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Cohen–Macaulay Multi-Rees Algebras

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Abstract. Let A be a local ring, and let $I_1, \ldots, I_r \subset A$ be ideals of positive height. In this article we compare the Cohen–Macaulay property of the multi-Rees algebra $R_A(I_1, \ldots, I_r)$ to that of the usual Rees algebra $R_A(I_1 \cdots I_r)$ of the product $I_1 \cdots I_r$. In particular, when the analytic spread of $I_1 \cdots I_r$ is small, this leads to necessary and sufficient conditions for the Cohen–Macaulayness of $R_A(I_1, \ldots, I_r)$. We apply our results to the theory of joint reductions and mixed multiplicities.

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Key words. multi-Rees algebra, Cohen–Macaulay, joint reduction, mixed multiplicity.

1. Introduction

Let $(A, \mathfrak{m})$ be a local ring, and let $I_1, \ldots, I_r \subset A$ be ideals of positive height. The multi-Rees algebra $R_A(I_1, \ldots, I_r) = A[I_1t_1, \ldots, I_rt_r] \subset A[t_1, \ldots, t_r]$ where $t_1, \ldots, t_r$ are variables. Multi-Rees algebras arise in successive blowing-up, which is a fundamental process in birational geometry. The purpose of this work is to investigate their Cohen–Macaulay property. In particular, we want to link the Cohen–Macaulayness of $R_A(I_1, \ldots, I_r)$ to the theory of joint reductions developed by D. Rees. We recall here from [15] that given $\mathbf{q} \in \mathbb{N}^r$, a set $\{a_{i,j} \in I_i \mid i = 1, \ldots, r;\ j = 1, \ldots, q_i\}$ is called a joint reduction of $I_1, \ldots, I_r$ of type $\mathbf{q}$ if

$$I_1^{n_1} \cdots I_r^{n_r} = \sum_{i=1}^{r} (a_{i,1}, \ldots, a_{i,q_i}) I_1^{n_1} \cdots I_{i-1}^{n_{i-1}} I_i^{n_i - 1} I_{i+1}^{n_{i+1}} \cdots I_r^{n_r} \qquad (\dagger)$$

for all $n_1, \ldots, n_r \gg 0$. In the case $r = 1$, this coincides with the notion of reduction of an ideal which was introduced by D. Rees and D. G. Northcott in [23] and has played a central role in the study of usual Rees algebras.

In investigating the Cohen–Macaulay property of multi-Rees algebras one of the main difficulties arises from the lack of interplay between the Rees algebra and the form ring which has proved itself very useful in the case of the Rees algebra of a single ideal. Our method is to compare the properties of a multi-Rees algebra to those of the corresponding diagonal subring $A[I_1 \cdots I_r t_1 \cdots t_r]$, which is the Rees algebra $R_A(I_1 \cdots I_r)$ of the product ideal $I_1 \cdots I_r$. It is already known by [12, Corollary 2.10] that the Cohen–Macaulayness of $R_A(I_1, \ldots, I_r)$ implies that of

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$R_A(I_1 \cdots I_r)$. The converse does not hold in general. We are therefore trying to find out the additional conditions which would make $R_A(I_1, \ldots, I_r)$ Cohen–Macaulay in this situation.

Our starting point is Theorem 3.1 where the Cohen–Macaulay property of an r-graded ring is characterized in terms of sheaf cohomology. In the case $r = 1$ this was done by J. Lipman in [20, Theorem 4.1]. The case $r = 2$ is [12, Theorem 2.5]. We then focus on the multi-Rees algebra $R_A(I_1, \ldots, I_r)$. Theorem 3.1 now enables us to utilize the fact that the corresponding multiprojective scheme is isomorphic to the usual blow-up of Spec$A$ along the product $I_1 \cdots I_r$. Our main tool is Lemma 4.2, which is a multigraded variant of the Castelnuovo–Mumford lemma ([22, p. 99]). In fact, we observe in Lemma 4.2 that a similar statement also holds for sheaf cohomology with supports.

Recall that the analytic spread $\ell(I_1 \cdots I_r) = \delta + 1$ where δ is the dimension of the closed fiber of the blow-up. The vanishing of sheaf cohomology above δ makes the situation especially easy to manage when the analytic spread is small. In the case $I_1, \ldots, I_r \subset A$ are ideals of positive grade with $\ell(I_1 \cdots I_r) = 2$ it comes out in Theorem 4.1 that $R_A(I_1, \ldots, I_r)$ is Cohen–Macaulay if and only if $R_A(I_1 \cdots I_r)$ is Cohen–Macaulay and the condition $(I_{j_1} \cdots I_{j_k}) : I_{j_l} = I_{j_1} \cdots I_{j_{l-1}} I_{j_{l+1}} \cdots I_{j_k}$ holds for all $1 \leqslant j_1 < \cdots < j_k \leqslant r$ and $1 \leqslant l \leqslant k$. When A is Cohen–Macaulay and $I, J \subset A$ are ideals of positive height, we are able to treat the case $\ell(IJ) = 3$, too (see Theorem 4.2). In particular, we prove in Theorem 4.3 that if A is excellent of equicharacteristic zero, then $R_A(I, J)$ has rational singularities if and only if $R_A(I)$, $R_A(J)$ and $R_A(IJ)$ have rational singularities. If the analytic spread is higher, then the situation becomes more subtle. However, when A is Cohen–Macaulay and $I, J \subset A$ are $\mathfrak{m}$-primary ideals such that the reduction number $r(IJ) \leqslant 1$, we prove in Theorem 4.4 that $R_A(I, J)$ is Cohen–Macaulay if and only if $IJ : J = I$ and $IJ : I = J$.

