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MAXIMAL IDEAL SPACE OF FUNCTION ALGEBRAS

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Abstract

We present a representation theory for the maximal ideal space of a real function algebra, endowed with the Gelfand topology, using the theory of uniform spaces. Application are given to algebras of differentiable functions in a normed space, improving and generalizing some known results.

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1. Introduction and notation

1.1. Recall that if B is a commutative algebra, Δ is the set of all real B-homomorphisms, and $x \in B$, the formula

$$\hat{x}(h) = h(x) \qquad (h \in \Delta),$$

defines the *Gelfand transform* of x. If we set $\hat{B} = \{\hat{x} : x \in B\}$, the *Gelfand topology* of Δ is the weak topology induced by $\hat{B}$; Δ equipped with the Gelfand topology is usually called the *maximal ideal* space of B.

Δ has been intensively studied when $B = C(X)$ for a completely regular Hausdorff space X (see [4]). Recently different papers have been devoted to study some subalgebras of $C(X)$ (see, for example [1, 2, 6, 7] and some of the references given there). Recent research in the theory of homomorphisms of algebras of differentiable functions have brought out a number of special facts related to algebras of continuous functions on a completely regular space. Attempts to obtain a suitable representation for the maximal ideal space, often use the Stone-Čech compactification βX of X as an intermediate step. This task is very difficult, as [6] shows, and depends on the actual

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space. The idea is to identify a homormorphism with evaluation at some point p of βX. This point is not unique; and a quotient topology is needed. Since quotients of completely regular space fail to be completely regular, each case calls for independent analysis.

1.2. Let X be a non-empty set. By a *function algebra* on X we mean a family of real valued functions on the set X forming a linear algebra with unit under the operations defined pointwise, which separates points and is closed under bounded inversion, that is, if $f \geq 1$ and $f \in A$, then $1/f \in A$. Since these are the only algebras to be discussed here, no misunderstanding should arise. The subalgebra of bounded functions in A will be denoted by A_b. It is easy to prove that A_b is a function algebra on X. We denote by $H(A)$ the family of all A-homomorphism, that is, all non-null real multiplicative linear functionals on A.

There are, among others, two interesting problems concerning $H(A)$: to obtain a suitable topological representation of $H(A)$, and to know whether or not $H(A) = X$, where the equality means that every homomorphism φ on A is the evaluation at some point $p \in X$; that is, φ is supported at $\{p\}$. This paper is related with the first problem, while the method presented here will be used to study the second problem in a separate paper. A general form of obtaining representations for the maximal ideal space of any prescribed algebra will be given in Theorem 2.2. The rest of the paper is devoted to showing how this general setting can be used to obtain representations in more familiar terms. In particular, some open problems, related to algebras of differentiable functions on a real Banach space, are solved (see Proposition 3.3).

Algebras on X, as defined above, can be considered as algebras of continuous functions on a completely regular Hausdorff space. There are different forms of obtaining a uniformity on X using the functions in A. This may be accomplished, for example, by identifying X with a (dense) subspace of $H(A)$, as in Isbell's paper [5], considering in X the weak topology induced by the isotone homomorphisms, and proving that $H(A)$ is the completion of X in the weak uniformity induced by A. That paper is a basis for our work, but we do not need the hypothesis 'A is closed under composition' that is heavily used there.

Here we present the following approach. Let U_A be the uniformity generated on X by A_b; that is, U_A is defined by the pseudometrics

$$d_{f_1,\dots,f_n}(x, y) = \max_{1\leq k\leq n}\{| f(x) - f(y) |; f_1, \dots , f_n \in A_b\}, \qquad x, y \in X.$$

We denote by τ_A the topology induced by U_A on X; that is, U_A is the weak uniformity in the notation of [5]. Since A separates points in X, (X, τ_A) is a completely regular Hausdorff space. In what follows, topological notions on X are relative to the τ_A-topology.

If A contains unbounded functions, it is possible to define another uniformity on

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X using A in place of A_b. Under the assumptions that A is closed under bounded inversion, these two uniformities induce the same topology on X (see [5], Theorem 1.4).

