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GRAPHS ASSOCIATED WITH TRIANGULATIONS OF LATTICE POLYGONS

DUANE DeTEMPLE and JACK M. ROBERTSON

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Abstract

Two graphs, the *edge crossing* graph E and the *triangle* graph T are associated with a simple lattice polygon. The maximal independent sets of vertices of E and T correspond to the triangulations of the polygon into fundamental triangles. Properties of E and T are derived including a formula for the size of the maximal independent sets in E and T. It is shown that T is a factor graph of edge-disjoint 4-cycles, which gives corresponding geometric information, and is a partition graph as recently defined by the authors and F. Harary.

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1. Introduction

If the values of a real valued function f are known at a sequence of integral points $1, 2, \dots, n$ on the real line there is only one function which is affine on $[j, j+1]$, $1 \le j \le n-1$, and agrees with f at each $j \in \{1, 2, \dots, n\}$. The two-dimensional analog is more interesting. Let P be a simple polygon in the plane with vertices at lattice points and let a given real valued function f defined on P and its interior have values $y_{ij} = f((i,j))$ at lattice points (i, j) inside and on the boundary of P. In general there will be many functions $\hat{f}$ which are piecewise affine approximations to f in the following sense:

(i) $\hat{f}((i,j)) = f((i,j)) = y_{ij}$ for all lattice points (i, j) inside and on the boundary of P, and

(ii) $\hat{f}$ is affine on fundamental triangles (those with exactly three lattice points on the boundary at the vertices and no interior lattice points) in P.

To emphasize the dependence of $\hat{f}$ on the triangulation of P, suppose the values y_{ij} are linearly independent in the vector space of real numbers over rational scalars. Let P be triangulated into fundamental triangles with node (i,j) having valence a_{ij} in the graph which the triangulation induces. By Pick's Theorem [1] each of the triangles has area $1/2$, so if $\hat{f}$ satisfies (i) and (ii) we have

$$\int_P \hat{f} = \frac{1}{6}\left(\sum_{(i,j)\text{ interior to }P} a_{ij}y_{ij} + \sum_{(i,j)\text{ on boundary of }P} (a_{ij}-1)y_{ij}\right).$$

Thus if $\hat{f}_1$ and $\hat{f}_2$ are determined by two triangulations of P, $\int_P \hat{f}_1 = \int_p \hat{f}_2$ if and only if each lattice point has the same valence in the two triangulations. A natural question is how many triangulations are there for a simple lattice polygon P?

In the following work we relate the problem of determining all possible triangulations of P to the problem of determining all maximal independent sets of vertices in each of two related graphs. Properties of the graphs are discussed. One of the graphs is a special intersection graph and this naturally introduces *partition graphs* which have been studied in [2], [3], and [4].

Some related problems are known to be difficult. Given inputs of a graph G and arbitrary integer k, determining whether or not G has an independent set with k or more vertices is NP-complete [5], and finding the number of maximal independent sets for an arbitrary graph is $\#P$-complete [8], so the results we give are likely more of theoretical rather than practical interest (except possibly in special instances). Also Gavril [6] has shown that determining whether or not a graph is the intersection graphs for a set of rectangles on an $m \times n$ grid is NP-complete. The graphs T we consider below are intersection graphs for triangles inside lattice polygons.

We first formalize the terminology and introduce graphs E and T. A segment joining two lattice points is *fundamental* if no other lattice point lies on the segment. A *fundamental triangle* is one which does not contain any lattice points in its interior and whose three sides are each a fundamental segment. A *fundamental parallelogram* is a parallelogram either of whose diagonals divides it into two fundamental triangles. A *lattice polygon* is one having all of its vertices at lattice points in the plane.

Let P be a simple lattice polygon, and suppose that all fundamental edges in P are drawn. The *edge crossing graph* E of P is constructed by letting each fundamental edge e' in P correspond to a vertex e in E, with e_1 and e_2 adjacent if and only if the corresponding fundamental segments e_1' and e_2' in

