A NUMERICAL APPROACH TO DYNAMICALLY CONSISTENT SPHERICAL DYNAMO MODELS

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Abstract. Models of spherical dynamos are considered which involve the full interaction between the magnetic field and the motion of an incompressible conducting fluid. In the basic equations magnetic field and fluid velocity are expanded in series of certain decay modes. In this way these equations are reduced to an infinite set of ordinary first-order differential equations for the coefficients of these expansions. The behaviour of dynamos can then be studied by integrating a finite set of these equations numerically. Some first results obtained in this way are presented for mean-field models in which the growth of the magnetic field due to the $\alpha-$effect is limited by large-scale motions generated by Lorentz forces.

Key words: MHD, dynamo models, mean-field electrodynamics

1. The models

The paper aims at contributing to the study of spherical dynamo models involving the full interaction between magnetic fields and fluid motions. The numerical approach proposed here applies to the dynamo problem in its original as well as in its mean-field formulation. We refer here to the mean-field equations (which turn into the original ones by cancelling some specific terms). In that sense we suppose the mean magnetic flux density, $\mathbf{B}$, to be governed by

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{u} \times \mathbf{B} + \mathcal{E}) + \eta \Delta \mathbf{B}, \qquad \operatorname{div} \mathbf{B} = 0, \tag{1}$$

inside the fluid body, and to continue in outer space as an irrotational solenoidal field vanishing at infinity. We further suppose the fluid body to be incompressible and its mean velocity, $\mathbf{u}$, to obey

$$\begin{aligned} \rho(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}) &= -\nabla p + \rho \nu \Delta \mathbf{u} - 2\rho \boldsymbol{\Omega} \times \mathbf{u} + \frac{1}{\mu}\operatorname{curl}\mathbf{B} \times \mathbf{B} + \mathbf{f} + \mathcal{G}, \\ \operatorname{div} \mathbf{u} &= 0, \end{aligned} \tag{2}$$

and a proper condition at the boundary of the fluid body. $\mathcal{E}$ and $\mathcal{G}$ are the electromotive and ponderomotive forces caused by fluctuations of the magnetic field and the fluid motion, which in general depend on $\mathbf{B}$ and $\mathbf{u}$. As usual, η is the magnetic diffusivity, ρ the mass density, p a modified pressure, ν the kinematic viscosity and μ the magnetic permeability, all assumed to be constant. We refer to a rotating frame of reference. $\boldsymbol{\Omega}$ means the angular velocity responsible for Coriolis forces, and $\mathbf{f}$ an external force.

2. The numerical approach

Our numerical approach is an extension of that used in earlier papers (Rädler et al. 1989, 1990). To explain it we redefine $\mathbf{B}, \mathbf{u}, \mathcal{E}, \mathbf{f}$ and $\mathcal{G}$ as dimensionless quantities, introduce dimensionless space and time coordinates and rewrite (1) and (2) into

$$\Delta \mathbf{B} - P_\eta \frac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl}\mathbf{F}, \qquad \operatorname{div}\mathbf{B} = 0, \tag{3}$$

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and

$$\Delta \mathbf{u} - P_\nu \frac{\partial \mathbf{u}}{\partial t} = \nabla P - \mathbf{G}, \qquad \mathrm{div}\mathbf{u} = 0, \tag{4}$$

with

$$\mathbf{F} = R_\eta \mathbf{u} \times \mathbf{B} + R'_\eta \mathcal{E}, \tag{5}$$

$$\mathbf{G} = -R_\nu (\mathbf{u} \cdot \nabla)\mathbf{u} - T_a^2 \, \mathbf{e} \times \mathbf{u} + N \, \mathrm{curl}\mathbf{B} \times \mathbf{B} + Q_f \, \mathbf{f} + Q_g \, \mathcal{G}. \tag{6}$$