Denote by X_A the completion of (X, U_A); then X_A is a compact Hausdorff space. X can be considered as a dense subspace of X_A. It is clear that (X, τ_A) is a compact space if and only if it is complete. If A contains unbounded functions, then (X, τ_A) is not complete. It is known that each $f \in A_b$ has a unique continuous extension to X_A; this extension will be denoted by $\hat{f}$ and $\hat{A} = \{\hat{f} : f \in A_b\}$. In Proposition 2.1 it will be proved that $\hat{A}$ separates points in X_A; then, by the Stone-Weierstrass theorem, $\hat{A}$ is a dense subspace of $C(X_A)$ in the uniform norm.

In order to show the advantage of using the topology τ_A, even when A is an algebra of continuous functions on X with respect to some topology τ in X, let us present the following example.

EXAMPLE. Let B be the algebra of 2π-periodic functions in $\mathbb{R}$ and A the restriction of functions in B to $(0, 2\pi]$. It can be proved that $(0, 2\pi]$ with the τ_A topology is a compact space.

If (X, τ) is a completely regular Hausdorff topological space and A is an algebra of τ-continuous functions in X, then $\tau_A = \tau$ if and only A weakly separates points and τ-closed set; that is, if $P \subset X$ is closed and $x \notin P$, there exists $f \in A$ such that $f(x) \notin \operatorname{cl} f(P)$. Let P be a subset of X_A, such that $X \subset P$ and every function $f \in A$ has a continuous (in the induced topology) extension to each point $q \in P$: this extension will be denoted by $\hat{f}(q)$. Since continuous extensions are unique, there is no confusion in using this notation without reference to P.

If (X, τ) is a topological space and $P \subset X$, $\operatorname{cl}_X P$ denotes the closure of P in X and $P^c = X \setminus P$. If $f : X \to \mathbb{R}$, let $Z(f) = \{x \in X : f(x) = 0\}$ and $\operatorname{Coz}(f) = X \setminus Z(f)$.

The notation φ_p for a homomorphism means that it is supported at $\{p\}$.

An algebra A on X will be called *inverse-closed* if for every $f \in A$ such that $Z(f) = \emptyset$, $1/f \in A$.

Finally, topological structures of algebras are, in general, not considered here: this matter requires another paper.

2. The maximal ideal space and pairs of subordinated algebras

PROPOSITION 2.1. *Let A be a function algebra on X. Then $\hat{A}$ separates point in X_A. Moreover $\hat{A}$ weakly separates points and closed sets in X_A.*

PROOF. Let us consider the spaces (X, U_A), (X_A, T), (X_A, V) and (Y, W), where (X, U_A) and X_A are defined as in Section 1, T is the uniformity of X_A as a completion of $X(X, U_A)$, V is the uniformity generated on X_A by the algebra of functions $\hat{A}$ as in Section 1, and (Y, W) is the uniform completion of (X_A, V).

In X_A, the (X_A, V)-induced topology is weaker than the (X_A, T)-induced topology. We know that X is dense in X_A in the second topology above. Thus X is dense in (Y, W). Taking into account that $V|_X = U_A$, (Y, W) is a uniform completion of (X, U_A), and thus is a Hausdorff uniformity. But this is possible if and only if $\hat{A}$ separates points on X_A.

That $\hat{A}$ separates points for closed sets follows from the fact that $\hat{A}$ separates points and X_A is compact.

REMARK. $\hat{A}$ is an algebra on X_A, because $\hat{A}$ weakly separates points in X_A. If we apply the method of Section 1 to $(X_A, \hat{A})$ we again obtain X_A; that is $(X_A)_{\hat{A}} = X_A$. In fact $\hat{A}$ weakly separates points and closed sets in X_A, thus the topology induced by $U_{\hat{A}}$ on X_A agrees with the original topology on X_A, and so X_A is a completion of $(X_A, U_{\hat{A}})$.

All of our results are based in the following Theorem.

THEOREM 2.2. *Let A be a function algebra on X, then*

(a) $\varphi \in H(A_b)$ *if and only if there exists a (unique)* $p \in X_A$ *such that* $\varphi(f) = \hat{f}(p)$ *for every* $f \in A$. *Moreover* X_A *is (homeomorphic to) the maximal ideal space of* A_b;

(b) $\varphi \in H(A)$ *if and only if there exists a point* $p \in X_A$ *such that, every* $f \in A$ *has a finite continuous extension* $\hat{f}(p)$ *to* p *and* $\varphi(f) = \hat{f}(p)$. *The set* $I(A)$ *of all such* p, *with the topology induced by* X_A, *is (homeomorphic to) the maximal ideal space of* A.

