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AN INTEGRAL REPRESENTATION FOR THE GENERALIZED BINOMIAL FUNCTION

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ABSTRACT. The generalized binomial function $\mathcal{B}_\alpha$ can be obtained as the solution of the equation $y = 1 + zy^\alpha$ which satisfies $y(0) = 1$ where $\alpha \neq 1$ is assumed to be real and positive. The technique of Lagrange inversion can be used to express $\mathcal{B}_\alpha$ as a series which converges for $|z| < \alpha^{-\alpha}|\alpha - 1|^{\alpha-1}$. We obtain a representation of the function as a contour integral and show that if $\alpha > 1$ it is an analytic function in the complex z plane cut along the nonnegative real axis. For $0 < \alpha < 1$ the region of analyticity is the sector $|\arg(-z)| < \alpha\pi$. In either case $\mathcal{B}_\alpha$ defined by the series can be continued beyond the circle of convergenece of the series through a functional equation which can be derived from the integral representation.

1. **Introduction.** The *generalized binomial function* or *hyperbinomial function* $\mathcal{B}_\alpha$ can be obtained as the solution of the equation

$$y(z) = 1 + zy(z)^\alpha \tag{1}$$

which satisfies $y(0) = 1$. We assume that z is an arbitrary complex number and that $\alpha \neq 1$ is real and positive. Throughout the paper α is considered to be a fixed parameter and $\mathcal{B}_\alpha$ is considered to be a function of z only. A property of the function derivable from the functional equation (1) can be used to deal with negative values of α. The restriction of α to real values will serve to cover the applications of the function which we give here, however extension to complex values would seem to be straightforward in view of the integral representation for the function which we obtain.

The method of Lagrange inversion [8, p. 133] can be applied to equation (1) to represent the solution in series form as

$$\mathcal{B}_\alpha(z) = \sum_{k=0}^{\infty} \binom{\alpha k + 1}{k} \frac{z^k}{\alpha k + 1}. \tag{2}$$

A simple application of the ratio test shows that the series converges for

$$|z| < \frac{|\alpha - 1|^{\alpha-1}}{\alpha^\alpha}. \tag{3}$$

Since the value on the right side of this equation occurs several times throughout the course of the paper, we set $A = \alpha^{-\alpha}|\alpha - 1|^{\alpha-1}$ in the following. For $\alpha = 1$ the series reduces to the geometric series and so converges for $|z| < 1$.

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In the following we show that the function defined by (1) can be represented by

$$\mathcal{B}_\alpha(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha s+1)\Gamma(-s)}{\Gamma\big((\alpha-1)s+2\big)}(-z)^s\, ds \tag{4}$$

The integral converges for $|\arg(-z)| \leq \pi$ if $\alpha > 1$ and for $|\arg(-z)| \leq \alpha\pi$ if $0 < \alpha < 1$. It represents an analytic function of z for $|\arg(-z)| \leq \pi-\delta$ if $\alpha > 1$ and for $|\arg(-z)| \leq \alpha\pi-\delta$ if $0 < \alpha < 1$ where $\delta > 0$. It is also shown that the continuation of (2) beyond the circle of convergence $|z| = A$ is given by the functional equation

$$\mathcal{B}_\alpha(z) = 1 - \mathcal{B}_{1/\alpha}\left(\frac{-1}{\sqrt[\alpha]{-z}}\right) \tag{5}$$

since the series represented by the right member converges for $|z| > A$. As an application of (5), we later demonstrate a simple differential equation whose solution on a given interval can be expressed by using the appropriate form of the generalized binomial function.

The generalized binomial function is introduced in the form (2) in [1] where it is used to obtain various identities involving binomial coefficients. This has been one of the primary uses of the function in the past, but our investigations indicate that there is a considerable range of applications for which this function is well suited as an analytical tool. In the following examples as well as others which we are currently developing, the utility of the generalized binomial function appears to be two-fold: Its functional properties can be used to simplify differential equations and non-elementary integral forms or they can be used to explicitly invert expressions which would otherwise remain implicit. The function has also appeared in previous work, but its form has gone unnoticed. An example of this occurs in [4] where the calculation of roots of certain polynomial equations is considered. Here, the generalized binomial function appears in the form of a generalized hypergeometric function. Application of the fundamental properties of the function to the calculations contained therein would have considerably simplified the final form.