P_η and P_ν are dimensionless parameters defined by $R^2/T\eta$ and $R^2/T\nu$, where R is the radius of the fluid body and T an arbitrary time unit. P is a modified pressure and $\mathbf{e}$ the unity vector $\mathbf{\Omega}/\Omega$. $R_\eta, R'_\eta, R_\nu, T_a, N, Q_f$ and Q_g are again dimensionless parameters whose meaning can easily be seen; e. g., R_η and R_ν are the magnetic and hydrodynamic Reynolds numbers UR/η and UR/ν, T_a the Taylor number $(2\Omega R^2/\nu)^{\frac{1}{2}}$, and N a modified Stuart number $B^2R/\mu\rho\nu U$, where U and B are arbitrary units of the fluid velocity and the magnetic flux density. Of course, these equations have to be completed by the above condition on the continuation of $\mathbf{B}$ in outer space and the boundary condition for $\mathbf{u}$.

Consider first the special case $\mathbf{F} = \mathbf{G} = 0$, which covers the two independent free decay problems for magnetic fields and for slow motions. As is well known there are solutions for $\mathbf{B}$ in the form $\mathbf{B}_i(\mathbf{x}) \exp[-\lambda_i^B t]$. Analogously, there are solutions for $\mathbf{u}$ of the form $\mathbf{U}_i(\mathbf{x}) \exp[-\lambda_i^U t]$. The $\mathbf{B}_i$ and the $\mathbf{U}_i$ define complete orthogonal sets which allow the expansion of solenoidal vector functions satisfying the continuation or boundary conditions for $\mathbf{B}$ or $\mathbf{u}$, respectively.

Returning to the original problem with non-vanishing $\mathbf{F}$ and $\mathbf{G}$ we represent $\mathbf{B}$ and $\mathbf{u}$ by

$$\mathbf{B}(x,t) = \sum_i b_i(t)\mathbf{B}_i(\mathbf{x}), \qquad \mathbf{u}(x,t) = \sum_i u_i(t)\mathbf{U}_i(\mathbf{x}), \tag{7}$$

and assume $\mathbf{B}_i$ and $\mathbf{U}_i$ normalized so that

$$\int \mathbf{B}_i \cdot \mathbf{B}_j dv = \delta_{ij}, \qquad \int \mathbf{U}_i \cdot \mathbf{U}_j dv = \delta_{ij}, \tag{8}$$

where the first integral is over all space, the second one over the fluid body only. Then equations (3) and (4) can be replaced by an infinite set of ordinary differential equations,

$$P_\eta \frac{db_i}{dt} = -\lambda_i^B b_i + f_i, \tag{9}$$

$$P_\nu \frac{du_i}{dt} = -\lambda_i^U u_i + g_i, \tag{10}$$

where

$$f_i = \int \mathbf{F} \cdot \mathrm{curl}\mathbf{B}_i dv, \qquad g_i = \int \mathbf{G} \cdot \mathbf{U}_i dv, \tag{11}$$

with integrals taken over the fluid body.

Changing the notation we write now $\mathbf{B}_{nl}^{\alpha m\beta}$, $\lambda_{nl}^{B\alpha m\beta}$, $\mathbf{U}_{nl}^{\alpha m\beta}$ and $\lambda_{nl}^{U\alpha m\beta}$ instead of $\mathbf{B}_i, \lambda_i^B, \mathbf{U}_i$ and λ_i^U . The index α takes the values P or T which denote poloidal and toroidal fields. Referring to spherical polar coordinates r, θ, φ with $r = 1$ at the boundary of the body and $\mathbf{r}$ being the radius vector, we have

$$\mathbf{B}_{nl}^{Pm\beta}(\mathbf{x}) = -N_{nl}^{BPm} \operatorname{curl}[\mathrm{j}_n(\mu_{n-1,l}\, r)\, \mathbf{r} \times \nabla \mathrm{Y}_n^{m\beta}(\theta,\varphi)], \tag{12}$$

