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Higher Order Tangents to Analytic Varieties along Curves

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Abstract. Let V be an analytic variety in some open set in $\mathbb{C}^n$ which contains the origin and which is purely k-dimensional. For a curve γ in $\mathbb{C}^n$, defined by a convergent Puiseux series and satisfying $\gamma(0) = 0$, and $d \geq 1$, define $V_t := t^{-d}\big(V - \gamma(t)\big)$. Then the currents defined by V_t converge to a limit current $T_{\gamma,d}[V]$ as t tends to zero. $T_{\gamma,d}[V]$ is either zero or its support is an algebraic variety of pure dimension k in $\mathbb{C}^n$. Properties of such limit currents and examples are presented. These results will be applied in a forthcoming paper to derive necessary conditions for varieties satisfying the local Phragmén-Lindelöf condition that was used by Hörmander to characterize the constant coefficient partial differential operators which act surjectively on the space of all real analytic functions on $\mathbb{R}^n$.

1 Introduction

A classical concept in analysis is to approximate a given analytic variety by simpler tangent varieties and to relate properties of these objects. For linear partial differential operators $P(D)$ with constant coefficients an important illustration of this method was given by Hörmander in his fundamental paper [7], in which he characterized the surjectivity of $P(D)$ on spaces $\mathcal{A}(\Omega)$ of all real analytic functions on a convex open set Ω in $\mathbb{R}^n$. He proved that this property depends only on the principal part P_m of P and is equivalent to a condition $\mathrm{HPL}(\Omega)$ of Phragmén-Lindelöf type for the zero variety $V(P_m)$ of P_m. Necessary for $\mathrm{HPL}(\Omega)$ and characterizing for $\mathrm{HPL}(\mathbb{R}^n)$ is that $V(P_m)$ satisfies the local Phragmén-Lindelöf condition $\mathrm{PL}_{\mathrm{loc}}(\xi)$ at each $\xi \in V(P_m) \cap \mathbb{R}^n$, $|\xi| = 1$. When $n = 3$, $\mathrm{PL}_{\mathrm{loc}}(\xi)$ is equivalent to the local hyperbolicity of P_m at ξ by Hörmander [7]. A different geometric characterization was given by Braun [2].

Other results in the same spirit but for different conditions of Phragmén-Lindelöf type for algebraic varieties were obtained by Meise, Taylor, and Vogt [8] and [9] and by the present authors in [3] and [4]. Through the latter work it became clear that more refined tangent objects were needed to investigate the behavior of the given variety near the singularities of the tangent variety since the approximation by the tangent variety in these regions is not good enough. Finally, our attempt to find a geometric characterization of the property $\mathrm{PL}_{\mathrm{loc}}(\xi)$ showed that the following extension of the concept of tangent varieties is useful (see [5]).

For a precise statement, let V be an analytic variety in some open set in $\mathbb{C}^n$ which contains the origin and which is purely k-dimensional, let $\gamma\colon B(0,\epsilon) \setminus [-\epsilon, 0] \to \mathbb{C}^n$

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be defined by $\gamma(t) = \sum_{j=q}^{\infty} a_j t^{j/q}$ as a convergent Puiseux series, and fix $d \geq 1$. Then the varieties $V_t := t^{-d}\big(V - \gamma(t)\big)$, $t \in B(0,\epsilon) \setminus [-\epsilon, 0]$, are defined in larger and larger open sets in $\mathbb{C}^n$ as t tends to zero. As main result of the present paper we show that the currents of integration over V_t converge to a limit current $T_{\gamma,d}[V]$ as t tends to zero and that $T_{\gamma,d}[V]$ is either zero or a holomorphic k-chain. In particular, its support is a purely k-dimensional analytic variety, which even is algebraic. An explicit description of $T_{\gamma,d}[V]$ in terms of algebraic equations can be derived using canonical defining functions. This description is applied to derive properties of the limit current. When $d = 1$, the limit current $T_{\gamma,1}[V]$ is supported by $T_0V - \gamma'(0)$, where T_0V denotes the tangent cone of V and $\gamma'(0) = \lim_{t \searrow 0} \gamma(t)/t$. When $d > 1$, then the limit current $T_{\gamma,d}[V]$ is invariant under shifts parallel to $\gamma'(0)$. We also give a precise statement concerning the approximation of V by $T_{\gamma,d}V$ and several examples.

The results of the present paper are used in [5] to derive a necessary geometric condition for a purely k-dimensional analytic variety V in $\mathbb{C}^n$ to satisfy $\mathrm{PL}_{\mathrm{loc}}(\xi)$ at $\xi \in V \cap \mathbb{R}^n$. The necessary condition of [5] is also sufficient in the case of analytic surfaces in $\mathbb{C}^3$ and it implies a geometric characterization of those polynomials $P \in \mathbb{C}[Z_1, \dots, Z_4]$ for which $P(D)$ is surjective on $\mathcal{A}(\mathbb{R}^4)$.

2 Preliminaries about Currents, k-Chains, and Convergence

In this section we introduce the basic notions and facts which are needed to introduce limit varieties and to investigate their properties.

We denote by $\mathbb{N}$ the set of positive integers and by $B^n(z,r)$ the ball $\{w \in \mathbb{C}^n : |w - z| < r\}$, where $|\cdot|$ denotes the euclidean norm. The exponent n may be omitted.

Definition 2.1 An *analytic variety* in $\mathbb{C}^n$ is defined as a closed analytic subset of some open set Ω in $\mathbb{C}^n$. If V is of pure dimension k, its *current of integration* $[V]$ is defined by

$$[V](\varphi) := \int_V \varphi,$$

where φ is any C^∞-form of bidegree (k,k) with compact support in Ω.

Definition 2.2 A *holomorphic k-chain* in an open set Ω in $\mathbb{C}^n$ is a locally finite sum

$$W = \sum n_i [V_i]$$

where $n_i \in \mathbb{Z}$ and $[V_i]$ is the current of integration over an irreducible analytic subvariety of Ω of pure dimension k. Recall that the support of W is equal to the union of those V_i for which $n_i \neq 0$.

Definition 2.3 The following definitions are taken from Chirka [6, 10.1, 11.1, 12.1, 12.2, and 11.3]. Fix an analytic set $V \subset \mathbb{C}^n$ of pure dimension k, an affine plane $L \subset \mathbb{C}^n$ of dimension $n-k$, and an isolated point z of $V \cap L$. Then there is a neighborhood U of z such that the projection $\pi_L \colon U \cap V \to \pi_L(U \cap L) \subset L^\perp$ along L is an analytic cover. Its sheet number in z is denoted by $\mu_z(\pi_L|_V)$.

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The minimum of the sheet numbers $\mu_z(\pi_L|_V)$ when L ranges over all $(n-k)$-dimensional affine subspaces for which z is an isolated point of $V\cap L$ is the *multiplicity* $\mu_z(V)$ of V at z.

If $D\subset\mathbb{C}^n$ is open and $D\cap V\cap L$ is finite, then the *intersection index* $i_D(V,L)$ is defined as
$$i_D(V,L):=\sum_{w\in D\cap L\cap V}\mu_w(\pi_L|_V).$$
If $W=\sum_{j=1}^m n_j[V_j]$ is a holomorphic k-chain and $D\cap L\cap \operatorname{Supp} W$ is finite, then
$$i_z(W,L):=\sum_{j=1}^m n_j\mu_z(\pi_L|_{V_j})\quad\text{and}\quad i_D(W,L):=\sum_{w\in D\cap L\cap \operatorname{Supp} W} i_w(W,L).$$
$i_z(W,L)$ is called *sheet number* of the holomorphic chain W in z.

If V is a purely k-dimensional algebraic subset or a holomorphic k-chain in $\mathbb{C}^n$ and $L\subset\mathbb{C}^n$ is an affine $(n-k)$-dimensional subspace such that $V\cap L$ is finite and such that the projective closures of V and of L do not have points at infinity in common, then $i_{\mathbb{C}^n}(V,L)$ is the *degree* of V.

Remark 2.4 Note that, in the setting of Definition 2.3,
$$\mu_z(\pi_L|_V)=\mu_z(V)$$
whenever L is an affine $(n-k)$-dimensional subspace of $\mathbb{C}^n$ which is transversal to V at z (see [6, 11.2, Proposition 2]).

In order to define convergence of holomorphic k-chains, we recall first the notion of convergence of a sequence of sets in a metric space (see Chirka [6, 15.5]).

Definition 2.5 A sequence of sets $(V_j)_{j\in\mathbb{N}}$ in a metric space is said to *converge* to a set V if

(i) V coincides with the limit set of the sequence, *i.e.*, consists of all points of the form $\lim_{\nu\to\infty}x_\nu$ where $x_\nu\in V_{j_\nu}$ for an arbitrary subsequence $(j_\nu)_{\nu\in\mathbb{N}}$ of $\mathbb{N}$, and
(ii) for any compact set $K\subset V$ and any $\epsilon>0$, there is an index $j(\epsilon,K)$ such that K belongs to the ϵ neighborhood of V_j for all $j>j(\epsilon,K)$.

Definition 2.6

(a) A sequence $(T_j)_{j\in\mathbb{N}}$ of currents of bidegree $(n-k,n-k)$ on some open set Ω in $\mathbb{C}^n$ is said to *converge* to the current T if $T(\varphi)=\lim_{j\to\infty}T_j(\varphi)$ for each C^∞-form φ of bidegree (k,k) with compact support in Ω.
(b) A sequence $(W_j)_{j\in\mathbb{N}}$ of holomorphic k-chains in Ω *converges* to a holomorphic k-chain W if
   (i) the supports of W_j converge to $V:=\operatorname{Supp} W$ as subsets of Ω in the sense of Definition 2.5, and

(ii) for each regular point $a \in V$ and each $(n-k)$-dimensional plane L through a, transversal to V at a, there is a neighborhood U of a such that $V \cap L \cap U = \{a\}$ and such that $i_U(W_j, L) = i_a(W, L)$ for all sufficiently large j.

The main facts about convergence of varieties are collected in the following theorem. For the proof of these facts, see Chirka [6, Proposition 16.1].

Notation 2.7 Let V be an analytic variety of pure dimension k in some open set Ω in $\mathbb{C}^n$. For $a \in \Omega$ and $\rho > 0$ satisfying $B(a, \rho) \subset \Omega$ let $\mathrm{vol}(\rho, V, a)$ denote the $2k$-dimensional Hausdorff measure of $V \cap B(a, \rho)$. If $W = \sum_i n_i [V_i]$ is a holomorphic k-chain with nonnegative n_i, then $\mathrm{vol}(\rho, W, a) := \sum_i n_i \,\mathrm{vol}(\rho, V_i, a)$.

We say that a sequence $(W_j)_{j \in \mathbb{N}}$ of analytic varieties or holomorphic k-chains in some open set $\Omega \subset \mathbb{C}^n$ has locally uniformly bounded volume if for all $a \in \Omega$ there are $\rho, C > 0$ such that $B(a, \rho) \subset \Omega$ and $\mathrm{vol}(\rho, W_j, a) \leq C$ for all $j \in \mathbb{N}$.

Theorem 2.8 *Let $(V_j)_{j \in \mathbb{N}}$ be a sequence of pure k-dimensional analytic varieties. Furthermore, let $(W_j)_{j \in \mathbb{N}}$ be a sequence of holomorphic k-chains in some open set $\Omega \subset \mathbb{C}^n$ with locally uniformly bounded volume. Then the following assertions hold:*

(a) *$(W_j)_{j \in \mathbb{N}}$ converges as a sequence of holomorphic k-chain if and only if it converges as a sequence of currents. In particular, the limit current is a holomorphic k-chain.*
(b) *If $([V_j])_{j \in \mathbb{N}}$ converges, then it has locally uniformly bounded volume and $(V_j)_{j \in \mathbb{N}}$ converges as sets in the sense of Definition 2.5.*
(c) *If $([V_j])_{j \in \mathbb{N}}$ has locally uniformly bounded volume, then it admits a convergent subsequence $([V_{j_\nu}])_{\nu \in \mathbb{N}}$.*

3 Existence of Limit Varieties

In this section we show that limit varieties of order $d \geq 1$ along curves γ, given by certain Puiseux series, always exist and that they are algebraic varieties in $\mathbb{C}^n$. To do this, we will use the following notions.