A nonlinear, second order differential equation which arises in the study of shocks was solved in [5]. There, the properties of the functional relation (1) were instrumental in obtaining the solution. It was also shown that a corresponding first order differential equation also has solutions which can be expressed in terms of generalized binomial functions. In an unpublished work [6], the function was obtained as the solution to a nonlinear recurrence relation. In the same paper a general form involving natural logarithms is shown to give rise to generalized binomial functions when it is inverted. The function has also been used to evaluate integrals of *hyperelliptic* type in [2, 3].

In the preceding discussion we have mentioned some of the applications of the generalized binomial function to problems arising in rather diverse fields and beyond these, we are currently developing additional applications of the function. The function appears to have very useful analytic properties, but does not appear to have been widely studied. Based on this, we believe that a systematic study of the function should be carried out and that, as this is done, further applications of the function will be uncovered. This paper is a first step in that process.

In Section 2 the actual form for the integral representation is obtained. It is based on the observation that when $\alpha = n$ is a positive integer greater than 1, the generalized binomial function is a generalized hypergeometric function of type ${}_n\mathcal{F}_{n-1}$. These functions have representations as contour integrals of Mellin-Barnes type and when the specific form applicable to the ${}_n\mathcal{F}_{n-1}$ hypergeometric function is considered, it is shown that the integrand reduces to a simple ratio of Gamma functions. Moreover, the same form holds when n is replaced by a general parameter α. We specifically note that if α is not a positive integer, then $\mathcal{B}_\alpha$ is not hypergeometric in nature.

In the next Section we show that the series for the evaluation of the integral for arbitrary values of the parameter α leads directly to the series (2) for the generalized binomial function. The analytic continuation of $\mathcal{B}_\alpha$ given by (5) and valid for $|z| > A$ is found in the last Section.

2. **Connection with the generalized hypergeometric function.** In [5] the generalized binomial function occurred in the solution of a particular second order, nonlinear ordinary differential equation. It was observed there that if $n \geq 2$ is an integer, then $\mathcal{B}_n$ is a generalized hypergeometric function of type ${}_n\mathcal{F}_{n-1}$. The particular form is

$$\mathcal{B}_n(z) = {}_n\mathcal{F}_{n-1}\left(\begin{matrix} \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} \\ \frac{2}{n-1}, \frac{3}{n-1}, \dots, \frac{n}{n-1} \end{matrix} \,\middle|\, \frac{n^n z}{(n-1)^{n-1}} \right). \tag{6}$$

The generalized hypergeometric function ${}_{q+1}\mathcal{F}_q$ can be expressed as a contour integral by (see *e.g.*, [7, p. 101])

$${}_{q+1}\mathcal{F}_q\left(\begin{matrix} a_1, a_2, \dots, a_{q+1} \\ b_1, b_2, \dots, b_q \end{matrix} \,\middle|\, z \right) = \frac{\Gamma}{2\pi i} \int_B \frac{\Gamma(-s)\prod_{j=1}^{q+1}\Gamma(a_j + s)}{\prod_{j=1}^{q}\Gamma(b_j + s)} (-z)^s \, ds \tag{7}$$

where

$$\Gamma = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{q+1}\Gamma(a_j)}$$

and B is a Barnes path of integration which separates the poles of $\Gamma(-s)$ to the right of the path from the poles of $\Gamma(a_j + s)$ to the left of the path. According to general theory (see *e.g.*, [7, p. 97]), the integral in (7) represents an analytic function of z in the region $|\arg(-z)| < \pi$.