PROOF. (a) Since each function $f \in A_b$ admits a continuous extension to X_A, it is clear that evaluations at points of X_A are in $H(A_b)$. Now suppose that $\varphi \in H(A_b)$. Set $M = \ker \varphi$. Since X_A is compact, for every $f \in M$ there exists $p \in X_A$ such that $\hat{f}(p) = 0$; in fact if $Z(f) = \emptyset$, there exists $\alpha > 0$ such that $f^2 \geq \alpha$. Since A is closed under bounded inversion, $f^{-2} \in A$; but

$$1 = \varphi(f^{-2} f^2) = \varphi(f^{-2})(\varphi(f))^2 = 0,$$

and we have a contradiction. Set $H = \{Z(\hat{f})\}$. The above argument says that H has the finite intersection property (any finite family $f_1, \ldots, f_n$ in M has a common zero). Because X_A is a compact set, then there exists $p \in X_A$ such that $p \in \bigcap_{f \in M} Z(\hat{f})$. A standard argument says that $\varphi(f) = \hat{f}(p)$ for every $f \in A$. The kernel of each homomorphism is a maximal ideal in A and $\hat{A}$ separates points in X_A; thus for each $\varphi \in H(A_b)$ there exists only one support point in X_A.

Let us prove the last assertion in (a). Denote by $\overline{\tau_A}$ the topology of X_A. Since every function $\hat{f}$ is $\overline{\tau_A}$-continuous and $(X_A, \overline{\tau_A})$ is compact, in order to prove that $\overline{\tau_A}$ is the Gelfand topology it is enough to see that the Gelfand topology on X_A is Hausdorff. Taking into account that $\hat{A}$ separates points, we have the proof.

(b) Let $\varphi \in H(A)$, then there exists $p \in X_A$ such that for $f \in A_b$, $\varphi(f) = \hat{f}(p)$. If $g \in A$, setting $h_g(x) = (g(x) - \varphi(g))^2/(1 + (g(x) - \varphi(g))^2)$, we have that $\varphi(h_g) = \hat{h}_g(p) = 0$. Take an arbitrary net $\{x_\lambda\}$ in X such that $x_\lambda \to p$. Since $\hat{h}_g \in C(X_A)$, $\hat{h}_g(x_\lambda) \to 0$, but this is possible if and only if $g(x_\lambda) \to \varphi(g)$. Thus g has a finite continuous extension $\hat{g}(p)$ to p and $\varphi(g) = \hat{g}(p)$.

If $p \in X_A$ and every function $f \in A$ has a finite continuous extension $\hat{f}(p)$ to p, by defining $\varphi(f) = \hat{f}(p)$ we obtain a homomorphism on A.

For every function $f \in A$, $\hat{f} \in C(I(A))$. Therefore the restriction of the $\overline{\tau_A}$ topology to $I(A)$ is finer than the Gelfand topology. In order to prove the assertion it is enough to show that if C is a closed subset of $I(A)$ for the induced topology and $p \in I(A) \setminus C$, then there exists $f \in A$ such that $\hat{f}(p) \notin \mathrm{cl}_{\mathbb{R}}\, \hat{f}(C)$. Take a closed set $D \subset X_A$ such that $D \bigcap I(A) = C$. Since $p \notin D$, for every $q \in D$, there exists $f_q \in A_b$ such that $\hat{f}_q(p) = 0$ and $\hat{f}_q(q) = 1$. Set $V_q = \{x \in X_A : \hat{f}_q(x) \geq 1/2\}$. Since D is compact we can take $q_1, \dots, q_n \in D$ such that $D \subset \bigcup_{k=1}^n \{x \in X_A : f_{q_k}^2(x) \geq 1/4\}$. Defining $f(x) = \sum_{k=1}^n f_{q_k}^2(x)$, we have that $\hat{f}(q) = 0$ and $\hat{f}(s) \geq 1/4$ for all $s \in C$; then $\hat{f}(q) \notin \mathrm{cl}_{\mathbb{R}}\, \hat{f}(C)$.

REMARKS. (a) Notice that if A is an algebra of bounded function on X and $f \in A$, then $\hat{f}$ is just the Gelfand transform of f. Since the Gelfand topology is initial, $I(A)$ has the initial topology generated by the extension of functions in A to $I(A)$. Thus the extension of functions in A to $I(A)$ weakly separates points and closed set.

