\upsilon)$.

Projection for the QASIP. By lemma A.1, the QASIP for the RYT

$$\sum_{k\leq n} \varphi_{\sigma^k\omega} \circ f_\omega^k \circ \pi_\omega - \bar{B}^\omega_{\sigma_n^2(\omega)} = O(n^e) \quad \mathbf{Q}_\omega\text{-a.s. for a.e. } \omega \in \Omega$$

implies that there is a function $H : \mathbb{R}^{\mathbb{N}} \times [0, 1] \to C[0, 1]$, a uniform distribution U on $[0, 1]$ and a Brownian motion $\hat{B}^\omega := H((\varphi_{\sigma^k\omega} \circ f_\omega^k \circ \pi_\omega)_{k\geq 1}, U)$ defined on $(\Delta_\omega \times [0, 1], \mu_\omega \times \mathrm{Leb}_{[0,1]})$ such that for a.e. $\omega \in \Omega$,

$$\mathbf{Q}_\omega\{((\varphi_{\sigma^k\omega} \circ f_\omega^k \circ \pi_\omega)_{k\geq 1}, \bar{B}^\omega) \in (\cdot, \cdot)\}$$
$$= (\mu_\omega \times \mathrm{Leb}_{[0,1]})\{((\varphi_{\sigma^k\omega} \circ f_\omega^k \circ \pi_\omega)_{k\geq 1}, \hat{B}^\omega) \in (\cdot, \cdot)\}.$$

This implies that for a.e. $\omega \in \Omega$,

$$\sum_{k\leq n} \varphi_{\sigma^k\omega} \circ f_\omega^k \circ \pi_\omega - \hat{B}^\omega_{\sigma_n^2(\omega)} = O(n^e) \quad \mu_\omega \times \mathrm{Leb}_{[0,1]}\text{ -a.s.}$$

Then $B^\omega := H((\varphi_{\sigma^k\omega} \circ f_\omega^k)_{k\geq 1}, U)$ is also a Brownian motion defined on $(M \times [0, 1], \upsilon_\omega \times \mathrm{Leb}_{[0,1]})$. Therefore, for a.e. $\omega \in \Omega$,

$$\sum_{k\leq n} \varphi_{\sigma^k\omega} \circ f_\omega^k - B^\omega_{\sigma_n^2(\omega)} = O(n^e) \quad \upsilon_\omega \times \mathrm{Leb}_{[0,1]}\text{ -a.s.}$$

and the extended probability space is $\mathbf{M}_\omega := (M \times [0, 1], \upsilon_\omega \times \mathrm{Leb}_{[0,1]})$. □

7. *Applications*

We will apply Theorem 6.1 to each of the following RDS, by verifying conditions (1)–(6) in Definition 6.1: i.i.d. translations of unimodal maps (satisfying the Collet–Eckmann conditions) in [**BBMD02**]; i.i.d. translations of non-uniformly expanding maps (with a slow recurrence to singularities) in [**AA03**, **AV13**]; i.i.d. perturbations of admissible S-unimodal maps (satisfying the Collet–Eckmann conditions or summability conditions of exponent 1) in [**Du15**]; and i.i.d. perturbations of random LSV maps with a neutral fixed point in [**BBR19**]. Here i.i.d. means that the randomness of $f_{\sigma^i\omega}$ only depends on ω_i; then for any $n \in \mathbb{N}$, $f_{\sigma^n\omega}$ is independent of $(f_{\sigma^i\omega})_{i\leq n-1}$. In Definition 6.1, conditions (1), (2), (4), (6.1) and (6.2) are satisfied when the RYT is constructed. Condition (3) is also satisfied since $\{R_\omega = n\}$ is constructed inductively in these papers (it only depends on $(f_{\sigma^i\omega})_{0\leq i\leq n-1}$), that is, $\{R_\omega = n\}$ only depends on $\omega_0, \omega_1, \ldots, \omega_{n-1}$. Thus it remains to verify condition (6.3):

$$d(f_\omega^k(x), f_\omega^k(y)) \leq C\mathcal{K}_{\sigma^k\omega} d(f_\omega^{R_\omega}(x), f_\omega^{R_\omega}(y)),$$

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and condition (6), that is,

$$\int m(x \in \Lambda : R_\omega(x) > n)\, d\mathbb{P} \leq C\rho_n \searrow 0. \tag{7.1}$$

7.1. *i.i.d. translations of unimodal maps.* Condition (6.3) is due to the proof of Lemma 9.1 of [**BBMD02**, pp. 123]. Condition (7.1) is due to Proposition 8.3 of [**BBMD02**]. The probability measure $\nu := \nu_\epsilon$ is defined as (2.1) of [**BBMD02**, pp. 82]. The QASIP convergence rate is $1/4 + \epsilon_0$ for any small $\epsilon_0 > 0$.

7.2. *i.i.d. perturbations of S-unimodal maps.* Condition (7.1) is due to Theorems 8.1.2 and 8.1.4 of [**Du15**]. We now verify (6.3).