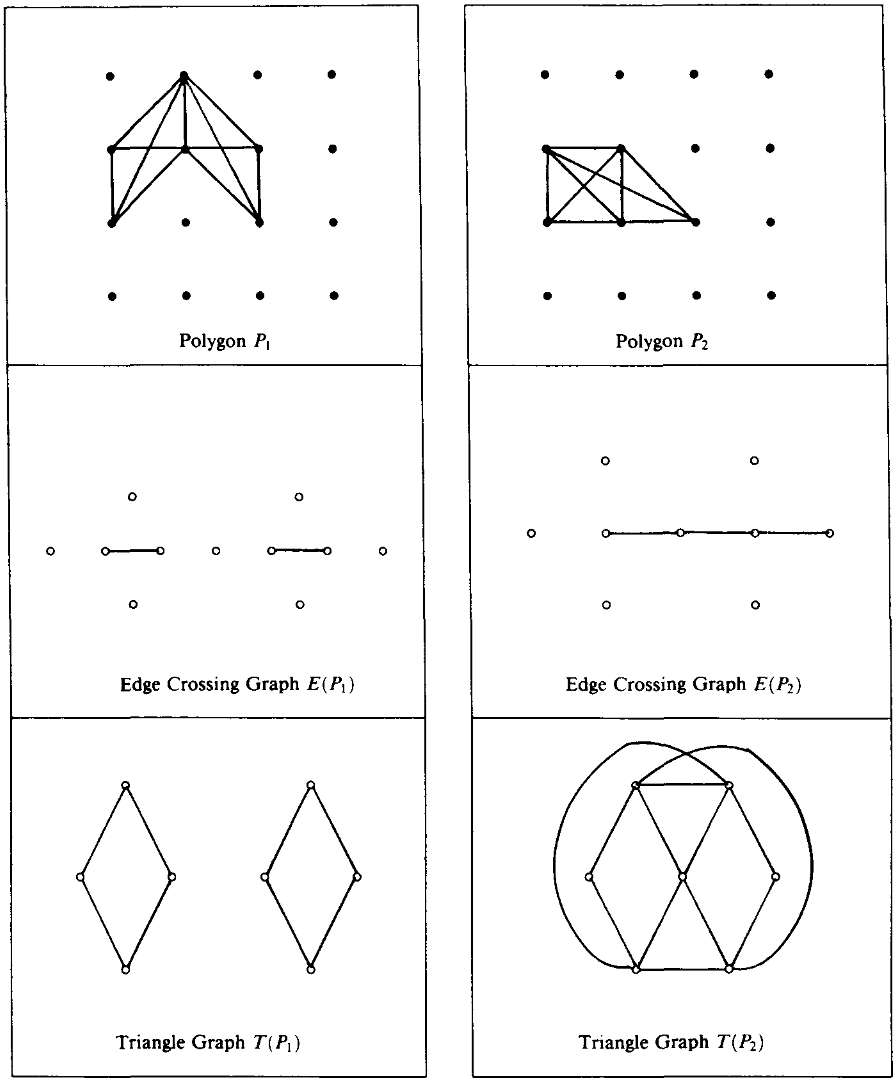

FIGURE 1. Two lattice polygons and their associated edge and triangle graphs

P intersect at a point interior to each segment. Figure 1 shows examples of edge crossing graphs for two different polygons. Note that all boundary edges correspond to isolated vertices in E; more generally e is isolated in E if and only if the corresponding segment e' in P is used in every triangulation of P. The examples show that E need not be connected even after isolated vertices are removed.

Recall that a subset M of the vertices of a graph is *independent* if no two are joined by an edge, and is *maximal independent* if it is not properly contained in a larger independent set of vertices. The following result is clear from the construction of the edge crossing graph, since each maximal independent set of vertices in E determines a unique triangulation of P, and conversely.

THEOREM 1. *If τ is the set of all triangulations P and $\mathcal{M}_E$ is the set of all maximal independent sets in E, then there is a natural bijection $\tau \leftrightarrow \mathcal{M}_E$.*

Next we define the *triangle graph* T associated with a given lattice polygon P. To each fundamental triangle t' in P corresponds a vertex t in T, with t_1 and t_2 adjacent if and only if the corresponding triangles t_1' and t_2' share common interior points (again see Figure 1). Noting that each maximal independent set in T corresponds to a unique triangulation of P and conversely we have the following.

THEOREM 2. *If τ is the set of all triangulations P and $\mathcal{M}_T$ is the set of all maximal independent sets of vertices in T, then there is a natural bijection $\tau \leftrightarrow \mathcal{M}_T$.*

2. Properties of E and T

Let P have b boundary lattice points and i interior lattice points. Pick's Theorem [1] states that the area A of P is given by $A = (b/2) + i - 1$.

THEOREM 3. *All maximal independent sets in E have the same cardinality: if $M_E \in \mathcal{M}_E$, then $|M_E| = 2b + 3i - 3$. All maximal independent sets in T have the same cardinality: if $M_T \in \mathcal{M}_T$, then $|M_T| = b + 2i - 2$.*

PROOF. By Pick's Theorem we have $2A = b + 2i - 2 = |M_T|$ since each fundamental triangle has area $1/2$. The expressions $3|M_T| + b = 2|M_E|$ count each edge twice so that $|M_E| = 3/2|M_T| + b/2 = 3/2(b + 2i - 2) + b/2 = 2b + 3i - 3$.