Definition 3.1

(a) A *simple curve* γ in $\mathbb{C}^n$ is a map $\gamma\colon B^1(0, \epsilon) \setminus \,]-\infty, 0] \to \mathbb{C}^n$ which, for some $q \in \mathbb{N}$, admits a convergent Puiseux series expansion

$$\gamma(t) = \xi_0 t + \sum_{j=q+1}^{\infty} \xi_j t^{j/q}, \quad |\xi_0| = 1,$$

where for a real number $d \geq 1$, t^d denotes the principal branch of the power function, *i.e.*, $t^d = |t|^d \exp\bigl(id \arg(t)\bigr)$, where $-\pi < \arg(t) < \pi$ for $t \in \mathbb{C} \setminus \,]-\infty, 0]$. The vector ξ_0 will be called the tangent vector to γ in the origin.
(b) For a pure k-dimensional variety V in $B^n(0, r)$ which contains the origin, a simple curve γ, and a real number $d \geq 1$ we let

$$V_t := V_{\gamma,d,t} := \{w \in \mathbb{C}^n : \gamma(t) + t^d w \in V\}, \quad t \in B^1(0, \epsilon) \setminus \,]-\epsilon, 0]\,.$$

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Remark Note that V_t is a pure k-dimensional variety in $B^n\big(-t^{-d}\gamma(t), |t|^{-d}r\big)$. Therefore, as t tends to zero in $B^1(0,r) \setminus\,]-r,0]$, the varieties V_t are defined in larger and larger open balls in $\mathbb{C}^n$. The following theorem shows that the currents $[V_t]$ have a limit.

Theorem 3.2 *Let V be a purely k-dimensional variety in $B^n(0,r)$ which contains the origin, let γ be a simple curve, and let $d \geq 1$. For the varieties V_t defined in Definition 3.1(b), the currents $[V_t]$ converge to a limit current W as t tends to zero in $\mathbb{C} \setminus\,]-\infty, 0]$. W is a holomorphic k-chain the support of which is an algebraic variety in $\mathbb{C}^n$.*

Definition 3.3 Under the hypotheses of Theorem 3.2 we define

$$T_{\gamma,d}[V] := \lim_{t\to 0}[V_t]$$

and call it the *limit current* of order d along the simple curve γ. Furthermore, $T_{\gamma,d}V :=$ Supp $T_{\gamma,d}[V]$ will be called the *limit variety* of V of order d along γ.

Remark Theorem 3.2 can be viewed as an extension of Chirka [6, 16.6, Proposition 2], where it is explained how the currents $[\frac{1}{t}V]$ converge to the current $T_0[V]$ as $t \to 0$, where Supp $T_0[V]$ is the tangent cone T_0V in the sense of [6, 8.1]. In fact, when $d = 1$, the variety $T_{\gamma,1}V$ is exactly equal to this tangent cone up to a translation by ξ_0, as we will show in Proposition 4.1.

To prove the theorem, we will show that the varieties V_t have locally bounded volume so that they form a relatively compact family of varieties. Therefore, the family of varieties will converge if we can prove that there is a unique limit variety in $\mathbb{C}^n$. This will be shown by studying the convergence of associated canonical defining functions for V_t. Lastly, it will be clear that the volume of the limit variety in a ball of radius r is $O(r^{2k})$, so that the limit is algebraic by Stoll's theorem. The first step is to find a bound for the volumes of the V_t.

Lemma 3.4 *Let V be an analytic variety of pure dimension k in $\mathbb{C}^n$ which contains the origin. Then there are constants $C, \epsilon > 0$ such that*

$$\operatorname{vol}(\rho, V, a) \leq C\rho^{2k}, \quad |a| + \rho < \epsilon, a \in \mathbb{C}^n.$$

Moreover, there are $R_0, r_0 > 0$ such that for each simple curve γ and each $d \geq 1$

$$\operatorname{vol}(r, V_t, 0) \leq Cr^{2k}$$

whenever $t \in \mathbb{C} \setminus\,]-\infty, 0]$, $0 < |t| < r_0$, and $r \leq R_0/|t|^d$.

Proof For the proof we can assume that V is irreducible at 0 since otherwise the lemma can be applied to each irreducible component of V. The first inequality of the lemma is well known and follows from Wirtinger's Theorem that the volume of an analytic variety is the sum of the volumes of its projections, counted with multiplicity, onto the coordinate subspaces [6, 13.3]. That is, if the multiplicity of V at the origin is m, then we can make an orthogonal choice of coordinates such that in some ball about the origin, say $B(0,\epsilon)$, the projection maps of V to the k-dimensional coordinate subspaces are proper and at most m-sheeted [6, 3.4 and 3.7]. Therefore, in any ball $B(a,\rho)$, the projection of V to a coordinate subspace covers that subspace at most m times, so the volume of the projection is at most $m\pi^k\rho^{2k}/k!$. Wirtinger's Theorem then shows that the first inequality holds with $C=\binom{n}{k}m\pi^k\rho^{2k}/k!$ whenever $B(a,\rho)\subset B(0,\epsilon)$.

The variety V_t is obtained from V by making the affine change of variables, $z=\gamma(t)+|t|^d w$. Therefore, $\operatorname{vol}(r,V_t,0)$ is equal to $|t|^{-2dk}$ times the volume of V in the ball $B\bigl(\gamma(t),|t|^d r\bigr)$, so the second inequality follows from the first whenever t is small enough that $|\gamma(t)|+|t|^d r<\epsilon$. ■

Corollary 3.5 *For V, γ, and $d\geq 1$ as in Theorem 3.2 let $(t_j)_{j\in\mathbb{N}}$ be a null-sequence in $\mathbb{C}\setminus(-\infty,0]$.*

(a) *There exists a subsequence $(t_{j_\nu})_{\nu\in\mathbb{N}}$ for which $[V_{t_{j_\nu}}]$ converges.*
(b) *If $\lim_{l\to\infty}[V_{t_l}]=W$ for some holomorphic k-chain W then* $\operatorname{Supp} W$ *is either empty or an algebraic variety of dimension k. Further, the degree of W is at most equal to the multiplicity of V at* 0.

Proof Part (a) is a consequence of the previous lemma and Theorem 2.8. The same theorem implies that the limit holomorphic chain W is either empty or of pure dimension k. The total mass in a current in a closed ball is a lowersemicontinuous function of the current. Therefore, if $[V_{t_j}]\to W$, then the second estimate of Lemma 3.4 shows that the volume of W in the ball of radius r about the origin is $O(r^{2k})$, so $\operatorname{Supp} W$ is algebraic by the Stoll-Bishop Theorem [10], [1] (see also [6, Theorem 17.2]). Because V_t is obtained from V by an affine transformation, the intersection of V_t with a generic $(n-k)$-dimensional subspace also contains at most m points. By [6, 12.2, Proposition 2], this implies that the degree of W does not exceed m. ■

In the sequel we will complete the proof of Theorem 3.2 by showing that there is a unique limit for the convergent subsequences of $(V_{t_j})_{j\in\mathbb{N}}$. That is, there exists a holomorphic k-chain W_0 such that for each null-sequence $(t_l)_{l\in\mathbb{N}}$ for which the sequence $([V_{t_l}])_{l\in\mathbb{N}}$ converges, the limit chain equals W_0. To prove this, fix V as in Theorem 3.2 and a null-sequence $(t_l)_{l\in\mathbb{N}}$ and assume that $W=\lim_{l\to\infty}[V_{t_l}]$ exists. To describe W in a way which shows that it does not depend on the sequence $(t_l)_{l\in\mathbb{N}}$ we will study the canonical defining function of V as defined in Whitney [11, Appendix V, Section 7]. For that purpose we choose excellent coordinates for the varieties V and W in the sense of [11, 7.7]. This means that we assume that for $\mathbb{C}^n=\mathbb{C}^{n-k}\times\mathbb{C}^k$ the projection $\pi\colon z=(z'',z')\mapsto(0,z')$ is proper when restricted to V and W and that

$$
\begin{aligned}
|z| &\leq C|\pi(z)|, && z\in V, |z|<\epsilon,\\
|z| &\leq C\bigl(1+|\pi(z)|\bigr), && z\in \operatorname{Supp} W.
\end{aligned}
\tag{3.1}
$$

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In the remainder of this section we will assume these hypotheses, even when they are not mentioned explicitly.

Note that in the above situation the $(n-k)$-dimensional subspace $\mathbb{C}^{n-k} \times \{0\}$ is transverse to V and $\operatorname{Supp} W$. To see the existence of such a subspace L, it suffices to construct an affine subspace L' of dimension $n-k-1$ in $\{1\} \times \mathbb{C}^{n-1}$ such that $L' \cap \{z \in T_0 V \cap W : z_1 = 1\} = \varnothing$. Such an L' exists by [6, Corollary 3.5]. The space L is then the cone over L'. If B is the branch locus of $\pi\colon V \to \mathcal{U} \subset \mathbb{C}^k$, then B and $\pi(B)$ are analytic varieties of dimension at most $k-1$ and

$$\pi\colon V \setminus B \to \mathcal{U} \setminus \pi(B)$$

is a covering map. The number of points in a fiber over $z' \in \mathcal{U} \setminus \pi(B)$ is a constant, say m, so we can write

$$\pi^{-1}(z') = \left\{ \big(\alpha_i(z'), z'\big) : 1 \le i \le m \right\}$$

where the $\alpha_i(z') = \alpha_i(z'\,;V)$ are all distinct. We will also use the same notation for $z' \in \mathcal{U} \cap \pi(B)$ by repeating each $\alpha_i(z')$ as many times as indicated by the sheet number $i_z(V, L)$, where $z := \big(\alpha_i(z'), z'\big)$ and $L := \mathbb{C}^{n-k} \times \{z'\}$.

For $u, w \in \mathbb{C}^{n-k}$, let $\langle u, w\rangle = u_1 w_1 + \cdots + u_{n-k} w_{n-k}$ denote the dot product. Let us call the *canonical defining function for* V:

$$P(z, \xi\,; V, \pi) = \prod_{i=1}^{m} \langle z'' - \alpha_i(z'), \xi\rangle. \tag{3.2}$$

We will write

$$P(z, \xi) = P(z, \xi\,; V) = P(z, \xi\,; V, \pi)$$

when the missing data are clear from the context. A point z belongs to V if and only if

$$P(z, \xi) = 0 \quad \text{for all } \xi \in \mathbb{C}^{n-k}.$$

Equivalently, one can expand P as a homogeneous polynomial in ξ,

$$P(z, \xi) = \sum_{|\beta| = m} P_\beta(z) \xi^\beta$$

and then $z \in V$ if and only if $P_\beta(z) = 0$ for all $|\beta| = m$.

Note that P is a polynomial of degree m in z'' and a homogeneous polynomial of degree m in $\xi \in \mathbb{C}^{n-k}$. It is defined at first for $z' \in \mathcal{U} \setminus \pi(B)$ but extends, by the Riemann removable singularity theorem, to be analytic on all of $\mathbb{C}^{n-k} \times \mathcal{U} \times \mathbb{C}^{n-k}$. With the convention made about counting the points $\alpha_i(z')$ with multiplicity when $z' \in \pi(B)$, the formula (3.2) is still valid.