In terms of the particular generalized hypergeometric function ${}_n\mathcal{F}_{n-1}$ which defines the generalized binomial function $\mathcal{B}_n$, the contour integral defined by (7) becomes

$${}_n\mathcal{F}_{n-1}\left(\begin{matrix} \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} \\ \frac{2}{n-1}, \frac{3}{n-1}, \dots, \frac{n}{n-1} \end{matrix} \,\middle|\, z \right) = \frac{\Gamma}{2\pi i} \int_B \frac{\Gamma(-s)\prod_{j=0}^{n-1}\Gamma\left(1 + s - \frac{j}{n}\right)}{\prod_{j=2}^{n}\Gamma\left(s + \frac{j}{n-1}\right)} (-z)^s \, ds$$

The integral appearing in this expression can be considerably simplified in form if we make use of the Gaussian multiplication formula [7, p. 26] for the Gamma function

$$\Gamma(nz) = (2\pi)^{\frac{1-n}{2}} n^{nz-\frac{1}{2}} \prod_{j=0}^{n-1} \Gamma\left(z + \frac{j}{n}\right) \tag{8}$$

Applying (8) to the products which appear in the integrand as well as to the factor Γ allows us to express the integral as

$$\frac{1}{2\pi i} \int_B \frac{\Gamma(-s)\Gamma(ns+1)}{\Gamma\big((n-1)s+2\big)} \left(-\frac{(n-1)^{n-1}}{n^n} z\right)^s ds$$

where we have used the recurrence relation $\Gamma(t+1) = t\Gamma(t)$ several times to simplify the form. Comparing this with (6), we obtain the contour integral representation for the generalized binomial function for the case when $n \geq 2$ is an integer as

$$\mathcal{B}_n(z) = \frac{1}{2\pi i} \int_B \frac{\Gamma(-s)\Gamma(ns+1)}{\Gamma\big((n-1)s+2\big)} (-z)^s \, ds. \tag{9}$$

The general form given in (4) is obtained by replacing n by α in (9) and is, in fact, valid for any $\alpha > 0$, although we do not specifically consider the case $\alpha = 1$. For this value, the integral reduces to the known form for the function $\mathcal{B}_1(z) = 1/(1-z)$ and is valid for $\arg(-z) < \pi$, but the following analysis is not directly applicable.

For the general integral, the path of integration is along the imaginary axis indented to the left at the origin to avoid the pole of $\Gamma(-s)$ at the origin but passing to the right of the pole of $\Gamma(\alpha s + 1)$ at $s = -1/\alpha$ on the negative real axis. On the contour for $s = iy$ and $\alpha > 1$, it is straightforward to show that

$$\left| \frac{\Gamma(\alpha s+1)\Gamma(-s)}{\Gamma\big((\alpha-1)s+2\big)} (-z)^s \right| \sim \frac{\sqrt{2\pi\alpha}}{(\alpha-1)^{3/2}} \frac{e^{-\pi|y|-y\arg(-z)}}{|y|^{3/2}}$$

as $|y| \longrightarrow \infty$. Hence, the integral is convergent for any $z \neq 0$ and can easily be shown (see *e.g.*, [8, p. 92]) to be uniformly convergent for $|\arg(-z)| < \pi$. Then in the complex z plane with the nonnegative real axis deleted, the above estimate shows that (4) represents an analytic function of z. Similarly, if $\alpha < 1$, we have

$$\left| \frac{\Gamma(\alpha s+1)\Gamma(-s)}{\Gamma\big((\alpha-1)s+2\big)} (-z)^s \right| \sim \frac{\sqrt{2\pi\alpha}}{(1-\alpha)^{3/2}} \frac{e^{-\alpha\pi|y|-y\arg(-z)}}{|y|^{3/2}}$$

as $|y| \longrightarrow \infty$. Here the integral is uniformly convergent for $|\arg(-z)| < \alpha\pi$ and represents an analytic function for z in this sector.

One final remark: The integral (4) has been developed for positive values of α. However, it can be shown from (1) that $\mathcal{B}_\alpha(z) = 1/\mathcal{B}_{1-\alpha}(-z)$ so we can easily obtain results relating to negative values of the parameter.