(b) If A is inverse-closed and closed under uniform convergence, then every function in A has a definite limit (finite or not) at each point of X_A; moreover A_b may be (algebraically) identified with $C(X_A)$. If (X, τ_A) is locally compact and σ-compact, X_A is the Stone-Čech compactification of (X, τ_A).

(c) If $A = A_b$, then $X = I(A)$ if and only if (X, τ_A) is compact.

In order to give some applications of the above results, let us study relations between two algebras on the same set. Given a non-empty set X, (A, B) is called a *pair of subordinated algebras* on X if:

(i) A and B are function algebras on X;
(ii) $B \subset A$;
(iii) Every homomorphism on B has an extension to a homomorphism on A.

If (A, B) is a pair of subordinated algebras on X, different homomorphisms on A may induce the same homomorphism on B. Then X_B can be obtained as a quotient set of X_A. In order to prove Theorem 2.12, let us consider the following notation. If $f \in B_b$, we denote by f_a its extension to X_A and by f_b its extension to X_B; f_a (respectively f_b) denotes the extension of f to X_A and to $I(A)$ (respectively to X_B and to $I(B)$).

PROPOSITION 2.3. *Let A and B be function algebras on X with $B \subset A$. Then for every homomorphism $\varphi \in H(B_b)$, there exists $\psi \in H(A_b)$ such that $\psi|_{B_b} = \varphi$.*

PROOF. Fix $\varphi \in H(B_b)$ and $p \in X_B$ such that for every $f \in B_b$, $\varphi(f) = f_b(p)$. If $p \in X$, evaluation at p is an extension of φ to A_b. If $p \in X_B \setminus X$ there exists a net $\{x_\lambda\}$ in X such that $x_\lambda \to p$ in X_B and for every $f \in B_b$, $f_b(x_\lambda) \to \varphi(f) = f_b(p)$. Since X_A is compact we can suppose, without loss of generality, that there exists $q \in X_A$ such that $x_\lambda \to q$. Evaluation at q is a homomorphism on A_b which extends φ.

If A and B are function algebras on X, $B \subset A$, given a homomorphism φ on B_b, there exists (a unique) $p \in X_B$ such that $\varphi(f) = f_b(p)$, for every $f \in B_b$. Fix any $q_p \in X_A$ such that the evaluation at q_p is an extension of φ to A_b. So there is defined a function $\Theta : X_B \to X_A$, for $z \in X_B$, $h_b(z) = h_a(\Theta(z))$.

Let R_B be the equivalence relation on X_A defined by

$$x R_B y \quad \text{if and only if} \quad \text{for all } f \in B_b,\ f_a(x) = f_a(y).$$

Define $\Psi : X_B \to X_A/R_B$ by $\Psi(p) = \pi(\Theta(p))$, where $\pi : X_A \to X_A/R_B$ is the quotient map. Notice that Ψ is one-to-one and onto. In X_A/R_B we consider the quotient topology.

PROPOSITION 2.4. *Let (A, B) be a pair of subordinated algebras on X. For $x, y \in I(A)$ set*

$$x R_B^* y \quad \textit{if and only if} \quad \textit{for all } f \in B,\ f_a(x) = f_a(y).$$

Then R_B and R_B^* determine the same equivalence relation on $I(A)$.

PROOF. Fix $x, y \in I(A)$.

If x and y are not R_B-related, there exists $f \in B_b$ such that $f_a(x) \neq f_a(y)$; then x and y are not R_B^*-related.

If x and y are not R_B^*-related, there exists $f \in B$ such that $f_a(x) \neq f_a(y)$. Defining $g(z) = (f(z) - f_a(y))^2/(1 + (f(z) - f_a(y))^2)$, $g \in B_b$ and $g_a(x) \neq g_a(x)$.

Let us present some notation: let $\pi^* : I(A) \to I(A)/R_B$ be the quotient map, where in $I(A)$ we consider the restriction of the equivalence relation R_B on X_A to

$I(A)$; $I(A)$ is endowed with the topology induced by X_A and $I(A)/R_B$ with the quotient topology. Let $\Omega : I(A)/R_B \to \pi(I(A))$ be defined by

$$\Omega(\pi^*(x)) = \pi(x), \qquad \text{for } x \in I(A).$$

$\pi I(A))$ is endowed with the topology induced by X_A/R_B.