To express the canonical functions of V_t in a useful form, write $\gamma(t) = \big(\gamma_1(t), \gamma_2(t)\big)$ where $\gamma_2(t) = \pi\big(\gamma(t)\big)$ and then

$$\begin{aligned} P\big(\gamma(t) + t^d w, \xi\big) &= \prod_{j=1}^{m} \big\langle \gamma_1(t) + t^d w'' - \alpha_j\big(\gamma_2(t) + t^d w'\big), \xi \big\rangle \\ &= t^{md} \prod_{j=1}^{m} \langle w'' - \beta_j(w', t), \xi \rangle, \end{aligned} \tag{3.3}$$

where

$$\beta_j(w', t) = \frac{\alpha_j\big(\gamma_2(t) + t^d w'\big) - \gamma_1(t)}{t^d}. \tag{3.4}$$

The last formula gives the canonical functions for the varieties V_t with respect to the projection π onto the z' coordinates up to the scale factor t^{md}.

The limit chain W of a sequence $([V_{t_j}])_{j\in\mathbb{N}}$ is not necessarily a current of integration over an analytic set, so its associated canonical defining function must take account of multiplicities. To fix the notation, let us suppose that

$$W = n_1[W_1] + \cdots + n_p[W_p] \tag{3.5}$$

where the W_j are the irreducible components of $\operatorname{Supp} W$, and

$$W_j = \big\{ \big(\beta_{j,i}(w'), w'\big) : w' \in \mathbb{C}^k, 1 \le i \le m_j \big\} \tag{3.6}$$

where m_j is the degree of W_j. Then

$$\nu := n_1 m_1 + \cdots + n_p m_p \tag{3.7}$$

is the degree of W, so $\nu \le m$ by Corollary 3.5. The canonical defining function of W is then

$$P(w, \xi\,; W) := P(w, \xi\,; W, \pi) := \prod_{j=1}^{p} \prod_{i=1}^{m_j} \langle w'' - \beta_{j,i}(w'), \xi \rangle^{n_j}. \tag{3.8}$$

Lemma 3.6 *The canonical defining function $P(w, \xi\,; W, \pi)$ is a polynomial.*

Proof The function P is clearly a polynomial in w'' and ξ. It grows at a polynomial rate as a function of w' by (3.1) and the definition of P. Therefore, it is also a polynomial in w' by Liouville's theorem. ∎

The degree of a limit chain will frequently be smaller than that of V_t. This occurs when some of the $\beta_j(w', t)$ go to infinity while the others converge to points in $\operatorname{Supp} W$.

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Lemma 3.7 *Suppose* $\lim_{l\to\infty}[V_{t_l}] = W$ *for some holomorphic k-chain W in* $\mathbb{C}^n$.

(i) *For all* $R, \epsilon > 0$ *there is* l_0 *such that for each* $l > l_0$ *and each* $w' \in \mathbb{C}^k$ *with* $|w'| \le R$ *there are exactly* ν *values of j for which the point* $\big(\beta_j(w', t_l), w'\big)$ *lies in the* ϵ*-neighborhood of* $\operatorname{Supp} W$.

(ii) *For each* $R > 0$ *there is* $M(R) > 0$ *such that for each* $M > M(R)$ *there is* l_0 *such that for each* $l > l_0$ *and each* $w' \in \mathbb{C}^k$ *with* $|w'| \le R$ *there are exactly* $m - \nu$ *values of j for which the* $|\beta_j(w', t_l)| \ge M$.

(iii) *For each* $w = (w'', w') \in \mathbb{C}^n$ *there is* ϵ_0 *such that for each* $0 < \epsilon < \epsilon_0$ *there is* l_0 *such that for each* $l > l_0$ *the number of j with* $|\beta_j(w') - w''| < \epsilon$ *is exactly equal to the sheet number of W in w, i.e.,*

$$\big|\{j : |\beta_j(w', t_l) - w''| < \epsilon\}\big| = \sum_{j=1}^{p} n_j |\{i : \beta_{j,i}(w') = w''\}| \quad \text{for } l > l_0.$$

This implies that for fixed w' *the* β_j *can be numbered in such a way that* $\lim_{l\to\infty}\beta_j(w', t_l)$ *exists for* $1 \le j \le \nu$ *and* $|\lim_{l\to\infty}\beta_j(w', t_l)| = \infty$ *for* $j > \nu$. *The sequence of functions* $w' \mapsto \beta_j(w', t_l)$ *converges uniformly on compact sets (although they may have discontinuities).*

Proof To prove (i), fix R and ϵ. Let $D \subset \mathbb{C}^n$ consist of all those points $w = (w'', w')$ for which there are j, i with $|\beta_{j,i}(w') - w''| < \epsilon$. For $w' \in \mathbb{C}^k$ define the affine subspace $L_{w'} := \mathbb{C}^{n-k} \times \{w'\}$. It suffices to show the existence of l_0 such that for each $l > l_0$ and each $w' \in \mathbb{C}^k$ with $|w'| \le R$ we have $i_D(V_{t_l}, L_{w'}) = \nu$. To prove this claim, assume for contradiction that for each j there are $l_j > j$ and w'_j with $|w'| \le R$ and $i_D(V_{t_{l_j}}, L_{w'_j}) \ne \nu$. We may assume the existence of $\lim_{j\to\infty} w'_j =: w'$. Then $\lim_{j\to\infty} L_{w'_j} = L_{w'}$ in the sense of Definition 2.5. Thus Chirka [6, 12.2, Proposition 2], implies that $i_D(V_{t_{l_j}}, L_{w'_j}) = i_D(W, L_{w'})$ for sufficiently large j. Since $i_D(W, L_{w'}) = \nu$ by (3.7) we have arrived at a contradiction.

The proof of (ii) is the same, except that now ϵ is replaced by $M > M(R)$, where

$$M(R) = 1 + \max\{|z''| : (z'', z') \in W, |z'| \le R\}.$$

For $w \in \mathbb{C}^n$ define

$$\epsilon_0 := \frac{1}{2}\min\{|w'' - \zeta''| : (\zeta'', w') \in \operatorname{Supp} W, \zeta'' \ne w''\}.$$

Then the proof of (i) implies (iii) for $\epsilon < \epsilon_0$. ∎

In order to derive a description of W from the canonical defining function $P(\cdot, \text{-}\,; V, \pi)$ of V we will use the following notation.

Definition 3.8 For $d \ge 1$, $q \in \mathbb{N}$, and $l \in \mathbb{N}_0$, a polynomial $p \in \mathbb{C}[w_1, \dots, w_n, t^{1/q}, \xi_1, \dots, \xi_l]$ is called *d-quasihomogeneous* in w and t of *d-degree* ω if

$$p(\lambda^d w, \lambda t, \xi) = \lambda^\omega p(w, t, \xi), \quad \lambda > 0.$$

It is easy to check that p is d-quasihomogeneous of d-degree ω if and only if p has the form

$$p(w,t,\xi) = \sum_{j+d|\beta|=\omega} \sum_{\alpha\in\mathbb{N}_0^l} a_{j,\beta,\alpha} w^\beta t^j \xi^\alpha,$$

where j runs through $q^{-1}\mathbb{N}_0$ and β through $\mathbb{N}_0^n$.

Remark 3.9 For P as in (3.2), γ as in Theorem 3.2, and $d \geq 1$, let

$$F(w,t,\xi) := P\big(\gamma(t)+w,\xi\,;V,\pi\big) = \sum_{j,\beta,\alpha} a_{j,\beta,\alpha} t^j w^\beta \xi^\alpha \tag{3.9}$$

where the sum is the power series expansion of the holomorphic function $F(w,s^q,\xi)$ in $s = t^{1/q}$, w, ξ in a neighborhood of the origin. Collecting all terms in (3.9) which have the same d-degree, we can regroup the series as

$$F(w,t,\xi) = F_{\omega_0}(w,t,\xi) + \sum_{\omega>\omega_0} F_\omega(w,t,\xi), \tag{3.10}$$

where F_ω is the d-quasihomogeneous part of d-degree ω of the series and

$$\omega_0 = \omega_0(d,V,\pi) = \min\{\omega : F_\omega \text{ does not vanish identically}\}. \tag{3.11}$$

Now note that for $t \in B(0,\epsilon)\setminus\,]-\infty,0]$ the quasihomogeneity property implies $F(t^d w,t,\xi) = t^{\omega_0}F_{\omega_0}(w,1,\xi) + \sum_{\omega>\omega_0} t^\omega F_\omega(w,1,\xi)$ and hence

$$\lim_{t\to 0} t^{-\omega_0} P\big(\gamma(t)+t^d w,\xi\,;V,\pi\big) = \lim_{t\to 0} t^{-\omega_0} F(t^d w,t,\xi) = F_{\omega_0}(w,1,\xi), \tag{3.12}$$

where the convergence is uniform on compact subsets of $\mathbb{C}^n\times\mathbb{C}^{n-k}$.

Lemma 3.10 *Assume that* $\lim_{l\to\infty} V_{t_l} = W$ *for some holomorphic k-chain* W. *Then there is a polynomial* Φ *on* $\mathbb{C}^k\times\mathbb{C}^{n-k}$ *such that*

$$F_{\omega_0}(\zeta'',\zeta',1,\xi) = P(\zeta'',\zeta',\xi\,;W)\Phi(\zeta',\xi)$$
$$\textit{for all } \zeta = (\zeta'',\zeta') \in \mathbb{C}^{n-k}\times\mathbb{C}^k,\ \xi\in\mathbb{C}^{n-k}.$$

Proof We work in the coordinate system of (3.1), and we let ν be as in (3.7). Then Lemma 3.7 implies that we may assume that for each ζ' the β_j are numbered in such a way that the sequence $\big(\beta_j(\zeta',t_l)\big)_{l\in\mathbb{N}}$ either converges to a finite limit or has no bounded subsequence and that the first alternative holds if and only if $1\leq j\leq\nu$. Fix $R>0$, set $B_R := \{(\zeta'',\zeta',\xi)\in\mathbb{C}^{n-k}\times\mathbb{C}^k\times\mathbb{C}^{n-k} : |\zeta'|<R\}$ and let $M(R)$ be as in (ii) of Lemma 3.7. Set $M = 2M(R)$ and let l_0 be as in (ii) of 3.7. For $l > l_0$ Riemann's theorem on removable singularities implies that the functions

$$f_l, g_l\colon B_R \to \mathbb{C},$$

$$f_l(\zeta'',\zeta',\xi) = \prod_{j=1}^{\nu} \langle \zeta''-\beta_j(\zeta',t_l),\xi\rangle,$$

$$g_l(\zeta'',\zeta',\xi) = t_l^{md-\omega_0} \prod_{j=\nu+1}^{m} \langle \zeta''-\beta_j(\zeta',t_l),\xi\rangle$$

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are holomorphic. Now Lemma 3.7 implies that

$$\lim_{l\to\infty} f_l(\zeta'',\zeta',\xi) = \prod_{j=1}^{p}\prod_{i=1}^{m_j}\langle \zeta'' - \beta_{j,i}(\zeta'),\xi\rangle^{n_j},$$

where convergence is uniform on B_R. On the other hand, $t_l^{-\omega_0}F(t_l^d\zeta,t_l,\xi) = f_l(\zeta,\xi)g_l(\zeta,\xi)$ and $\lim_{l\to\infty} t_l^{-\omega_0}F(t_l^d\zeta,t_l,\xi) = F_{\omega_0}(\zeta,1,\xi)$, where convergence is uniform on compact sets. Hence, also the sequence $(g_l)_{l\in\mathbb{N}}$ converges to a function g, which satisfies

$$F_{\omega_0}(\zeta,1,\xi) = g(\zeta,\xi)\prod_{j=1}^{p}\prod_{i=1}^{m_j}\langle \zeta'' - \beta_{j,i}(\zeta'),\xi\rangle^{n_j}.$$

Since g does not depend on R, it is entire. In fact, g is a polynomial, since it divides F_{ω_0}.