$$\mathbf{B}_{nl}^{Tm\beta}(\mathbf{x}) = -N_{nl}^{BTm}\, \mathrm{j}_n(\mu_{n\,l}\, r)\, \mathbf{r} \times \nabla \mathrm{Y}_n^{m\beta}(\theta,\varphi), \tag{13}$$

$$\lambda_{nl}^{BPm\beta} = \mu_{n-1,l}^2, \qquad \lambda_{nl}^{BTm\beta} = \mu_{nl}^2, \tag{14}$$

and, introducing now the no-slip condition $\mathbf{u} = 0$ at the boundary of the fluid body,

$$\mathbf{U}_{nl}^{Pm\beta}(\mathbf{x}) = -N_{nl}^{UPm} \operatorname{curl}[(\mathrm{j}_n(\mu_{n+1,l}\, r) - \mathrm{j}_n(\mu_{n+1,l})r^n)\, \mathbf{r} \times \nabla \mathrm{Y}_n^{m\beta}(\theta,\varphi)], \tag{15}$$

$$\mathbf{U}_{nl}^{Tm\beta}(\mathbf{x}) = -N_{nl}^{UTm}\, \mathrm{j}_n(\mu_{n\,l}\, r)\, \mathbf{r} \times \nabla \mathrm{Y}_n^{m\beta}(\theta,\varphi), \tag{16}$$

$$\lambda_{nl}^{UPm\beta} = \mu_{n+1,l}^2, \qquad \lambda_{nl}^{UTm\beta} = \mu_{nl}^2. \tag{17}$$

The N are normalization constants, the j_n spherical Bessel functions and $Y_n^{m\beta}$ spherical harmonics, and the relations apply for $n \geq 1$ and $0 \leq m \leq n$. The μ_{nl} are the positive zeros of j_n, that is, $\mathrm{j}_n(\mu_{nl}) = 0$, and β takes two values distinguishing between spherical harmonics with $\cos m\varphi$ and $\sin m\varphi$.

For our numerical approach the expansions (7) are truncated so that (9) and (10) turn into a finite set of equations for b_i and u_i. This set is integrated by a modified second order Runge-Kutta method. In general, each f_i and g_i depends on all b_i and u_i. However, apart from very special cases, the integrals in (11), which define this dependence, cannot be solved analytically. Therefore, in each time step the numerical values of f_i and g_i have to be determined by numerical evaluation of these integrals.

3. Some preliminary results

The numerical approach described so far has been applied to particular α-effect models as investigated already in a former paper (Rädler et al 1990). To define them we specify equations (1) and (2) by

$$\mathcal{E} = \alpha\mathbf{B}, \qquad \alpha = \alpha_0 a(r,\theta), \qquad \mathbf{f} = \mathcal{G} = 0, \tag{18}$$

where α_0 is a constant and a a dimensionless function given by

$$a = 30r^2(1-r^2)^2 \cos\theta. \tag{19}$$

Proceeding to equations (3) to (6) for the dimensionless quantities we write then R_α instead of R'_η and define it as $\alpha_0 R/\eta$. On this level $\mathcal{E}$ is equal to $a\mathbf{B}$.

The results presented here have been obtained with the further specification $R_\nu = R_\eta = T_a^2 = 10$, $N = 40$, and several values of R_α. For a certain range of R_α steady axisymmetric solutions have been found with magnetic fields antisymmetric or symmetric about the equatorial plane, i.e., A0 or S0 fields. The solutions can be roughly characterized by the magnetic and kinetic energies. For a few examples

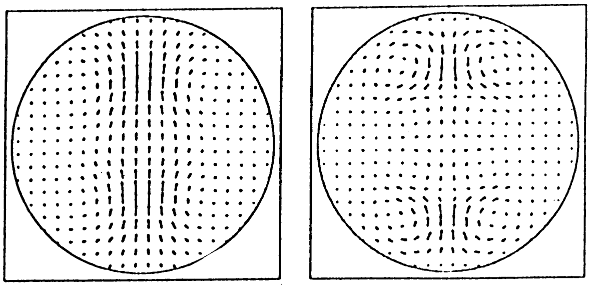

Fig. 1. The steady magnetic A0 field (left) and the corresponding velocity field (right) in a meridional plane, $R_\alpha = 6.0$.