Let (A, B) be a pair of subordinated algebras on X. Then B can be realised as an algebra of functions on each one of the spaces defined above as follows: For $f \in B$, denote by $L(f)$ the extension of f to $I(B)$; that is, $L(f) = f_b$; denote by $M(f)$ the extension f_a of f to $I(A)$; $N(f)(\pi^*(x)) = L(f)(x)$ for $x \in I(A)$ and $P(f)(\pi(x)) = M(f)(x)$ for $x \in I(A)$. It is clear that all these functions are continuous in the corresponding topologies defined above. Set

$$B_1 = \{L(f) : f \in B\}, \qquad B_2 = \{M(f) : f \in B\},$$
$$B_3 = \{N(f) : f \in B\}, \qquad B_4 = \{P(f) : f \in B\}.$$

The mappings L, M, N and P are one-to-one and onto considered from B into B_1, B_2, B_3 and B_4 respectively.

PROPOSITION 2.5. *Let (A, B) be a pair of subordinated algebras on X. The maximal ideal space X_B of B_b is (homeomorphic to) X_A/R_B.*

PROOF. It is sufficient to prove that Ψ is a continuous open mapping.

Let $P \subset X_A/R_B$ be closed in the quotient topology. Fix $x \in X_B \setminus \Psi^{-1}(P)$. Set $y = \Theta(x)$ and $z = \pi(y)$. It is clear that $z \notin P$. Since $\pi^{-1}(P)$ is R_B-saturated, for each $v \in \pi^{-1}(P)$, there exists $f^v \in B_b$ such that $f_a^v(y) = 0$ and $0 \notin \text{cl}\, f_a^v(V_v)$ for some open neighbourhood V_v of v. Taking into account that $\pi^{-1}(P)$ is compact, there exists $h \in B_b$ such that $h_a(y) = 0$ and $h_a(\pi^{-1}(P)) \subset [\epsilon, \infty)$ for some $\epsilon > 0$. Now $h_a(y) = h_b(x) = 0$ and for $z \in \Psi^{-1}(P)$, $h_a(\Theta(z)) = h_b(z) \geq \epsilon$. Thus there exists in X_B an open neighborhood of x which does not meet $\Psi^{-1}(P)$. This says that Ψ is a continuous map.

Let us prove that Ψ is open. Fix a proper open subset D of X_B and $x \in D$. If $\Psi(x)$ is not an inner point of $\Psi(D)$, there exits a net $\{x_\lambda\}$ in $\Psi(D)^c$ such that $x_\lambda \to \Psi(x)$. Take $\{y_\lambda\}$ in X_B such that $\Psi(y_\lambda) = x_\lambda$. Without loss of generality, since X_B is compact, we may suppose that $y_\lambda \to s$ for some $s \in X_B$.

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For every $f \in B_b$, $L(f)(y_\lambda) \to L(f)(s)$ and

$$\begin{aligned}L(f)(y_\lambda) &= f_b(y_\lambda) = f_a(\Theta(y_\lambda) \\ &= N(f)(\pi(\Theta(x)) = N(f)(\Psi(y_\lambda) \\ &= N(f)(x_\lambda) \\ &\to N(f)(\Psi(x)) \\ &= N(f)(\pi(\Theta(x)) = f_a(\Theta(x)) \\ &= f_b(x).\end{aligned}$$

Then for every $f \in B_b$, $f_b(x) = f_b(s)$; since B_b separates points in X_B, $x = s$; that is, $y_\lambda \to x$.

Now taking into account that D is X_B-open there exists λ_0 such that for $\lambda > \lambda_0$, $y_\lambda \in D$. Thus, for such λ, $\Psi(y_\lambda) = x_\lambda \in \Psi(D)$, a contradiction.

PROPOSITION 2.6. *Let (A, B) be a pair of subordinated algebras on X. Then quotient topology on $I(A)/R_B$ and the B_3-initial topology agree.*

PROOF. Denote by τ_1 the quotient topology, and by τ_2 the B_3-initial topology. Since for every $f \in B$, $N(f)$ is τ_2-continuous (π^* is a quotient map), $\tau_2 \subset \tau_1$. Let us prove the other inclusion.

Let Q be a closed set in $\mathbb{R}$ and fix $f \in B$. $M(f)^{-1}(Q)$ is a saturated closed set in $I(A)$; thus $\pi^*(M(f)^{-1}(Q)$ is closed in $I(A)/R_B$ in the quotient topology and $(N(f)^{-1}(Q) = \pi^*(M(f)^{-1}(Q))$. Since the family of set $N(f)^{-1}(Q)$, $f \in B$ and Q a closed set on $\mathbb{R}$, is a base for the closed sets in the B_3-initial topology, the proof is complete.