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3. **Evaluation of the Integral for $\mathcal{B}_\alpha$.** In the following we shall establish that (2) and (4) are representations of the same function by showing that they agree on a region in the complex z plane. Recalling that A is defined by (3), we obtain the following:

THEOREM. *Let either*
(1) $\alpha > 1$ *and* $z \neq 0$ *or*
(2) $0 < \alpha < 1$ *and* $|\arg(-z)| < \alpha\pi$.
with $|z| < A$. *Then*

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha s+1)\Gamma(-s)}{\Gamma\big((\alpha-1)s+2\big)}(-z)^s\,ds = \sum_{k=0}^{\infty}\binom{\alpha k+1}{k}\frac{z^k}{\alpha k+1}.$$

Furthermore, if
(1) $\alpha > 1$ *and* $|\arg(-z)| \leq \pi-\delta$ *or*
(2) $0 < \alpha < 1$ *and* $|\arg(-z)| \leq \alpha\pi-\delta$.
where $\delta > 0$, *the integral represents an analytic function of* z.

PROOF. By the argument at the end of the previous Section we can easily establish that the integral is uniformly convergent for $|\arg(-z)| \leq \pi-\delta$ or $|\arg(-z)| \leq \alpha\pi-\delta$ according to whether α is greater than or less than 1. Hence, in either case, it represents an analytic function of z on the indicated region. It remains to show that (2) and (4) are representations of the same function.

The analysis is slightly different according to whether $\alpha > 1$ or $\alpha < 1$. In the following we assume that $\alpha > 1$ and indicate, at an appropriate point, what modifications must be made to deal with the case $\alpha < 1$.

Suppose $|z| < A$ with $z \neq 0$. Let N be a positive integer and B_N be a contour along the imaginary axis running from $-(N+\frac{1}{2})i$ to $(N+\frac{1}{2})i$. We suppose that it is indented to the left at the origin to pass between the poles of $\Gamma(-s)$ and $\Gamma(\alpha s+1)$ at $s=0$ and $s=-1/\alpha$ respectively. Close the contour to the right by taking a semicircular arc C_N having radius $N+\frac{1}{2}$ and centred at the origin. Set

$$\varphi(s) = \frac{\Gamma(\alpha s+1)\Gamma(-s)}{\Gamma\big((\alpha-1)s+2\big)}(-z)^s.$$

The only poles of $\varphi(s)$ inside the the closed contour B_N+C_N are those of $\Gamma(-s)$ occurring at $s=k$ where $k=0,1,2,\ldots$ is an integer. So, by the residue Theorem,

$$\frac{1}{2\pi i}\int_{B_N}\varphi(s)\,ds + \frac{1}{2\pi i}\int_{C_N}\varphi(s)\,ds = \sum_{k=0}^{N}\frac{\Gamma(\alpha k+1)}{\Gamma\big((\alpha-1)k+2\big)}\frac{z^k}{k!}$$

where we have used the fact that the residue of the Gamma function is $(-1)^{k+1}/k!$. Then as $N \longrightarrow \infty$, B_N approaches the Barnes' path B indicated earlier and all we must show is that the contribution over C_N tends to zero.

On C_N it is convenient to rewrite the integrand using the reflection principle of the Gamma function as

$$\varphi(s) = \frac{\Gamma(\alpha s+1)}{\Gamma\big((\alpha-1)s+2\big)\Gamma(s+1)}\frac{-\pi}{\sin\pi s}(-z)^s.$$

Then for $s = Re^{i\theta}$ where $R = N + \frac{1}{2}$ and $-\pi/2 \leq \theta \leq \pi/2$, we easily see that

$$\left|\frac{\Gamma(\alpha s + 1)}{\Gamma\big((\alpha - 1)s + 2\big)\Gamma(s + 1)}\right| \sim \frac{\sqrt{\alpha}}{\sqrt{2\pi}(\alpha - 1)^{3/2}} \frac{A^{-R\cos\theta}}{R^{3/2}}$$

for large values of R. Also, it is straightforward to establish that

$$\frac{1}{|\sin \pi s|} = \mathrm{O}\big(e^{-\pi R|\sin\theta|}\big)$$

and

$$|(-z)^s| = |z|^{R\cos\theta} e^{-R\arg(-z)\sin\theta}.$$

Then on C_N, $|\varphi(s)|$ is of order

$$\frac{1}{R^{3/2}}\left[\frac{|z|}{A}\right]^{R\cos\theta} e^{-R(\pi|\sin\theta|+\arg(-z)\sin\theta)} \leq \frac{1}{R^{3/2}}.$$