To complete the proof, we have to show that g does not depend on ζ''. Fix $\zeta'\in\mathbb{C}^k$. There is a a subsequence $(t_{l_\kappa})_{\kappa\in\mathbb{N}}$ for which the limits

$$\lim_{\kappa\to\infty}\frac{\beta_j(\zeta',t_{l_\kappa})}{|\beta_j(\zeta',t_{l_\kappa})|} =: \xi_j$$

exist for $\nu < j \le m$. Let A be the set of all $\xi\in\mathbb{C}^{n-k}$ such that $\langle \xi_j,\xi\rangle = 0$ for at least one j. Since A is a proper algebraic subset of $\mathbb{C}^{n-k}$, it suffices to show the claim for $\xi\notin A$. Let such a $\xi\notin A$ be fixed. Then $|\langle\beta_j(\zeta',t_{l_\kappa}),\xi\rangle|\to\infty$ as $\kappa\to\infty$, and the claim can be deduced in the following way

$$\begin{aligned}\frac{g(\zeta'',\zeta',\xi)}{g(0,\zeta',\xi)} &= \lim_{\kappa\to\infty}\prod_{j=\nu+1}^{m}\frac{\langle\zeta''-\beta_j(\zeta',t_{l_\kappa}),\xi\rangle}{\langle-\beta_j(\zeta',t_{l_\kappa}),\xi\rangle}\\ &= \prod_{j=\nu+1}^{m}\langle-\xi_j,\xi\rangle^{-1}\lim_{\kappa\to\infty}\left\langle\frac{\zeta''}{|\beta_j(\zeta',t_{l_\kappa})|}-\frac{\beta_j(\zeta',t_{l_\kappa})}{|\beta_j(\zeta',t_{l_\kappa})|},\xi\right\rangle = 1. \qquad\blacksquare\end{aligned}$$

Lemma 3.11 *Suppose that* $\lim_{l\to\infty}[V_{t_l}] = W$ *for some holomorphic k-chain* W*. For each* $w'\in\mathbb{C}^k$ *the set* $\{\xi\in\mathbb{C}^{n-k} : \Phi(w',\xi)\neq 0\}$ *is open and dense in* $\mathbb{C}^{n-k}$*.*

Proof Fix $w\in\mathbb{C}^n$, set $L := \mathbb{C}^{n-k}\times\{w'\}$, and let $\mu = i_w(W,L)$. Lemma 3.7 implies that we can arrange the $\beta_j(w',t_l)$ in such a way that

$$(3.13)\qquad \big(\beta_j(w',t_l)\big)_{l\in\mathbb{N}}\begin{cases}\text{converges to } w'', & \text{for } 1\le j\le\mu,\\ \text{converges, but not to } w'', & \text{for } \mu<j\le\nu,\\ \text{has no bounded subsequence,} & \text{for } \nu<j.\end{cases}$$

Next, find a subsequence $(t_{l_\kappa})_{\kappa\in\mathbb{N}}$ of $(t_l)_{l\in\mathbb{N}}$ such that

$$\lim_{\kappa\to\infty}\frac{\beta_j(w',t_{l_\kappa})}{|\beta_j(w',t_{l_\kappa})|} =: \xi_j$$

exists for $\nu < j \le m$. We assume in the sequel that the passage to subsequences was unnecessary, *i.e.*, that $t_{l_\kappa} = t_\kappa$. Now define three sets A, B, C, all open and dense in $\mathbb{C}^{n-k}$, by

$$A := \{\xi \in \mathbb{C}^{n-k} : \langle \xi_j, \xi \rangle \neq 0 \text{ for } \nu < j \le m\},$$
$$B := \{\xi \in \mathbb{C}^{n-k} : \langle w'' - \zeta'', \xi \rangle \neq 0 \text{ whenever } (\zeta'', w') \in W \text{ and } \zeta'' \neq w''\},$$
$$C := \{\xi \in \mathbb{C}^{n-k} : F_{\omega_0}(\cdot, 1, \xi) \text{ does not vanish identically}\}.$$

Let $\xi_0 \in A \cap B \cap C$ be arbitrary. To keep notations simple, we assume $\xi_0 = (1, 0, \dots, 0)$. Since this amounts to a rotation in $\mathbb{C}^{n-k}$ only, the property (3.1) of the coordinate system remains valid. Let $\beta_j^{(1)}$ denote the first coordinate of β_j, *i.e.*, $\beta_j^{(1)}(\zeta', t) = \langle \beta_j(\zeta', t), \xi_0 \rangle$. Then $\lim_{l\to\infty} |\beta_j^{(1)}(w', t_l)| = \infty$ for $j > \nu$ because $\xi_0 \in A$. Since $\xi_0 \in B$, (3.13) implies that $\bigl(\beta_j^{(1)}(w', t_l)\bigr)_{j \in \mathbb{N}}$ converges to w_1 for $1 \le j \le \mu$, and converges, but not to w_1 for $\mu < j \le \nu$. For $l \in \mathbb{N}$ consider the polynomial in one variable

$$g_l \colon \tau \mapsto t_l^{-\omega_0} F(t_l^d \tau, t_l^d w_2, \dots, t_l^d w_n, t_l, \xi_0) = t_l^{md-\omega_0} \prod_{j=1}^{m} \bigl(\tau - \beta_j^{(1)}(w', t_l)\bigr).$$

We have just seen that for an arbitrarily small neighborhood U of w_1 there is $M \in \mathbb{N}$ such that g_l has exactly μ zeros in U, counting multiplicities, provided $l > M$. Since $\xi_0 \in C$, (3.12) and Rouché's theorem imply that

$$\tau \mapsto F_{\omega_0}(\tau, w_2, \dots, w_n, 1, \xi_0) \text{ has a zero of order exactly } \mu \text{ in } \tau = w_1. \tag{3.14}$$

By Lemma 3.10 we have that $F_{\omega_0}(\zeta, 1, \xi) = P(\zeta, \xi \;;\; W)\Phi(\zeta', \xi)$. Since $P(\cdot, w_2, \dots, w_n, \xi_0 \;;\; W)$ has a zero of order at least μ in $\tau = w_1$, it follows that $\Phi(w', \xi_0) \neq 0$. ■

The following proposition allows us to recover $P(\cdot, \text{-} \;;\; W)$ from F_{ω_0}. Hence it shows the independence of $\lim_{l\to\infty}[V_{t_l}]$ from the null-sequence $(t_l)_{l \in \mathbb{N}}$ and thus completes the proof of the main Theorem 3.2.

Proposition 3.12 *Suppose that* $\lim_{l\to\infty}[V_{t_l}] = W$ *for some holomorphic k-chain* W*. Let*

$$F_{\omega_0}(w, 1, \xi) = \prod_{a=1}^{A} F_a(w, \xi)^{\lambda_a}$$

be the decomposition of F_{ω_0} *into powers of mutually nonproportional irreducible factors. Let* I *be the set of all* a *for which there is* $w \in \mathbb{C}^n$ *with* $F_a(w, \xi) = 0$ *for all* ξ*. Then there is* $c \neq 0$ *such that*

$$P(w, \xi \;;\; W) = c \prod_{a \in I} F_a(w, \xi)^{\lambda_a}, \quad w \in \mathbb{C}^n, \xi \in \mathbb{C}^{n-k}.$$

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Proof Recall that $F_{\omega_0}(w, 1, \xi) = P(w, \xi\,; W)\Phi(w', \xi)$ by Lemma 3.10. If $a \in I$, then F_a must be a factor of P, since by Lemma 3.11 it cannot be a factor of Φ.

For the proof of the other direction fix a such that F_a is a factor of P. Choose $w = (w'', w') \in \mathbb{C}^n$ and $\xi_0 \in \mathbb{C}^{n-k}$ with $F_a(w, \xi_0) \neq 0$. Consider $F_a(\zeta'', w', \xi)$ as a polynomial in $\mathbb{C}[\zeta'', \xi]$. Then it is a factor of

$$P(\zeta'', w', \xi) = \prod_{j=1}^{p} \prod_{i=1}^{m_j} \langle \zeta'' - \beta_{j,i}(w'), \xi \rangle^{n_j}.$$

In particular, there is a pair (j, i) such that $\langle \zeta'' - \beta_{j,i}(w'), \xi \rangle$ divides $F_a(\zeta'', w', \xi)$ in $\mathbb{C}[\zeta'', \xi]$. Then $F_a\big(\beta_{j,i}(w'), w', \xi\big) = 0$ for all $\xi \in \mathbb{C}^{n-k}$ and hence $a \in I$. ■

Proof of Theorem 3.2 For each null-sequence $(t_l)_{l\in\mathbb{N}}$ the sequence $([V_{t_l}])_{l\in\mathbb{N}}$ has an accumulation point W by Corollary 3.5(a). This accumulation point is unique and does not depend on the null-sequence $(t_l)_{l\in\mathbb{N}}$ by Proposition 3.12. Hence W is the limit. Its support is either empty or algebraic of pure dimension k by Corollary 3.5(b). ■

In [5] we will apply the following corollary of Theorem 3.2, which is obvious from the proof of this theorem.

Corollary 3.13 *Let $V \subset \mathbb{C}^n$ be an analytic variety of pure dimension k which contains the origin, let γ be a simple curve, and let $d \geq 1$, $R > 0$, and a null-sequence $(t_j)_{j\in\mathbb{N}}$ in $\mathbb{C} \setminus\,]-\infty, 0]$ be given. Then the varieties V_{t_j} converge to $T_{\gamma,d}V$ in the sense of Meise, Taylor, and Vogt [8, 4.3], as j tends to infinity.*

It is possible to determine the sheet number of $T_{\gamma,d}[V]$ at each point. This provides a purely geometric description of $T_{\gamma,d}[V]$. If p is a polynomial in one variable, we denote by $\operatorname{ord}_0 p$ the vanishing order of p at the origin.

Proposition 3.14 *Let $w = (w'', w')$ be excellent coordinates for V and for $T_{\gamma,d}V$ as in* (3.1). *For $w' \in \mathbb{C}^k$ set $L_{w'} := \mathbb{C}^{n-k} \times \{w'\}$. For $w \in \mathbb{C}^n$ and $\xi \in \mathbb{C}^{n-k}$ consider the polynomial $p_{w,\xi} \colon \tau \mapsto F_{\omega_0}(w'' + \tau\xi, w', 1, \xi)$, $\tau \in \mathbb{C}$. Then*

$$i_w(T_{\gamma,d}[V], L_{w'}) = \min\{\operatorname{ord}_0 p_{w,\xi} : \xi \in \mathbb{C}^{n-k}\}.$$

Proof We start with the proof of "$\leq$". Choose $\xi_0 \in \mathbb{C}^{n-k}$ with $\min\{\operatorname{ord}_0 p_{w,\xi}\} = \operatorname{ord}_0 p_{w,\xi_0}$. Assume for convenience $\xi_0 = (1, 0, \dots, 0)$. By Proposition 3.12, p_{w,ξ_0} is a multiple of

$$P(w_1 + \tau, w_2, \dots, w_n, \xi_0\,; W) = \prod_{j=1}^{p} \prod_{i=1}^{m_j} \big(w_1 + \tau - \beta_{j,i}^{(1)}(w')\big)^{n_j},$$

where $\beta_{j,i}^{(1)}$ denotes the first coordinate of $\beta_{j,i}$ as in (3.8). The definition of $\beta_{j,i}$ implies that the order of this polynomial is not smaller than $i_w(T_{\gamma,d}[V], L_{w'})$.