Let (X, τ) be a topological space, $Y \subset X$ and R an equivalence relation on X. Denote by T the restriction of R to Y and let π and π^* be the respective quotient maps $\pi : X \to X/R$ and $\pi^* : Y \to Y/T$. It is well known that, in general, Y/T and $\pi(Y)$ are not homeomorphic when Y/T is endowed with the quotient topology and $\pi(Y)$ with the topology induced by the quotient topology of X/R (see [3, Examples 2.4.16 and 2.4.17]). Let us prove that, in the case of a pair of subordinated algebras (A, B), $I(A)/R_B$ and $\pi(I(A))$ are homeomorphic.

We were not able to find a reference for the following result. However, since it seems to be known, it is presented without proof.

PROPOSITION 2.7. *Let Y, Z be non-empty sets, $C \subset \mathbb{R}^Y$ and $D \subset \mathbb{R}^Z$. Let $\delta : Y \to Z$ and $\Delta : C \to D$ be one-to-one onto mappings such that for every $x \in Y$ and each $f \in C$, $f(x) = \Delta(f)(\delta(x))$. If Y and Z are endowed with the initial topology for C and D respectively, then δ is a homeomorphism.*

As a consequence of the above proposition, we have:

PROPOSITION 2.8. *Let (A, B) be a pair of subordinated algebras on X. Then $I(B)$ and $I(A)/R_B$ are homeomorphic.*

PROOF. Let (A, B) be a pair of subordinated algebras on X. Fix $y \in I(B)$. Since evaluation at y is a homomorphism on B it has a continuous extension to A, therefore it can be seen as the evaluation at some point $z_y \in I(A)$. Thus the function $\delta : I(B) \to I(A)/R_B$ given by $\delta(y) = \pi^*(y_z)$ is well defined. Since B_1 separates points in $I(B)$ and (A, B) is a subordinated pair δ is one-to-one and onto.

Define $\Delta : B_1 \to B_3$ as $\Delta(f) = f o \delta^{-1}$. If $y \in X$, $f(y) = \Delta(f)(\delta(x))$; then setting $Y = I(B)$, $Z = I(A)/R_B$, $C = B_1$ and $D = B_2$, by the above Proposition, $I(B)$ and $I(A)/R_B$ are homeomorphic.

PROPOSITION 2.9. *Let (A, B) be a pair of subordinated algebras on X. Then $I(A)/R_B$ and $\pi(I(A))$ are homeomorphic, considering in $I(A)/R_B$ the quotient topology and in $\pi(I(A))$ the topology induced by the quotient topology of X_A/R_B.*

PROOF. It is sufficient to note that: (a) Ω is one-to-one and onto; (b) Ω is continuous; (c) Ω is closed.

(a) is easy to prove.

(b) Fix $K \subset \pi(I(A))$ closed, take $J \subset X_A/R_B$ closed such that $J \bigcap \pi(I(A)) = K$. Then $\pi^{-1}(J)$ is closed in X_A and R_B-saturated. Therefore $\pi^{-1}(J) \bigcap I(A)$ is closed in $I(A)$ and R_B^*-saturated (see Proposition 2.4). Thus $\pi^*(\pi^{-1}(J) \bigcap I(A))$ is closed in $I(A)/R_B^*$. But $\Omega^{-1}(K) = \pi^*(\pi^{-1}(J) \bigcap I(A))$.

(c) Let Q be a closed subset of $I(A)/R_B$; then $S = (\pi^*)^{-1}(Q)$ is closed in $I(A)$. There exists $D \subset X_A$ closed such that $D \bigcap I(A) = S$.

Set $E = \pi^{-1}(\mathrm{cl}_{X_A/R_B} \pi(D))$. We have that $E \bigcap I(A) = D \bigcap I(A)$. In fact, take $x \in E \bigcap I(A)$ and a net $\{x_\lambda\}$ in D such that $\pi(x_\lambda) \to \pi(x)$. Since $P(f)$ is continuous $P(f)(\pi(x_\lambda)) = M(f)(x_\lambda) \to P(f)(\pi(x)) = M(f)(x)$ for every $f \in B_b$. The same can be proved for every $f \in B$. Then $N(f)(\pi^*(x_\lambda)) \to N(f))\pi^*(x)$ for all $f \in B$. According to Remark (a) of Proposition 2.2, B_1 weakly separates points and closed sets in $I(B)$. By Proposition 2.9, B_3 has the same property in $I(A)/R_B$; thus $\pi^*(x)$ is an adherent point of Q. Since P is closed, $\pi^*(x) \in D$; this says that $x \in D \bigcap I(A)$.