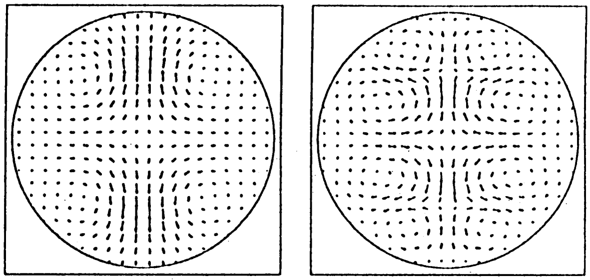

Fig. 2. The steady magnetic S0 field (left) and the corresponding velocity field (right) in a meridional plane, $R_\alpha = 6.0$.

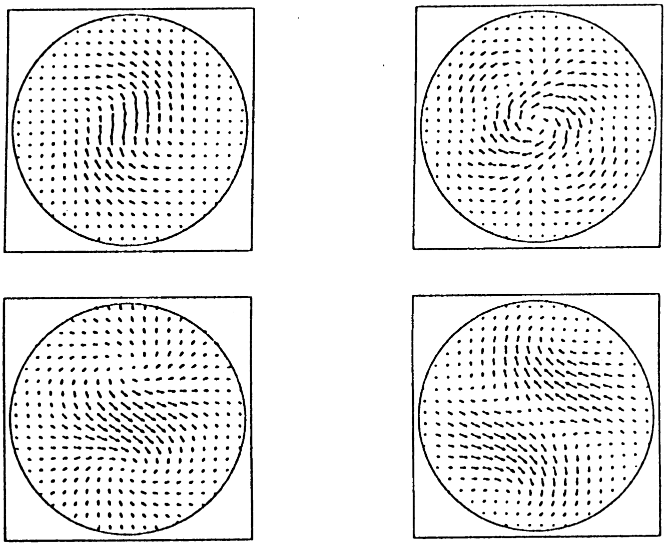

Fig. 3. Magnetic field configuration of A1 type (left) and corresponding velocity fields (right) in planes parallel to the equatorial plane at latitudes of 10° (upper panel) and 45° (lower panel), $R_\alpha = 6.0$.

these energies are given in Tables I and II , and magnetic and velocity field configurations are depicted in Figures 1 and 2. Moreover, non-axisymmetric solutions have been found, which are no longer steady in a strict sense but have the form of waves travelling in azimuthal direction. In the example chosen for Figure 3 the energies are independent of time but the magnetic field configuration, which is of A1 type, as well as the velocity field show an eastward rigid rotation with a rotation rate of about $4.7\eta/R^2$. The examples mentioned demonstrate that the growth of a magnetic field due to α-effect can well be limited by the fluid motion caused by this field. This seems, however, be restricted to a range of not too large R_α. Of course, the stability of these solutions remains to be checked.

TABLE I

Magnetic energy E_B, kinetic energy E_U and their poloidal and toroidal parts for solutions with magnetic fields of A0 type in units of $\rho U^2 R^3$, $R_\alpha = 6.0$.

R_α	E_B	E_{BP}	E_{BT}	E_U	E_{UP}	E_{UT}
5.5	0.52	0.28	0.24	0.015	0.002	0.013
5.7	1.01	0.59	0.42	0.053	0.015	0.048
6.0	1.68	1.00	0.68	0.14	0.01	0.13

TABLE II

The same as Table I but magnetic fields of S0 type.

R_α	E_B	E_{BP}	E_{BT}	E_U	E_{UP}	E_{UT}
5.5	0.84	0.44	0.40	0.0086	0.0032	0.0054
5.7	2.60	1.36	1.24	0.051	0.017	0.034
6.0	5.28	2.64	2.64	0.14	0.04	0.10

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