To prove the converse inequality, use Lemma 3.11 to choose $\xi_0 \in \mathbb{C}^{n-k}$ such that $\Phi(w', \xi_0) \neq 0$ and such that $\langle \zeta'', \xi_0 \rangle \neq \langle w'', \xi_0 \rangle$ whenever $(\zeta'', w') \in W$ and $\zeta'' \neq w''$. Again, we may assume $\xi_0 = (1, 0, \dots, 0)$. Then

$$p_{w,\xi_0}(\tau) = P(w_1 + \tau, w_2, \dots, w_n, \xi_0 \,;\, W) = \prod_{j=1}^{p} \prod_{i=1}^{m_j} \bigl(w_1 + \tau - \beta_{j,i}^{(1)}(w') \bigr)^{n_j} \Phi(w', \xi_0).$$

This shows that $\operatorname{ord}_0 p_{w,\xi_0} = i_w(T_{\gamma,d}[V], L_{w'})$ for the special choice of ξ_0. The proposition is proved. ∎

Remark Proposition 3.14 holds under the general hypothesis that the coordinates are excellent for V and for $T_{\gamma,d}V$ (see (3.1)). When investigating examples, one wants to be able to see from F_{ω_0} that a given system of coordinates is excellent. So let us assume that the standard coordinate system is excellent for V, *i.e.*, the first inequality in (3.1) is valid. Then it is possible to define the canonical defining function $P(z, \xi \,;\, V, \pi)$ as in (3.2) and, for a given simple curve γ and some $d \geq 1$, the expansion of $P\bigl(\gamma(t) + z, \xi \,;\, V, \pi\bigr)$ into d-quasihomogeneous terms as in (3.10) exists. Hence

$$Z := \{w \in \mathbb{C}^n : F_{\omega_0}(w, 1, \xi) = 0 \text{ for all } \xi \in \mathbb{C}^{n-k}\}$$

is defined, and the following holds:

Assume that $\dim Z = k$ *and that the standard coordinate system is also excellent for* Z*, i.e., there is* $C > 0$ *such that* $|w| \leq C(1 + |w'|)$ *for all* $w = (w'', w') \in Z$*. Then it is excellent for* $T_{\gamma,d}V$*. In particular,* $Z = T_{\gamma,d}V$ *and Proposition 3.14 holds in these coordinates.*

Proof Since (3.1) is inherited by subvarieties of the same dimension, it suffices to show $T_{\gamma,d}V \subset Z$. So fix $w \in T_{\gamma,d}V$ and an arbitrary null-sequence $(t_n)_{n\in\mathbb{N}}$. By Definition 2.5 there is a sequence $(z_n)_{n\in\mathbb{N}}$ such that $z_n \in V_{\gamma,d,t_n}$ and $\lim_{n\to\infty} z_n = w$. Fix $\xi \in \mathbb{C}^{n-k}$. Then

$$\begin{aligned} 0 &= t^{-\omega_0} P\bigl(\gamma(t_n) + t_n^d z_n, \xi \,;\, V, \pi\bigr) \\ &= F_{\omega_0}(z_n, 1, \xi) + \sum_{\omega > \omega_0} t_n^{\omega - \omega_0} F_\omega(z_n, 1, \xi) \xrightarrow{n\to\infty} F_{\omega_0}(w, 1, \xi). \end{aligned}$$

Hence $w \in Z$, and the claim is shown. ∎

Definition 3.15 For $q \in \mathbb{N}$ and $d \geq 1$ let

$$h(w, t) = \sum_{\omega \in \mathbb{N}/q} h_\omega(w, t)$$

be a convergent series in w and $t^{1/q}$ such that h_ω is zero or d-quasihomogeneous of d-degree ω in w and t. Then for $\omega_0 := \min\{\omega : h_\omega \neq 0\}$ the term h_{ω_0} is called the *d-quasihomogeneous localization* of h at the origin.

If h does not depend on t then the d-quasihomogeneous localization of h at 0 coincides with the localization in the classical sense. Recall that the localization of a holomorphic function f at θ is defined as the lowest degree homogeneous polynomial in the Taylor series expansion of f at θ.

In the case of a hypersurface the vanishing ideal is principal, and its generator replaces the canonical defining function as indicated in the next statement.

Corollary 3.16 *Let $U \subset \mathbb{C}^n$ be a neighborhood of zero, let $p\colon U \to \mathbb{C}$ be holomorphic, let $V := \{z \in U : p(z) = 0\}$. Furthermore, assume that there is a dense open subset A of V with* grad $p(z) \neq 0$ *for all $z \in A$. Let γ be a simple curve and set $f(w,t) := p\bigl(\gamma(t) + w\bigr)$. For $d \geq 1$ let f_{ω_0} be the d-quasihomogeneous localization of f at the origin. Then $T_{\gamma,d}V = \{w \in \mathbb{C}^n : f_{\omega_0}(w,1) = 0\}$. Let $W_1, \ldots, W_p$ be the irreducible components of $T_{\gamma,d}V$, and let $n_j := \mu_{w_j}(W_j)$ for an arbitrary regular point of W_j. Then*

$$T_{\gamma,d}[V] = \sum_{j=1}^{p} n_j [W_j].$$

Proof We may assume that coordinates are chosen in such a way that $p(\tau, 0, \ldots, 0)$ does not vanish identically. Replacing U by a smaller neighborhood of zero if necessary, we may also assume that U is a polycylinder and that $V \cap (\mathbb{C} \times \{0\}) \cap U = \{0\}$. Then the Weierstraß preparation theorem states that there are $m \in \mathbb{N}$ and a holomorphic and zero-free function φ on U such that

$$p = W\varphi \quad \text{where } W(z) := z_1^m + c_1(z')z_1^{m-1} + \cdots + c_k(z')z_1^0.$$

To determine P as in (3.2), let B denote the branch locus of $\pi\colon V \to \mathfrak{U} \subset \mathbb{C}^{n-1}$. For $z' \in \mathfrak{U} \setminus B$ there are a neighborhood U_1 of z', holomorphic functions $\alpha_j\colon U_1 \to \mathbb{C}$, and $m_j \in \mathbb{N}$, $j = 1, \ldots, m$, such that

$$W(z) = \prod_{j=1}^{m} \bigl(z_1 - \alpha_j(z')\bigr)^{m_j}, \quad z' \in U_1.$$

The hypothesis concerning the gradient means that p is square-free, *i.e.*, $m_j = 1$ for all j. We have shown that $P(z, \xi\,;V) = \xi^m W(z)$ for $z \in U$, $\xi \in \mathbb{C}$. If F is defined as in (3.9), then

$$\begin{aligned} F(w,t,\xi) = P\bigl(\gamma(t) + w, \xi\,; V\bigr) &= \frac{\xi^m}{\varphi\bigl(\gamma(t)+w\bigr)}\, p\bigl(\gamma(t)+w\bigr) \\ &= \frac{\xi^m}{\varphi\bigl(\gamma(t)+w\bigr)} \sum_{j,\beta} a_{j,\beta} t^j w^\beta. \end{aligned}$$

Since $\varphi(0) \neq 0$, division by φ does not affect the lowest order term of the expansion into terms with constant values of $j + d|\beta|$, except for a multiplication by $1/\varphi(0)$, *i.e.*,

$$F_{\omega_0}(w,t,\xi) = \frac{\xi^m}{\varphi(0)} \sum_{j+d|\beta|=\omega_0} a_{j,\beta} t^j w^\beta.$$

Now the claim follows immediately from Proposition 3.14. ■

If f defines the hypersurface V geometrically without generating the corresponding ideal (*i.e.*, if f is not square-free), then it is still possible to determine the tangent variety $T_{\gamma,d}V$.

Corollary 3.17 *Let $U \subset \mathbb{C}^n$ be a neighborhood of zero, let $A\colon U \to \mathbb{C}$ be holomorphic, let $V := \{z \in U : A(z) = 0\}$. Let γ be a simple curve and define $f(w,t) = A\big(\gamma(t) + w\big)$. For $d \geq 1$ let f_{ω_0} be the d-quasihomogeneous localization of f at 0. Then $T_{\gamma,d}V = \{w \in \mathbb{C}^n : f_{\omega_0}(w,1) = 0\}$.*

Proof Decompose A in the ring of germs in the origin. Reducing the size of U if necessary, we may assume that all these factors live on U. Thus $A = A_1^{m_1} \cdots A_l^{m_l}$ with mutually nonproportional holomorphic irreducible functions A_j. Then $r := A_1 \cdots A_l$ satisfies the hypotheses of Corollary 3.16, hence $T_{\gamma,d}V = \{w \in \mathbb{C}^n : g_{\sigma_0}(w) = 0\}$ where g_{σ_0} is the d-homogeneous localization of $g(w,t) := r\big(\gamma(t) + w\big)$. Let f_{j,ω_j} be the d-homogeneous localization of $f_j(w,t) := A_j\big(\gamma(t) + w\big)$. It is easy to see that d-homogeneous localizations are multiplicative, hence $g_{\sigma_0} = f_{1,\omega_1} \cdots f_{l,\omega_l}$ and $f_{\omega_0} = f_{1,\omega_1}^{m_1} \cdots f_{l,\omega_l}^{m_l}$. Thus the zero sets of g_{σ_0} and of f_{ω_0} coincide. ■

4 Properties of the Limit Varieties

It is convenient to record some simple properties of the limit varieties before studying specific examples in Section 6. Invariance properties of $T_{\gamma,d}[V]$ are studied in Proposition 4.1, while in Proposition 4.3 the influence of d is discussed. The main tool in the proof of the latter is the Newton polygon.

Proposition 4.1 *Let V be an analytic variety in a neighborhood of the origin in $\mathbb{C}^n$. Let γ be a simple curve as in Definition 3.1, $d \geq 1$, and $T_{\gamma,d}[V]$ the limit current defined in Definition 3.3.*

(i) *If $\widetilde{\gamma}(t)$ is another simple curve and if $\widetilde{\gamma}(t) = \gamma(t) + o(|t|^d)$, then $T_{\gamma,d}[V] = T_{\widetilde{\gamma},d}[V]$.*

(ii) *If $d = 1$, then $T_{\gamma,1}V = T_0V - \gamma'(0)$; more precisely, if $j(w) = \gamma'(0) + w$, then $j_*(T_{\gamma,1}[V]) = T_0[V]$.*

(iii) *If $d > 1$ and $\lambda \in \mathbb{C}$, then $w \in T_{\gamma,d}V$ if and only if $w + \lambda\gamma'(0) \in T_{\gamma,d}V$; or in terms of the currents, if $j(w) = w + \lambda\gamma'(0)$, then $j_*(T_{\gamma,d}[V]) = T_{\gamma,d}[V]$.*

(iv) *$T_{\gamma,d}V$ is empty if and only if for every relatively compact open set $\Omega \Subset \mathbb{C}^n$, there exists $r_0 > 0$ so small that the conoid with core γ, opening exponent d, and profile Ω truncated at r_0,*

$$\Gamma(\gamma, d, \Omega, r_0) = \bigcup_{0<t<r_0} \big(\gamma(t) + t^d\Omega\big) \tag{4.1}$$

has empty intersection with V.

(v) *If $\widetilde{\gamma}$ is another simple curve with the same trace as γ, then $T_{\gamma,d}[V] = T_{\widetilde{\gamma},d}[V]$ for each $d \geq 1$.*

Proof Choose coordinates as in Section 3 and recall the canonical defining function $P(w,\xi) = P(w,\xi\,;V,\pi)$ associated to this choice of coordinates along with the functions F and F_{ω_0} as in (3.9) and (3.11). It follows from the hypotheses about γ and $\widetilde{\gamma}$, (3.10), and the quasihomogeneity property of the F_ω that

$$\begin{aligned}
&\lim_{t\to 0} t^{-\omega_0} P\bigl(\widetilde{\gamma}(t) + t^d w, \xi\bigr)\\
&\quad = \lim_{t\to 0} t^{-\omega_0} P\Bigl(\gamma(t) + t^d\bigl(w + o(1)\bigr), \xi\Bigr)\\
&\quad = \lim_{t\to 0} F_{\omega_0}\Bigl(\bigl(w+o(1)\bigr), 1, \xi\Bigr) + \lim_{t\to 0} \sum_{\omega>\omega_0} t^{\omega-\omega_0} F_\omega\Bigl(\bigl(w+o(1)\bigr), 1, \xi\Bigr)\\
&\quad = F_{\omega_0}(w,1,\xi).
\end{aligned}$$

Therefore, under the hypothesis of (i), ω_0 and the function F_{ω_0} are unchanged if γ is replaced by $\widetilde{\gamma}$. By Proposition 3.12 this yields (i).