E is R_B-saturated and closed; thus $\pi(E)$ is closed in X_A/R_B. This implies $\pi(E) \bigcap \pi(I(A))$ is closed in the induced topology. On the other hand,

$$\begin{aligned}\Omega(Q) &= \Omega(\pi^*((\pi^*)^{-1}(Q)) = \Omega(\pi^*(D \bigcap I(A)) \\ &= \Omega * \pi^*(E \bigcap I(A)) = \pi(E \bigcap I(A)) \\ &= \pi(E) \bigcap \pi(I(A)).\end{aligned}$$

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PROPOSITION 2.10. *Let* (A, B) *be a pair of subordinated algebras on* X*; then* $I(B)$ *and* $\pi(I(A))$ *are homeomorphic.*

PROOF. This follows from Propositions 2.8 and 2.9.

If (A, B) is a pair of subordinated algebras on X, consider

$$H(A, B) = \{g \in \mathbb{R}^{\pi(I(A))} : \exists f \in B, f_a(x) = (g o \pi)(x) \text{ for all } x \in (I(A))\}.$$

PROPOSITION 2.11. *Let* (A, B) *be a pair of subordinated algebras on* X. *Then the* $H(A, B)$*-initial topology on* $\pi(I(A))$ *agrees with the topology induced by the quotient topology of* X_A/R_B.

PROOF. It is clear that the topology induced by the quotient one is finer than the initial. Now take P closed in the topology induced by the quotient topology of X_A/R_B in $\pi(I(A))$ and $x \in \pi(I(A)) \setminus P$. Take $z \in I(A)$ such that $\pi(z) = y$ and set $Q = \pi^{-1}(P) \subset X_A$. As in the proof of Proposition 2.3 there exists $f \in B_b$, such that $f_a(y) = 0$ and $f_a(\pi^{-1}(P)) \subset [\epsilon, \infty)$, for some $\epsilon > 0$. Let $g \in H(A, B)$ be such that $f_a = g o \pi$, then $g(x) \notin \text{cl}\, \mathbb{R} g(P)$.

THEOREM 2.12. *Let* (A, B) *be a pair of subordinated algebras on* X. *Then the following spaces are homeomorphic:*

(i) $H(B)$ *with the Gelfand topology;*
(ii) $\pi(I(A))$ *with the topology induced by* X_A/R_B*;*
(iii) $I(A)/R_B$ *with the quotient topology.*

PROOF. This can be derived from Propositions 2.4, 2.6 and 2.7.

The relations given above are transitive in the following sense:

PROPOSITION 2.13. *Let* (A, B) *and* (B, C) *be two pairs of subordinated algebras on* X. *Then*

$$I(C) \cong I(B)/R_C \cong I(A)/R_C.$$

3. Applications

Theorem 2.2 gives a method for obtaining the maximal ideal space of A. This can be used to determinate conditions for the equality $X = I(A)$. This will be accomplished in another paper by the same authors. Now let us see how Theorem 2.2 and 2.10 can be used to obtain representations of the maximal ideal space of A in more familiar terms.

PROPOSITION 3.1. *Let (X, d) be a metric space with an unbounded metric and A an algebra of continuous functions on X such that:*

(a) *A separates points and d-closed sets;*
(b) *There exists $x_0 \in X$ such that $f(x) = d(x, x_0) \in A$.*
(c) *For every $f \in A$ and each bounded set $S \subset X$, there exists $g \in A_b$ such that $f|_S = g|_S$.*

Fix $x_0 \in X$ and set $B_n = \{x \in X : d(x, x_0) \leq n\}$. Then the maximal ideal space of A is $Y = \bigcup_{n=1}^{\infty} cl_{X_A} B_n$ with the X_A induced topology.

PROOF. If $p \in X_A \setminus Y$ and $\{x_\lambda\}$ is a net in X with $x_\lambda \to p$, since $\{x_\lambda\}$ is an unbounded net, from (b) we have that there exists a function $f \in A$ which has no continuous extension to p; thus $p \notin I(A)$.