If $\alpha < 1$, we also apply the reflection principle to the factor $\Gamma\big((\alpha - 1)s + 2\big)$ to obtain the corresponding order estimate as

$$\frac{1}{R^{3/2}}\left[\frac{|z|}{A}\right]^{R\cos\theta} e^{-R(\alpha\pi|\sin\theta|+\arg(-z)\sin\theta)} \leq \frac{1}{R^{3/2}}$$

if $|\arg(-z)| \leq \alpha\pi$. Hence, in either case, we have

$$\int_{C_N} |\varphi(s)|\,|ds| = \mathrm{O}\Big(\frac{1}{\sqrt{R}}\Big) \text{ as } R = N + \frac{1}{2} \longrightarrow \infty$$

and this leads immediately to the desired result

$$\begin{aligned}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha s + 1)\Gamma(-s)}{\Gamma\big((\alpha - 1)s + 2\big)}(-z)^s\,ds &= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha k + 1)}{\Gamma\big((\alpha - 1)k + 2\big)} \frac{z^k}{k!} \\ &= \sum_{k=0}^{\infty} \binom{\alpha k + 1}{k} \frac{z^k}{\alpha k + 1}\end{aligned}$$

when we convert the Gamma functions to their binomial coefficient equivalent.

4. **Analytic continuation of $\mathcal{B}_\alpha$.** The analytic continuation of the hypergeometric series beyond the circle of convergence $|z| = 1$ can be obtained by closing the contour to the left of the imaginary axis rather than to the right. A similar technique can also be used to extend the series (2) for $\mathcal{B}_\alpha(z)$ beyond $|z| = A$. If these straightforward and somewhat tedious calculations are carried out, the appropriate result is obtained. However, in place of this we present the following simple Proof[1] of the analytic continuation.

[1] Proof suggested by the anonymous referee.

THEOREM. *If*

$$\mathcal{B}_\alpha(z) = \int_B \frac{\Gamma(\alpha s+1)\Gamma(-s)}{\Gamma\big((\alpha-1)s+2\big)}(-z)^s\,ds$$

then

$$\mathcal{B}_\alpha(z) = 1 - \mathcal{B}_{1/\alpha}\Big(\frac{-1}{\sqrt[\alpha]{-z}}\Big).$$

PROOF. In the integral B is a Barnes' path up the imaginary axis and indented to the left at the origin to separate the poles of $\Gamma(-s)$ and $\Gamma(\alpha s+1)$ at $s=0$ and $s=-1/\alpha$. Set $w=-\alpha s$ and let B' be a path which proceeds down the imaginary axis and which is indented to the right in such a fashion that $B+B'$ is a negatively oriented closed curve which encircles only the pole of the Gamma function corresponding to $s=w=0$.