To prove (ii), we can assume that $\gamma(t) = \xi_0 t$ because of part (i). Then $V_t = \{w \in \mathbb{C}^n : \xi_0 + w \in \frac{1}{t}V\}$, so $[V_t]$ is the translate of the current $[\frac{1}{t}V]$ by $-\xi_0$. Consequently, the same is true of the limit varieties.

To prove (iii), Proposition 3.12 implies that it suffices to show that $F_{\omega_0}\bigl(w + \lambda\gamma'(0), 1, \xi\bigr) = F_{\omega_0}(w,1,\xi)$. By analytic continuation, it is enough to prove this equation for $\lambda > 0$. Set $\widetilde{t} = t + \lambda t^d$ so that $t = \widetilde{t} - \lambda\widetilde{t}^d + o(\widetilde{t}^d)$. Then since $d > 1$,

$$\begin{aligned}
F_{\omega_0}(w,1,\xi) &= \lim_{\widetilde{t}\to 0} \widetilde{t}^{-\omega_0} P\bigl(\gamma(\widetilde{t}) + \widetilde{t}^d w, \xi\bigr)\\
&= \lim_{t\to 0}\bigl(t + o(t)\bigr)^{-\omega_0} P\Bigl(\gamma(t) + t^d\lambda\gamma'(0) + o(t^d) + \bigl(t+o(t)\bigr)^d w, \xi\Bigr)\\
&= \lim_{t\to 0} t^{-\omega_0} P\Bigl(\gamma(t) + t^d\bigl(w + \lambda\gamma'(0) + o(1)\bigr), \xi\Bigr) + o(1)\\
&= \lim_{t\to 0} F_{\omega_0}\bigl(w + \lambda\gamma'(0) + o(1), 1, \xi\bigr) + o(1)\\
&= F_{\omega_0}\bigl(w + \lambda\gamma'(0), 1, \xi\bigr)
\end{aligned}$$

so part (iii) is proved.

Part (iv) is a consequence of Lemma 3.7. The same lemma and Proposition 3.12 show that $T_{\gamma,d}V$ is nonempty except when all the points $\bigl(\beta_j(w',t), w'\bigr) \in V_t$ diverge to ∞ as $t \to 0$. From the definition of the β's in (3.4), we see that this means exactly that V_t has no points in any conoid $\Gamma(\gamma, d, \Omega, r_0)$ with relatively compact profile when r_0 is sufficiently small.

For the proof of part (v) let ξ_0 denote the tangent vector to γ. Since $\xi_0 = \lim_{t\to 0}\gamma(t)/|\gamma(t)|$, the tangents of γ and $\widetilde{\gamma}$ coincide. We may assume $\xi_0 = (0,\dots,0,1)$. Let γ_n and $\widetilde{\gamma}_n$ denote the last component of γ and $\widetilde{\gamma}$, respectively. Since both are injective, we can define $\rho := \gamma_n^{-1} \circ \widetilde{\gamma}_n$. Then $\widetilde{\gamma} = \gamma \circ \rho$. Both have the same tangent, so it is immediate that $\lim_{t\to 0}\rho(t)/t = 1$. Set $\widetilde{F}(w,t,\xi) := P\bigl(\widetilde{\gamma}(t) + w, \xi\bigr)$, and let $\widetilde{\omega}_0$ be defined by (3.11), but with F replaced by $\widetilde{F}$. Then

$$\begin{aligned}\widetilde{F}_{\omega_0}(w,1,\xi) &= \lim_{t\to 0} t^{-\omega_0} P\big(\widetilde{\gamma}(t),\xi\big)\\ &= \lim_{t\to 0}\left(\frac{t}{\rho(t)}\right)^{-\omega_0}\rho(t)^{-\omega_0}P\Big(\gamma\big(\rho(t)\big),\xi\Big)\\ &= F_{\omega_0}(w,1,\xi).\end{aligned} \tag{4.2}$$

Since the right hand side does not vanish, this shows that $\widetilde{\omega}_0 \le \omega_0$. Interchanging γ and $\widetilde{\gamma}$ in the preceding argument, we conclude that $\omega_0 = \widetilde{\omega}_0$. Now (4.2) completes the proof. ■

Remark 4.2 Part (iv) of Proposition 4.1 implies that $T_{\gamma,1}V$ is never empty and that $T_{\gamma,d}V$ is empty whenever $d > 1$ and $\gamma'(0) \notin T_0V$.

Proposition 4.3 *Let V be an analytic variety in a neighborhood of the origin in $\mathbb{C}^n$. Let $m = \mu_0(V)$ be its multiplicity in the origin, let $\gamma(t)$ be a simple curve as in Definition 3.1, $d \ge 1$, and $T_{\gamma,d}[V]$ the limit current defined in Definition 3.3.*

(i) *There are rational numbers $1 = d_1 < d_2 < \cdots < d_p$, where $1 \le p \le m+1$, such that $T_{\gamma,d}[V] = T_{\gamma,d'}[V]$ whenever $d_i < d \le d' < d_{i+1}$ for $1 \le i < p$ or $d_p < d \le d'$.*

We assume in the sequel that the set $\{1 = d_1, \dots, d_p\}$ is minimal, i.e., that (i) holds for no proper subset.

(ii) *If $d_i < d < d_{i+1}$, $1 \le i < p$, then $T_{\gamma,d}V$ is homogeneous and nonvoid.*
(iii) *If $d > d_p$, then $T_{\gamma,d}V$ is homogeneous or empty.*
(iv) *If $p = m+1$, then $T_{\gamma,1}[V] = T_{\gamma,d}[V]$ for $1 \le d < d_2$.*

Proof The proof relies on the Newton polygon for the function F defined in (3.9), *i.e.*, of the series $F(w,t,\xi) = \sum_{j,\beta,\alpha} a_{j,\beta,\alpha} w^\beta t^j \xi^\alpha$. Let M be the *support* of that series, *i.e.*,

$$M := \{(j,l) : qj \in \mathbb{N}_0, l \in \mathbb{N}_0, a_{j,\beta,\alpha} \neq 0 \text{ for some } \beta \text{ with } |\beta| = l \text{ and } |\alpha| = m\}.$$

For $\theta \in \mathbb{R}^2 \setminus \{0\}$ and $b \in \mathbb{R}$ define the closed half plane

$$H_{\theta,b} := \{x \in \mathbb{R}^2 : \langle x,\theta\rangle \ge b\}.$$

We call it admissible if $\theta \in [0,\infty[^2$ and $M \subset H_{\theta,b}$. The Newton polygon N is the intersection of all admissible half planes. Note that all vertices of N are elements of M. In particular, if (j,l) is a vertex of N, then $l \in \mathbb{N}_0$ and $l \le m$ since $(0,m) \in M$. Hence N has at most $m+1$ vertices and at most m edges between them (plus two unbounded edges). Let $1 = d_1 < d_2 < \cdots < d_p$ be an enumeration of

$$\{1\} \cup \left\{-\frac{1}{s} : s \text{ is the slope of a bounded edge of } N\right\}.$$

Then $p \leq m + 1$ is obvious, and if $p = m + 1$, then there is no edge with slope -1.

For $d \geq 1$ let $\omega_0(d) := \omega_0(d, V, \pi)$ be as in (3.11). Then the line

$$\partial H_{(1,d),\omega_0(d)} = \{(j, l) : j + dl = \omega_0(d)\}$$

has nonempty intersection with M. Fix i with $1 \leq i < p$. Then there is a pair $\big(j(i), l(i)\big)$ (a vertex of the Newton polygon) such that $M \cap \partial H_{(1,d),\omega_0(d)} = \big\{ \big(j(i), l(i)\big) \big\}$ for each $d \in \,]d_i, d_{i+1}[$. Hence

$$F_{\omega_0}(w, 1, \xi) = \sum_{|\beta|=l(d)} \sum_{|\alpha|=m} a_{j(d),\beta,\alpha} w^\beta \xi^\alpha. \tag{4.3}$$

By Proposition 3.12 this shows the part of (i) dealing with $d, d' < d_p$. The identity (4.3) also implies that $F_{\omega_0}(w, 1, \xi)$ is homogeneous and thus so is $T_{\gamma,d}V$. If $T_{\gamma,d}V$ were empty, then $l(d) = 0$, since otherwise $0 \in T_{\gamma,d}V$. However, the construction of the Newton polygon shows that then $T_{\gamma,d'}V = \varnothing$ for each $d' \geq d$, thus contradicting the minimality of the set $\{d_1, \ldots, d_p\}$. This completes the proof of (ii).

To show (iii) and finish the proof of (i), fix $d > d_p$. Then again there is a vertex $\big(j(p), l(p)\big)$ of the Newton polygon such that $M \cap \partial H_{(1,d),\omega_0(d)} = \big\{ \big(j(p), l(p)\big) \big\}$ for each $d > d_p$. This shows the independence of $T_{\gamma,d}[V]$ on $d > d_p$ and thus completes the proof of (i). The homogeneity of $T_{\gamma,d}V$ follows as before.

If $p = m + 1$ there is no edge with slope -1, hence the proof of (ii) applies also in this case. ■

The following result is a partial converse to Proposition 4.3(iii). We do not know whether it also holds in the case of arbitrary codimension.

Corollary 4.4 *Let $U \subset \mathbb{C}^n$ be a neighborhood of zero, let $A \colon U \to \mathbb{C}^n$ be holomorphic, let $V := \{z \in U : A(z) = 0\}$ and let γ be a simple curve. Let $d_1, \ldots, d_p$ be as in Proposition 4.3. If $2 \leq i \leq p$, then $T_{\gamma,d_i}V$ is not homogeneous.*

Proof Set $f(w, t) := A\big(\gamma(t) + w\big)$ and let $f_{\omega_0}(w, t) = \sum_{j+d\beta=\omega_0} a_{j,\beta} t^j w^\beta$ be the d-quasihomogeneous localization of f at 0. The proof of Proposition 4.3 shows that it suffices to show that $T_{\gamma,d}V$ is inhomogeneous if $-1/d$ is the slope of an edge of the Newton polygon. In that case, there are at least two pairs $(j_1, l_1) \neq (j_2, l_2)$ such that, for $i = 1, 2$, $j_i + dl_i = \omega_0$ and $a_{j_i,\beta_i} \neq 0$ for some β_i satisfying $|\beta_i| = l_i$. Then $f_{\omega_0}(w, 1)$ contains at least the terms $a_{j_i,\beta_i} w^{\beta_i}$, $i = 1, 2$. Since they have different degrees, Corollary 3.17 yields the claim. ■

5 Limit Varieties as Approximations

The aim of this section is to show that limit varieties $T_{\gamma,d}V$ approach V like $o(|z|^d)$ in conoids around γ that open like $|z|^d$, $d > 1$. This is analogous to the well known result that the classical tangent cone approaches V like $o(|z|)$.

In this section, let V be an analytic variety of pure dimension k in a neighborhood of zero.

For $z \in \mathbb{C}^n$ denote the n-th coordinate by z_n and the n-th coordinate of a simple curve γ by γ_n.

Definition 5.1 Let $\gamma \colon B(0,\epsilon)\setminus]-\infty,0] \to \mathbb{C}^n$ be a simple curve satisfying $\gamma_n(t) = t$ for all t, and let $d > 1$. Define

$$W_{\gamma,d} := \{\gamma(t) + t^d a : t \in B(0,\epsilon) \setminus \,]-\infty,0]\,, a \in T_{\gamma,d}V, a_n = 0\}.$$

We are going to prove that $W_{\gamma,d}$ approximates V of order d in conoids $\Gamma(\gamma,d,\Omega,r)$ as in (4.1). We begin with a preparatory lemma.