Now fix $z \in Y$ and $f \in A$. There exists $g \in A_b$ such that $g|_{B_n} = f|_{B_n}$. Defining $\hat{f}(p) = \hat{g}(p)$ (this value does not depend on g) we have that f has a continuous finite extension to p. In fact, if $\{x_\lambda\}$ is a net in X that converges to p, then it is bounded, that is, there exists a positive integer k such that $d(x_\lambda, x_0) \leq k$. Taking $h \in A$ such that $h|_{B_{k+1}} = f|_{B_{k+1}}$, we have that $\lim f(x_\lambda) = \lim h(x_\lambda) = \hat{f}(p)$. Since X is dense in Y, $\hat{f}$ is continuous on Y.

The above arguments say that $Y = I(A)$.

As an application of the above proposition we will extend and generalize some results of [6].

If E is a real Banach space, then $C_b(E)$ denotes the space of all continuous real functions in E which are bounded on bounded subset of E. Set $B_n = \{x \in E : \| x \| \leq n\}$ and let $E_X = \bigcup_{n=1}^{\infty} E_n$, where E_n is the closure of B_n in the Stone-Čech compactification of E.

C_b^m is the set of all functions $f : E \to \mathbb{R}$ of class C^m, such that f and its differentials $df, \dots, d^m f$,

$$d^k f : E \to L(^k E, \mathbb{R})$$

(for this notation see [8]), are bounded on bounded sets of E. We endow C_b^m with the topology generated by all seminorms $(p_n)_{n \in \mathbb{N}}$, where

$$f \in C_b^m(E) \to p_n(f) = \sup_{\|x\| \leq n} \left\{ |f(x)| + \sum_{k=1}^{m} \|d^k f(x)\| \right\}.$$

With this topology, $C_b^m(E)$ is a real Frechet algebra.

In [6] was proved that: (Theorem 3) $H(C_b(E)) = E_X$, and (Theorem 13) if E is a super-reflexive Banach space, then the spectrum of $C_b^1(E)$, endowed with the Gelfand topology agrees with $E_X / R_{C_b^1(E)}$ endowed with the quotient topology, where $R_{C_b(E)}$ is the following equivalence relation on E_X:

$$xRy \quad \text{if and only if} \quad x \text{ and } y \text{ determine the same homomorphism on } C_b^1.$$

As Jaramillo and Llavona quoted in [6], the technique used there cannot be extended to the case $C_b^m(E)$ for $m \geq 2$. Using the techniques of Theorem 2.1 we extend the above result. It is not necessary for the Banach spaces to be super-reflexives.

PROPOSITION 3.2. *Let E be a real Banach space. Then the maximal ideal space of $C_b(E)$ is $Y = \bigcup_{n=1}^{\infty} \mathrm{cl}_{\beta E} B_n$, where $B_n = \{x \in E : \|x\| \leq n\}$ and βE is the Stone-Čech compactification of E.*

PROOF. If $x_0 \in X$ and $P, Q \subset E$ are disjoint closed sets, the functions

$$f(x) = d(x, x_0) \quad \text{and} \quad g(x) = d(x, P)/(d(x, P) + d(x, Q))$$

are in $C_b(E)$. Thus conditions (a), (b) and (c) in Proposition 3.1 hold (we take $x_0 = 0$). On the other hand, it is clear that $C_b(E)$ is closed under bounded inversion and contains the constant functions.

Since all metric continuous bounded functions in E are in $C_b(E)$, $E_{C_b(X)}$ is the Stone-Čech compactification of E.

PROPOSITION 3.3. *Let E be a real Banach space. The maximal ideal space of $C_b^m(E)$ is $Y/R_{C_b^m(E)}$, where Y is defined as in Proposition 3.2 and $R_{C_b^m(E)}$ is the following equivalence relation on E_X:*

$$x R_{C_b^m(E)} \quad \textit{if and only if} \quad f(x) = f(y) \textit{ for all } f \in C_b^m(E).$$

PROOF. It is sufficient to prove that $(C_b(E), C_b^m(E))$ is a pair of subordinated algebras on E. For this it is enough to prove that each homomorphism in $C_b^m(E)$ has an extension to a homomorphism in $C_b(E)$). This last assertion can be proved as in [6, Theorem 8].

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