(1) For S-unimodal maps satisfying the Collet–Eckmann conditions in [**Du15**], (6.3) is due to Proposition 8.3.5 of [**Du15**].

(2) For S-unimodal maps on interval I satisfying summability conditions of exponent 1 in [**Du15**], we will verify (6.3) with $\mathcal{K} \in L^\infty(\Omega)$, that is, there is a constant $C > 0$ independent of $\omega \in \Omega$ such that

$$d(f_\omega^k(x), f_\omega^k(y)) \leq C d(f_\omega^{R_\omega}(x), f_\omega^{R_\omega}(y)),$$

where $k \leq R_\omega = n,\ x, y \in U_\omega(z, n) := (f_\omega^n)^{-1}(\widetilde{B}(\delta)) \bigcap J_{z,n}^\omega$ and n is a θ-good return time of (ω, z) into $\widetilde{B}(\delta)$. Moreover, $f_\omega^n : U_\omega(z, n) \to \widetilde{B}(\delta)$ is a diffeomorphism, $n-k$ is also a θ-good return time of $(\sigma^k\omega, f_\omega^k(z))$ into $\widetilde{B}(\delta)$, and $f_{\sigma^k\omega}^{n-k}$ is a diffeomorphism from $f_\omega^k(U_\omega(z, n)) \subseteq U_{\sigma^k\omega}(f_\omega^k(z), n-k)$ into $\widetilde{B}(\delta)$ (These properties can be found in Lemma 8.2.1 and Propositions 8.2.3 and 8.2.4 of [**Du15**], and see Definition 8.2.2 of [**Du15**] for θ-good return times.) Equivalently, we will prove (6.3) by showing

$$|Df_{\sigma^k\omega}^{n-k}|_{f_\omega^k(U_\omega(z,n))}| \geq C^{-1}.$$

By Lemma 8.2.1 of [**Du15**], for any $z_1, z_2 \in U_{\sigma^k\omega}(f_\omega^k(z), n-k)$,

$$e^{-1/2} \leq \frac{|Df_{\sigma^k\omega}^{n-k}(z_1)|}{|Df_{\sigma^k\omega}^{n-k}(z_2)|} \leq e^{1/2}.$$

Then for any $z_1 \in f_\omega^k(U_\omega(z, n))$,

$$|Df_{\sigma^k\omega}^{n-k}(z_1)| \geq e^{-1/2} \frac{|f_{\sigma^k\omega}^{n-k}[U_{\sigma^k\omega}(f_\omega^k(z), n-k)]|}{|U_{\sigma^k\omega}(f_\omega^k(z), n-k)|} \geq e^{-1/2} \frac{|\widetilde{B}(\delta)|}{|I|} =: C^{-1}.$$

The probability measure $\nu := \nu_\epsilon$ is defined in §8.1.1 of [**Du15**, pp. 80]. The QASIP convergence rates for these two RDS are $1/4 + \epsilon_0$ for any small $\epsilon_0 > 0$.

7.3. *i.i.d. translations of non-uniformly expanding maps.* Condition (7.1) is due to Proposition 5.1, Theorem 2.9, and §5.2.2 of [**AV13**]. Condition (6.3) is due to Proposition 4.9 of [**AV13**]. The probability measure $\nu := \theta_\epsilon$ is defined on p. 687 of [**AV13**]. The QASIP convergence rate is $1/4 + \epsilon_0$ for any small $\epsilon_0 > 0$.

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7.4. *i.i.d. perturbations of LSV maps with a neutral fixed point.* Condition (6.3) follows since LSV maps have derivatives no less than 1, that is, $d(f_\omega^k(x), f_\omega^k(y)) \leq d(f_\omega^{R_\omega}(x), f_\omega^{R_\omega}(y))$. Condition (7.1) is due to Proposition 5.3 and (5.5) in [**BBR19**] with $\alpha_0^{-1} > 6$, that is, the QASIP holds for $\Omega = [\alpha_0, \alpha_1]^{\mathbb{Z}}$ where $0 < \alpha_0 < 1/6$ and $\alpha_1 < 1$. The probability measure ν can be different distributions; see §5.2 of [**BBR19**]. The QASIP convergence rate is $1/4 + \epsilon_D + \epsilon_0$ where $D = \alpha_0^{-1}$ and $\epsilon_0 > 0$ is any small number.

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A. *Appendix*

LEMMA A.1. (See [**Kal02**, Theorem 6.10]) *For any measurable space S and Borel space T, let ξ, ξ' be random elements in S and η be a random element in T such that $\xi =_d \xi'$. Then there is a random element η' in T such that $(\eta, \xi) =_d (\eta', \xi')$. More precisely, there exists a measurable function $f : S \times [0, 1] \to T$ such that $\eta' = f(\xi', U)$ where $U \sim U(0, 1)$ and ξ' are independent.*

Indeed, to guarantee the independence above, we can simply extend the probability space by multiplying ([0, 1], Leb).

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