Lemma 5.2 *Let γ be a simple curve satisfying $\gamma_n(t) = t$ for all t, let $d > 1$, and let $\Omega \Subset \mathbb{C}^n$ be a relatively compact open subset. Then there are $A, r_0 > 0$ such that, whenever $r < r_0$ and $z \in \Gamma(\gamma,d,\Omega,r)$, then*

(a) $z_n \notin \,]-\infty,0]$,
(b) $\left|\big(z-\gamma(z_n)\big)\, z_n^{-d}\right| \le A$,
(c) $|z_n| \le |z| \le 2|z_n|$.

Proof Recall first that Definition 3.1 of a simple curve implies that its tangent vector ξ_0 has length 1. Hence the hypotheses of the present lemma imply $\xi_0 = (0,\dots,0,1)$. Thus claims (a) and (c) are obvious.

For the proof of (b), set $A := 1 + 2\sup\{|z| : z \in \Omega\}$ and let $z \in \Gamma(\gamma,d,\Omega,r)$ be given. Then there are $0 < t < r$ and $a \in \Omega$ with $z = \gamma(t) + t^d a$. In particular, $|z_n| = |t + t^d a_n| \le 2|t|$ provided that $r_0 > 0$ is sufficiently small. Then

$$\begin{aligned}\left|\big(z-\gamma(z_n)\big)\, z_n^{-d}\right| &= \left|\Big(\big(\gamma(t)-\gamma(z_n)\big) + t^d a\Big)\, z_n^{-d}\right| \\ &\le |t-z_n| \sup_{|\tau|\le r_0} |\gamma'(\tau)|\, |z_n|^{-d} + |a| \left|\frac{t}{z_n}\right|^d \\ &= \Big(|a_n| \sup_{|\tau|\le r_0} |\gamma'(\tau)| + |a|\Big) \left|\frac{t}{z_n}\right|^d .\end{aligned}$$

Since $\sup_{|\tau|\le r_0} |\gamma'(\tau)|$ and $|t z_n^{-d}|$ are arbitrarily close to 1 if r_0 is chosen sufficiently small, the claim is shown. ∎

Proposition 5.3 *For γ as in Definition 5.1 and each relatively compact open subset Ω of $\mathbb{C}^n$ there is $\delta > 0$ such that $W_{\gamma,d} \cap \Gamma(\gamma,d,\Omega,\delta)$ is a closed analytic subset of $\Gamma(\gamma,d,\Omega,\delta)$.*

For a precise description, let F_{ω_0} be as in (3.11)*, and fix $w \in \Gamma(\gamma,d,\Omega,\delta)$. Then $w \in W_{\gamma,d}$ if and only if $F_{\omega_0}\big(w-\gamma(w_n), w_n, \xi\big) = 0$ for all $\xi \in \mathbb{C}^{n-k}$.*

Proof Note that $\gamma(w_n)$ exists by Lemma 5.2. Obviously, $w \in W_{\gamma,d}$ if and only if $w = \gamma(t) + t^d a$ for suitable $t \in B(0,r) \setminus \,]-\infty,0]$ and $a \in T_{\gamma,d}V$. The special form of γ implies $t = w_n$. Hence $a = \big(w - \gamma(w_n)\big)\, w_n^{-d}$. By Proposition 3.14, $a \in T_{\gamma,d}V$ if and only if $F_{\omega_0}(a, 1, \xi) = 0$ for all $\xi \in \mathbb{C}^{n-k}$. Quasi-homogeneity of F_{ω_0} implies

$$F_{\omega_0}(a, 1, \xi) = F_{\omega_0}\Big(\big(w - \gamma(w_n)\big)\, w_n^{-d}, 1, \xi\Big) = w_n^{-\omega_0} F_{\omega_0}\big(w - \gamma(w_n), w_n, \xi\big). \qquad \blacksquare$$

Proposition 5.4 *Let V be an analytic set in $\mathbb{C}^n$ which contains the origin, let γ be a simple curve satisfying $\gamma_n(t) = t$ for all t, let $d > 1$, and let Ω be a relatively compact open subset of $\mathbb{C}^n$.*

(a) *For each $\epsilon > 0$ there is $\delta > 0$ such that for each $z \in V \cap \Gamma(\gamma, d, \Omega, \delta)$ there is $w \in W_{\gamma,d}$ with $|z - w| < \epsilon |z|^d$.*
(b) *For each $\epsilon > 0$ there is $\delta > 0$ such that for each $w \in W_{\gamma,d}$ with $|w| < \delta$ there is $z \in V$ with $|w - z| < \epsilon |w|^d$.*

Proof To show part (a) by contradiction, fix Ω and ϵ and assume that for each $l \in \mathbb{N}$ there is $z_l \in \Gamma(\gamma, d, \Omega, 1/l)$ with $|z_l - w| \geq \epsilon |z_l|^d$. Let t_l be the n-th coordinate of z_l. Then $t_l \in B(0, 1/l) \setminus\,]-\infty, 0]$ and $|t_l| \leq |z_l| \leq 2|t_l|$ for sufficiently large l by Lemma 5.2. Let

$$a_l := \big(z_l - \gamma(t_l)\big)\, t_l^{-d} \in V_{t_l},$$

then $|a_l| \leq A$ for A as in Lemma 5.2. Hence a subsequence $(a_{l_\nu})_{\nu \in \mathbb{N}}$ converges to some $a \in T_{\gamma,d}V$. Note that the n-th coordinate of each a_l vanishes, hence the same is true for a. This implies

$$w_l := \gamma(t_l) + t_l^d a \in W_{\gamma,d}.$$

Fix ν_0 so large that $|a_{l_\nu} - a| < \epsilon 2^{-d}$ for all $\nu > \nu_0$. Then for $\nu > \nu_0$

$$|w_{l_\nu} - z_{l_\nu}| = \big|\gamma(t_{l_\nu}) + t_{l_\nu}^d a - \big(\gamma(t_{l_\nu}) + t_{l_\nu}^d a_{l_\nu}\big)\big| = |t_{l_\nu}|^d\, |a_{l_\nu} - a| < \frac{\epsilon}{2^d} |t_{l_\nu}|^d \leq \epsilon |z_{l_\nu}|^d.$$

Since this contradicts the assumption, part (a) is shown.

To show part (b) by contradiction, fix $\epsilon > 0$ and assume that for each $l \in \mathbb{N}$ there is $w_l \in W_{\gamma,d}$ satisfying $|w_l| < 1/l$ such that for each $z \in V$ we have $|w_l - z| \geq \epsilon |w_l|^d$. Choose $t_l \in B(0, r) \setminus\,]-\infty, 0]$ and $a_l \in T_{\gamma,d}V$ such that $w_l = \gamma(t_l) + t_l^d a_l$ and such that the n-th coordinate of a_l vanishes. Then $|t_l| \leq 1/l$. By Lemma 3.7(i) there is l_0 such that for each $l > l_0$ there is $\zeta_l \in V_{t_l}$ with $|\zeta_l - a_l| < \epsilon$. Let $z_l := \gamma(t_l) + \zeta_l t_l^d$. Then $z_l \in V$ and

$$|z_l - w_l| = |a_l - \zeta_l|\, |t_l|^d \leq |a_l - \zeta_l|\, |w_l|^d < \epsilon |w_l|^d.$$

Since this contradicts the assumption, the claim is shown. $\blacksquare$

6 Examples

In this section we provide some examples to illustrate the results that we obtained so far. To keep things simple we concentrate on the case of hypersurfaces in $\mathbb{C}^n$, mainly even on surfaces in $\mathbb{C}^3$. First we use Corollary 3.17 to compute $T_{\gamma,d}V$ for a straight line γ.

For a compact statement of the results set $V(P) := \{z \in \mathbb{C}^n : P(z) = 0\}$ for a polynomial P. The first lemma reinterprets the construction in the proof of Proposition 4.3. It is shown that if γ is a straight line, then a subset of the support of the Taylor series expansion of $f\big(\gamma(t)+z\big)$ is sufficient to determine $T_{\gamma,d}V$.

Lemma 6.1 *Let $f\colon B^n(0,r) \to \mathbb{C}$ be a holomorphic function which has the expansion $f = \sum_{k\in I} Q_k$, where Q_k is a nonvanishing homogeneous polynomial of degree k. Assume further that $m := \min I \geq 2$ and that $Q_m(\theta) = 0$ for $\theta = (0,\dots,0,1)$. For $k \in I$ expand*

$$Q_k(z', z_n) = \sum_{\nu=0}^{a_k} p_{k,\nu}(z') z_n^{\nu}, \quad (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C},$$

where $p_{k,a_k} \not\equiv 0$. Let $\gamma\colon t \mapsto t\theta$. Then there are $p \in \mathbb{N}$, $1 = d_1 < d_2 < \cdots < d_p \in \mathbb{Q}$, $k_1 < \cdots < k_p \in \mathbb{N}$, and $L_1, \dots, L_p \subset \mathbb{N}$ such that

$$T_{\gamma,d}V = \begin{cases} V(Q_m) - \theta, & \text{if } d = 1,\\ V(\sum_{k\in L_j} p_{k,a_k}), & \text{if } d = d_j,\ j = 2,\dots,p,\\ V(p_{k_j,a_{k_j}}), & \text{if } d_j < d < d_{j+1},\ j = 1,\dots,p-1,\\ V(p_{k_p,a_{k_p}}), & \text{if } d > d_p. \end{cases}$$

All data can be calculated recursively in the following way: $d_1 := 1$, $L_1 := \{m\}$. Assume that $d_1,\dots,d_j$, $L_1,\dots,L_j$, and $k_1,\dots,k_{j-1}$ are already known. Then $k_j := \max L_j$. Further, define

$$K_j := \{k > k_j : a_k > a_{k_j}, k - a_k < k_j - a_{k_j}\}.$$

(a) *If $K_j = \varnothing$, then $p = j$.*
(b) *If $K_j \neq \varnothing$, then*

$$d_{j+1} := \left(\max_{k\in K_j} \frac{(k_j - a_{k_j}) - (k - a_k)}{a_k - a_{k_j}}\right)^{-1}$$

and

$$L_{j+1} := \{k_j\} \cup \left\{k \in K_j : \frac{k_j - a_{k_j} - (k - a_k)}{a_k - a_{k_j}} = \frac{1}{d_{j+1}}\right\}.$$

Proof To apply Corollary 3.17 we consider the expansion

$$\begin{aligned} f\big(\gamma(t)+z\big) &= f(z', t+z_n) = \sum_{k\in I} Q_k(z', t+z_n)\\ &= \sum_{k\in I}\sum_{j=0}^{a_k} p_{k,j}(z')(t+z_n)^j \qquad (6.1)\\ &= \sum_{k\in I}\sum_{j=0}^{a_k} p_{k,j}(z') \sum_{l=0}^{j} \binom{j}{l} t^l z_n^{j-l}. \end{aligned}$$

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Hence the support of the Taylor series expansion of $f\big(\gamma(t)+z\big)$ is contained in $\{(l,k-l) : k \in I, l \le a_k\}$. On the other hand, if $k \in I$, then the terms in the expansion (6.1) whose degree in t is a_k and whose degree in z is $k-a_k$ constitute the polynomial $p_{k,a_k} \not\equiv 0$. This implies in particular that each pair $(a_k, k-a_k)$ with $k \in I$ is in the support. Hence the Newton polygon consists of two unbounded edges parallel to the axes, possibly an edge of slope -1 through $(a_m, m-a_m)$, and possibly several more edges passing through two or more points of the form $(a_k, k-a_k)$ with $k \in I$. The slopes of these edges are related to the d_j as in the proof of Proposition 4.3. The formula for d_j is a consequence of the classical construction of the Newton polygon as a convex hull. Once the d_j are known, we proceed as in the proof of Proposition 4.3. ■

Remark Note that an analogue of Lemma 6.1 also holds if $\gamma(t)$ is any simple curve; of course then the coefficients in the Puiseux series expansion of γ also play a role.