The examples of Figure 1 suggest that the edge crossing graph E may in general be considerably simpler than the triangle graph T. That is in fact the case.

THEOREM 4. *There is a mapping f which assigns to each edge of the graph E a unique 4-cycle in T. Furthermore the collection of these 4-cycles forms a disjoint cover of all edges in T.*

PROOF. The result depends, of course, on the geometry of the lattice structure inside P. We first construct a function f which associated edges of the graph E with well defined four cycles in T. It is then established that f covers each edge of T once and only once.

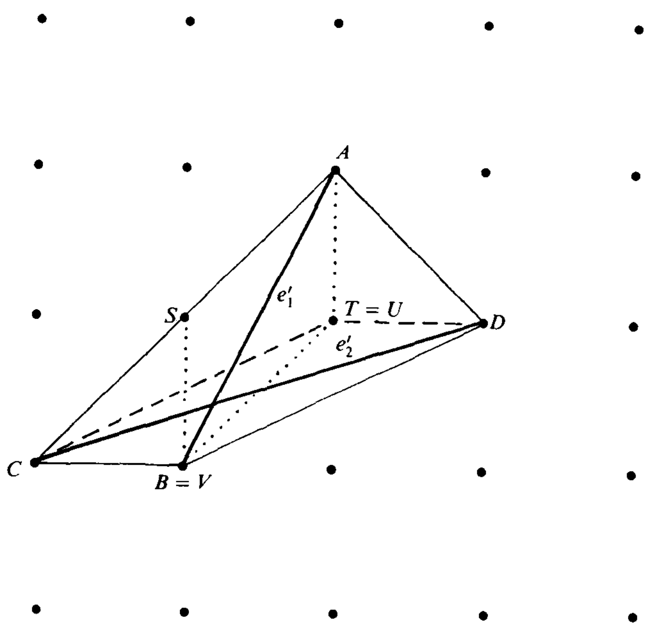

FIGURE 2. The kite $k(e_1', e_2')$ for two intersecting fundamental edges

Let $e_1 e_2$ be an edge in E indicating that fundamental segments e_1' joining lattice points A and B and e_2' joining C and D share a common interior point in P (Figure 2). The convex hull of $e_1' \cup e_2'$ is a quadrilateral we call the *kite* $k(e_1', e_2')$. From among the lattice points lying in or on triangle ABC let S be the one nearest but not on e_1'. Triangle ABS has area $1/2$ since it is fundamental by the choice of S. Furthermore S is uniquely determined in triangle ABC since it must fall on the next line parallel to e_1' which contains lattice points and the distance between successive lattice points on this parallel line is the same as the distance from A to B.

Similarly lattice points T, U and V are uniquely determined in the kite so that triangles ABT, CDU and CDV are all fundamental. In fact $ASBT$ and $CUDV$ are fundamental parallelograms since vector $\overline{SA}$ must equal vector $\overline{BT}$ because of the way lattice points fall on the two parallel lines containing lattice points nearest to AB. Also these fundamental parallelograms must lie in the original polygon P. To justify this we note that S, for example, must

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lie in P because it falls on the line nearest AB which contains lattice points, and segment CD is given to lie in P.

Let f assign to the pair e_1', e_2' the four cycle $f(e_1', e_2')$ in T given by the intersection of triangles ABS, CDU, ABT and CDV taken in that order. The function f is well defined by what precedes and it remains to be shown that each edge in T is covered once and only once by f. To see that edges in T are covered at most once, suppose the four cycle $f(e_1', e_2')$ contains the edge $t_1 t_2$ in T generated by the intersection of triangles t_1' and t_2'. Assume these triangles and edges are labeled as in Figure 2 where $t_1' = \triangle ABS$ and $t_2' = \triangle CDU$. If $f(e_1'', e_2'')$ also contains $t_1 t_2$ for a second pair of crossing fundamental edges e_1'' and e_2'', then e_1'' must be a side of t_1' and e_2'' must be a side of t_2' because of the way f is defined. But if e_1' is replaced by e_1'' or e_2' is replaced by e_2'' then one of A or B is not an endpoint of e_1'' or one of C or D is not an endpoint of e_2''. Let us assume A is not an endpoint of e_1''. Then the kite $k(e_1'', e_2'')$ does not contain A and any triangle with A as one of its vertices, in particular t_1', cannot correspond to an endpoint of edges in $f(e_1'', e_2'')$.