Then

$$\begin{aligned}
\mathcal{B}_\alpha(z) &= \frac{1}{2\pi i}\int_{B'} \frac{\Gamma(-w+1)\Gamma\big(\frac{w}{\alpha}\big)}{\Gamma\big((\alpha-1)\frac{w}{-\alpha}+2\big)}(-z)^{-w/\alpha}\Big(-\frac{dw}{\alpha}\Big)\\
&= \frac{-1}{2\pi i}\int_{-B'} \frac{\Gamma(-w)\big(\frac{w}{\alpha}\big)\Gamma\big(\frac{w}{\alpha}\big)}{\Gamma\big((\frac{1}{\alpha}-1)w+2\big)}\Big(-\frac{-1}{\sqrt[\alpha]{-z}}\Big)^w dw\\
&= \frac{-1}{2\pi i}\oint_{-(B+B')} \frac{\Gamma(-w)\big(\frac{w}{\alpha}\big)\Gamma\big(\frac{w}{\alpha}\big)}{\Gamma\big((\frac{1}{\alpha}-1)w+2\big)}\Big(-\frac{-1}{\sqrt[\alpha]{-z}}\Big)^w dw\\
&\quad - \frac{1}{2\pi i}\int_B \frac{\Gamma(-w)\Gamma\big(\frac{w}{\alpha}+1\big)}{\Gamma\big((\frac{1}{\alpha}-1)w+2\big)}\Big(-\frac{-1}{\sqrt[\alpha]{-z}}\Big)^w dw\\
&= 1-\mathcal{B}_{1/\alpha}\Big(\frac{-1}{\sqrt[\alpha]{-z}}\Big)
\end{aligned}$$

since the residue of $\Gamma(-w)$ at $w=0$ is -1. This functional equation can be used to continue $\mathcal{B}_\alpha(z)$ defined by (2) beyond its circle of convergence $|z|=A$.

This functional equation for the generalized binomial function can be shown to have an immediate application to the solution of a simple differential equation which has been obtained previously in the study of one-dimensional shocks. In [6] the equation

$$\lambda(2x+A-2)\frac{dx}{dt} = 2x-x^2 \text{ for } 0<x<2 \tag{10}$$

where A and λ are positive constants with $A>2$ was considered. Although the solution of this type of equation is normally expressed in terms of natural logarithms, it was shown there that it could be explicitly given by

$$x(t) = 2\mathcal{B}_\beta\big(-2^{\beta-1}e^{q(t-t_0)}\big) \tag{11}$$

where $q=-2/[\lambda(A+2)]$, $p=2/[\lambda(A-2)]$ and $\beta=-q/p$ where $0<\beta<1$.

The solution is normalized, as can easily be seen from the functional equation, so that $x(t_0) = 1$.

In (11), if we associate β and $-2^{\beta-1}e^{q(t-t_0)}$ with $1/\alpha$ and the argument of $\mathcal{B}_{1/\alpha}$ in (5) respectively, then $z = -2^{\alpha-1}e^{p(t-t_0)}$ and we obtain two convergent series representations of the solution valid on different t intervals. We have

$$x(t) = 2 - 2\sum_{k=1}^{\infty}(-1)^{k+1}\binom{\beta k+1}{k}\frac{2^{k(\beta-1)}e^{kq(t-t_0)}}{\beta k+1} \quad \text{valid for} \quad t - t_0 > \frac{1}{q}\left((1-\beta)\ln\left(\frac{2}{1-\beta}\right) - \beta\ln\beta\right) \tag{12}$$

and

$$x(t) = 2\sum_{k=1}^{\infty}(-1)^{k+1}\binom{\alpha k+1}{k}\frac{2^{k(\alpha-1)}e^{kp(t-t_0)}}{\alpha k+1} \quad \text{valid for} \quad t - t_0 < \frac{1}{p}\left((\alpha-1)\ln\left(\frac{\alpha-1}{2}\right) - \alpha\ln\alpha\right). \tag{13}$$

By (5), one of these expressions represents the continuation of the other so the limiting t values are equal. Of particular interest is the fact that they also directly give the asymptotic behaviour of the solution as $t \longrightarrow \pm\infty$ which is necessary information within the context of the original problem. Finally, we note that (12) and (13) can be obtained directly from the differential equation by assuming exponential series of this type about the stationary values $x = 0$ and $x = 2$ and solving the resulting nonlinear recurrence relations using generating functions.

The differential equation (10) is the lowest order equation of a family of nonlinear differential equations derivable from a general integro-differential equation. Each of these equations can also serve as a shock model, but are generally, with the noted exception in [5], not explicitly solvable. However, they do seem to have convergent exponential type series solutions which can be generated by knowledge of the asymptotic behaviours. The form is similar to (12) and (13) although the coefficients are not readily identifiable.

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