To draw a corollary from Lemma 6.1, we recall the following fact from [3, Lemma 3.9].

Lemma 6.2 *Let $P \in \mathbb{C}[z_1,\dots,z_n]$ be homogeneous of degree $m \ge 1$ and assume that $P(\theta) = 0$ for $\theta = (0,\dots,0,1)$. Then there is $1 \le \nu \le m$ such that*

$$P(z', z_n) = \sum_{k=\nu}^{m} p_k(z') z_n^{m-k}, \quad (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C},$$

for suitable polynomials $p_k \in \mathbb{C}[z_1,\dots,z_{n-1}]$, $\nu \le k \le m$, where p_k either vanishes identically or is homogeneous of degree k and where $p_\nu \not\equiv 0$. Moreover, for the localization P_θ of P at θ we have

$$P_\theta(z', z_n) = p_\nu(z').$$

Corollary 6.3 *Under the hypotheses of Lemma 6.1, denote by f_0 the localization of f at zero and by $(f_0)_\theta$ the localization of f_0 at θ. Then*

$$T_{\gamma,d} V = V\big((f_0)_\theta\big) \quad \textit{for } d \in \,]1,\infty[\, \textit{ if } p = 1 \textit{ and for } d \in \,]d_1, d_2[\, \textit{ if } p > 1.$$

Proof In the notation of Lemma 6.1 we have $f_0 = Q_m$. From Lemma 6.2 we get for $P = Q_m$ that

$$Q_m(z', z_n) = \sum_{k=\nu}^{m} p_\nu(z') z_n^{m-k} = \sum_{j=0}^{m-\nu} p_{m,j}(z') z_n^j,$$

where $p_{m,m-\nu} = p_\nu \not\equiv 0$. Hence we have $a_m = m-\nu$ and $(f_0)_\theta = p_\nu = p_{m,m-\nu}$. Therefore Lemma 6.1 implies that

$$T_{\gamma,d} V = V(p_{k_1,a_{k_1}}) = V(p_{m,m-\nu}) = V\big((f_0)_\theta\big)$$

for d as stated above. ■

Remark Corollary 6.3 shows that the localization of f_0 in θ determines $T_{\gamma,d}V$ for $1 < d < d_2$. Since such localizations have been used already in applications to partial differential equations, the lemma shows that—at least implicitly—limit varieties have already had a useful role to play in the literature.

Example 6.4 Define $P \in \mathbb{C}[x, y, z]$ by

$$P(x, y, z) := y(x^2 - y^2) + x^2z^2 + yz^6 + z^{11}$$

and let $V := V(P)$. Then

$$T_0V = \{(x, y, z) \in \mathbb{C}^3 : y(x^2 - y^2) = 0\}$$

and $\theta = (0, 0, 1)$ is a singular point of T_0V. Let γ denote the simple curve $\gamma\colon t \mapsto t\theta$. Then we have in the notation of Proposition 4.3

$$d_1 = 1, \quad d_2 = 2, \quad d_3 = 4, \quad d_4 = 5,$$

and the following limit varieties:

$$\begin{gathered}
T_{\gamma,d}V = T_0V \quad \text{for } d \in [1, 2[,\\
T_{\gamma,2}V = \{(x, y, z) \in \mathbb{C}^3 : -y^3 + (y + 1)x^2 = 0\},\\
T_{\gamma,d}V = \{(x, y, z) \in \mathbb{C}^3 : x^2 = 0\}, \quad \text{for } d \in]2, 4[,\\
T_{\gamma,4}V = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y = 0\},\\
T_{\gamma,d}V = \{(x, y, z) \in \mathbb{C}^3 : y = 0\}, \quad \text{for } d \in]4, 5[,\\
T_{\gamma,5}V = \{(x, y, z) \in \mathbb{C}^3 : y + 1 = 0\},\\
T_{\gamma,d}V = \varnothing, \quad \text{for } d > 5.
\end{gathered}$$

This follows recursively from Lemma 6.1.

Remark Note that by Example 6.4 the bound for p in Proposition 4.3 is sharp.

Example 6.5 Define $P \in \mathbb{C}[x, y, z]$ by

$$P(x, y, z) = (2x^2 - y^2)(x^2 - y^2) + xyz^2 + yz^4 + z^7$$

and let $V = V(P)$. Then

$$T_0V = \{(x, y, z) \in \mathbb{C}^3 : (2x^2 - y^2)(x^2 - y^2) + xyz^2 = 0\}$$

and $\theta = (0, 0, 1)$ is in T_0V. Let γ denote the simple curve $\gamma\colon t \mapsto t\theta$. Then we have

$$d_1 = 1, \quad d_2 = 2, \quad d_3 = 3$$

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and the following limit varieties:

$$\begin{gathered} T_{\gamma,1}V = T_0V - \theta, \\ T_{\gamma,d}V = \{(x,y,z) \in \mathbb{C}^3 : xy = 0\}, \quad \text{for } d \in\,]1,2[, \\ T_{\gamma,2}V = \{(x,y,z) \in \mathbb{C}^3 : (x+1)y = 0\}, \\ T_{\gamma,d}V = \{(x,y,z) \in \mathbb{C}^3 : y = 0\}, \quad \text{for } d \in\,]2,3[, \\ T_{\gamma,3}V = \{(x,y,z) \in \mathbb{C}^3 : y+1 = 0\}, \\ T_{\gamma,4}V = \varnothing, \quad \text{for } d > 3. \end{gathered}$$

As in the previous example, this follows from Lemma 6.1.

Example 6.6 Define $P \in \mathbb{C}[x,y,z]$ by

$$P(x,y,z) = y(x^2 - y^2) - yz^3 + z^5$$

and let $V = V(P)$. Then

$$T_0V = \{(x,y,z) \in \mathbb{C}^3 : y(x^2 - y^2) = 0\}$$

and $\theta = (0,0,1)$ is a singular point of T_0V. Define $\gamma\colon t \mapsto t\theta$. Then we have

$$d_1 = 1, \quad d_2 = \frac{3}{2}, \quad d_3 = 2$$

and the following limit varieties

$$\begin{gathered} T_{\gamma,d}V = T_0V, \quad \text{for } d \in \left[1, \tfrac{3}{2}\right[, \\ T_{\gamma,\frac{3}{2}}V = \{(x,y,z) \in \mathbb{C}^3 : y(x^2 - y^2 - 1) = 0\}, \\ T_{\gamma,d}V = \{(x,y,z) \in \mathbb{C}^3 : y = 0\}, \quad \text{for } d \in \left]\tfrac{3}{2}, 2\right[, \\ T_{\gamma,2}V = \{(x,y,z) \in \mathbb{C}^3 : -y+1 = 0\}, \\ T_{\gamma,d}V = \varnothing, \quad \text{for } d > 2. \end{gathered}$$

These statements follow from Lemma 6.1 as in the previous examples. Note that the limit variety $T_{\gamma,3/2}V$ has $(1,0,\lambda)$ and $(-1,0,\lambda)$, $\lambda \in \mathbb{C}$, as singular points. Define $\sigma\colon t \mapsto (t^{3/2},0,t)$ and let us check whether limit varieties $T_{\sigma,d}V$ exist for $d > 3/2$. To do so, expand

$$\begin{aligned} P\big(\sigma(t) + (x,y,z)\big) &= y\big((x + t^{3/2})^2 - y^2\big) + y(t+z)^3 + (t+z)^5 \\ &= y(x^2 - y^2) + z^5 + (5z^4 - 3yz^2)t + 2xyt^{3/2} \\ &\quad + (10z^3 - 3yz)t^2 + 10z^2t^3 + 5zt^4 + t^5. \end{aligned} \tag{6.2}$$

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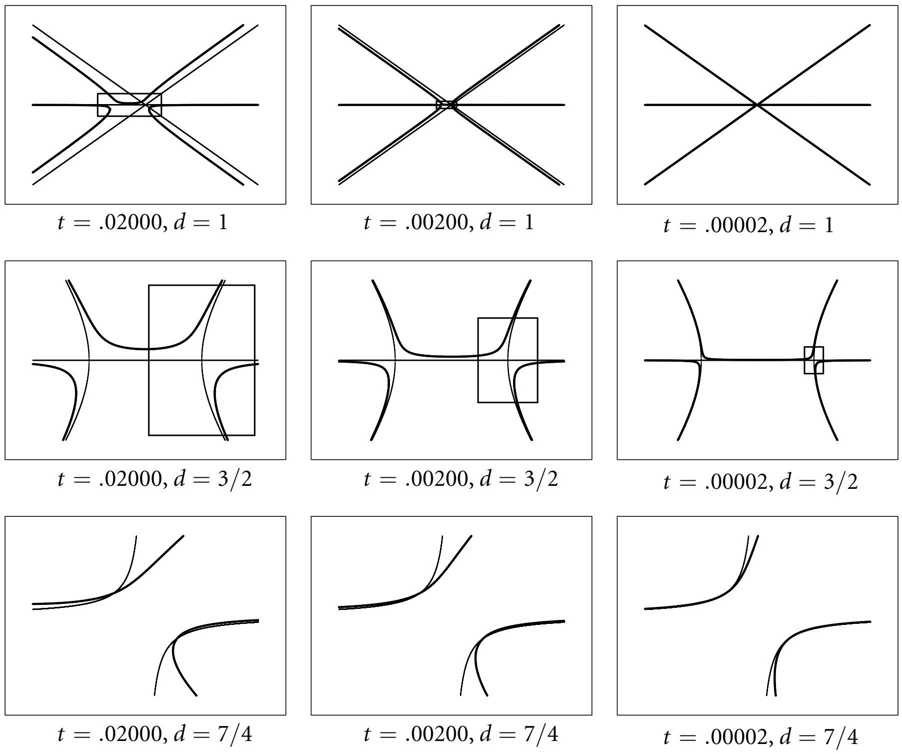

Figure 1: Intersection of $V_{\sigma,d,t}$ with $\{z = 0\}$ for V and σ as in Example 6.6 and different values of t and d. The thin line is the limit variety, the box indicates the area shown in the figure below. The scaling on both axes may differ.

The relevant points for the Newton polygon are now $(0, 3)$, $(\frac{3}{2}, 2)$, and $(0, 5)$. From this it follows that at the values $d = 3/2$ and $d = 7/4$ the behavior of the limit varieties $T_{\sigma,d}V$ changes. It is easy to check that $T_{\sigma,3/2}V$ is obtained from $T_{\gamma,3/2}V$ by a shift. Hence it does not give new information. From (6.2) and Corollary 3.17 we get for all limit varieties along σ:

$$T_{\sigma,\frac{3}{2}}V = \Big\{(x, y, z) \in \mathbb{C}^3 : y\big((x + 1)^2 - y^2 - 1\big) = 0\Big\},$$
$$T_{\sigma,d}V = \{(x, y, z) \in \mathbb{C}^3 : 2xy = 0\}, \quad \text{for } d \in \left]\tfrac{3}{2}, \tfrac{7}{4}\right[,$$
$$T_{\sigma,\frac{7}{4}}V = \{(x, y, z) \in \mathbb{C}^3 : 2xy + 1 = 0\},$$
$$T_{\sigma,d}V = \varnothing, \quad \text{for } d > \tfrac{7}{4}.$$

In Figure 1, we show $V_{\sigma,d,t} \cap (\mathbb{R}^2 \times \{0\})$ for $t = .02, 0.002$, and $.00002$ and $d = 1$, $\frac{3}{2}$, and $\frac{7}{4}$ together with the corresponding limit variety. Furthermore, the figure for the next d and the same t is indicated by a rectangle. For $d = 1$, the rectangle $[-t, t]^2$

is shown, for $d = \frac{3}{2}$ it is $[-3t^d, t^d] \times [-t^d, t^d]$, and for $d = \frac{7}{4}$ it is $[-\frac{5}{2}t^d, \frac{5}{2}t^d]^2$. In particular, squares do not plot as squares.

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