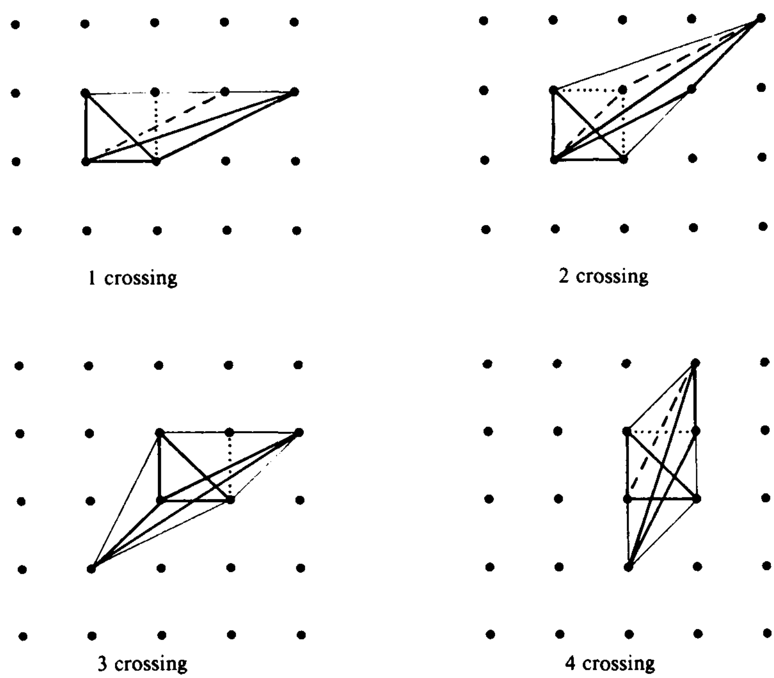

FIGURE 3. The four types of intersections of pairs of fundamental triangles

Finally we must observe that any edge $t_1 t_2$ in T is part of a four cycle $f(e_1', e_2')$. This requires that we identify the proper intersecting sides e_1' of t_1' and e_2 of t_2' so that both t_1' and t_2' lie in $k(e_1', e_2')$. For a pair t_1', t_2' there

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may be one, two, three, or four possibilities for the pair e'_1, e'_2 as illustrated in Figure 3. In every case, any two fundamental triangles sharing interior points lie in a unique kite determined by two of their intersecting sides, e'_1 and e'_2, making edge t_1t_2 lie on the four cycle $f(e'_1, e'_2)$. That the convex hull of two intersecting fundamental triangles is a kite determined by two intersecting edges can be shown directly, or by observing that those two triangles are affine images of a variation of those shown in Figure 3.

COROLLARY 1. *The graph T is a factor graph of edge-disjoint* 4*-cycles and isolated vertices.*

COROLLARY 2. *For a given simple lattice polygon, T has four times as many edges as E.*

COROLLARY 3. *If P is a lattice polygon in the plane and all fundamental triangles in P are considered, the number of pairs of such triangles sharing interior points is a multiple of* 4.

The graph T is an example of a partition graph as recently investigated in [2], [3] and [4]. A graph G is a *partition graph* if to each vertex v of G there can be assigned a set $S_v \neq \emptyset$ so that the following properties hold:

(i) distinct vertices u, v of G are assigned distinct sets S_u, S_v;

(ii) uv is an edge of G if and only if $S_u \cap S_v \neq \emptyset$;

(iii) every maximal independent set of vertices, M, of G gives a partition of $S = \bigcup S_u$; that is, $S = \dot{\bigcup}_{w \in M} S_w$ [$\dot{\bigcup}$ denotes disjoint union].

Properties (i) and (ii) mean G is an intersection graph [7], and so a partition graph is a special type of intersection graph.

THEOREM 5. *The triangle graph T for a polygon is a partition graph.*

PROOF. The system of all fundamental edges in P divides P into disjoint regions x_i. If $t \in T$, let $S_t = \{x_i : x_i \subset t'\}$, where t' is the corresponding fundamental triangle inside P. It is clear the graph T is a partition graph for the family $\{S_t : t \in T\}$.

Theorems 3 and 5 and Corollary 1 give three properties of triangle graphs T. Conversely one can ask: if the graph G is a partition graph and a factor graph of edge-disjoint four-cycles and isolated vertices with all maximal independent sets of the same cardinality, is it the triangle graph for some lattice polygon P? It is interesting to note there are no such graphs with 5 or 6 vertices, so that after C_4 the next graph with the three properties is the one shown in Figure 1.

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Department of Pure and
Applied Mathematics
Washington State University
Pullman, Washington 99164-2930
U.S.A.