Let $\{a_{i,j} \in I_i \mid i = 1, \ldots, r; j = 1, \ldots, q_i\}$ be a joint reduction of $I_1, \ldots, I_r$ of type $\mathbf{q}$ where $\mathbf{q} \in \mathbb{N}^r$ with $|\mathbf{q}| = \ell(I_1 \cdots I_r)$. It now turns out in Corollary 5.1 that if $R_A(I_1, \ldots, I_r)$ is Cohen–Macaulay, then the formula (†) already holds for all $n_1 \geqslant q_1, \ldots, n_r \geqslant q_r$. When $r = 1$, this reduces to the well-known result of Johnston and Katz in [14] saying that if $I \subset A$ is an ideal of positive height and analytic spread ℓ, then the Cohen–Macaulayness of the Rees algebra $R_A(I)$ implies that $I^\ell = JI^{\ell-1}$ for every ℓ-generated reduction $J \subset I$. Finally, we give in Theorem 6.1 a formula for mixed multiplicities which generalizes the one proved by J. Lipman in [19, Corollary 3.7] (see also [28, Corollary 3.3]).

2. Preliminaries

In this section we fix some notation and recall some basic facts about multigraded rings. First, we always assume that all rings and schemes are Noetherian. We also assume all schemes and morphisms to be separated. The norm of a multi-index $\mathbf{n} \in \mathbb{Z}^r$ is $|\mathbf{n}| = n_1 + \ldots + n_r$. If $\mathbf{n}, \mathbf{n}' \in \mathbb{Z}^r$ and $n_j < n'_j$ for $j = 1, \ldots, r$, we write

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$\mathbf{n} < \mathbf{n}'$. Let $\mathbf{1}_j = (0, \dots, 0, 1, 0, \dots, 0)$ $(j = 1, \dots, r)$ be the canonical base elements of $\mathbb{Z}^r$. Moreover, we set $\mathbf{1} = (1, \dots, 1)$.

Let $S = \bigoplus_{\mathbf{n}\in\mathbb{N}^r} S_{\mathbf{n}}$ be a standard r-graded ring defined over the ring $A = S_{\mathbf{0}}$. By the word 'standard' we mean that S is finitely generated over A by elements in degrees $\mathbf{1}_1, \dots, \mathbf{1}_r$. The diagonal subring of S is the graded ring $S^\Delta = \bigoplus_{n\in\mathbb{N}} S_{n,\dots,n}$. The irrelevant ideal of S is $S^+ = \bigoplus_{n_1,\dots,n_r>0} S_{\mathbf{n}}$. The r-projective scheme $\operatorname{Proj} S$ associated to S is defined analogously to the usual graded case by using r-homogeneous prime ideals $P \subset S$ which do not contain S^+. Set $Z = \operatorname{Proj} S$. Recall from [12, Lemma 1.2]) the formula

$$\dim Z = \max\{\dim S/P \mid P \in \operatorname{Min} S \cap \operatorname{Proj} S\} - r.$$

In particular, when S^+ has positive height, this gives $\dim Z = \dim S - r$.

The theory of multiprojective schemes is similar to the theory of projective schemes, which can be found in [6]. In fact, every multiprojective scheme is projective: if $Z^\Delta = \operatorname{Proj} S^\Delta$, then the inclusion $S^\Delta \longrightarrow S$ induces an isomorphism $f: Z \longrightarrow Z^\Delta$. When $\mathbf{n} \in \mathbb{Z}^r$, the invertible quasi-coherent sheaf corresponding to $S(\mathbf{n})$ is denoted by $\mathcal{O}_Z(\mathbf{n})$. Note that in the isomorphism $f: Z \longrightarrow Z^\Delta$ $f^*(\mathcal{O}_{Z^\Delta}(n)) = \mathcal{O}_Z(n, \dots, n)$ for all $n \in \mathbb{Z}$. As usual we have $\mathcal{O}_Z(\mathbf{m} + \mathbf{n}) = \mathcal{O}_Z(\mathbf{m}) \otimes \mathcal{O}_Z(\mathbf{n})$ for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r$.

We can look at the scheme Z also from another point of view. Let $j \in \{1, \dots, r\}$. Consider S as a usual graded ring by writing

$$S = \bigoplus_{k\geqslant 0}\Big(\bigoplus_{\mathbf{m}\in\mathbb{N}^{r-1}} S_{m_1,\dots,m_{j-1},k,m_j,\dots,m_{r-1}}\Big).$$

Let Y denote the $(r-1)$-projective scheme $\operatorname{Proj} S_0$. For every $k \in \mathbb{N}$, let $\mathcal{S}_k$ be the quasi-coherent $\mathcal{O}_Y$-module corresponding to the $(r-1)$-graded S_0-module S_k. Then $\mathcal{S} = \bigoplus_{k\geqslant 0} \mathcal{S}_k$ is a quasi-coherent graded $\mathcal{O}_Y$-algebra so that we have an associated projective scheme $\mathit{Proj}\,\mathcal{S}$. It is easily checked that it is possible to identify $\mathit{Proj}\,\mathcal{S}$ with Z. The corresponding canonical invertible sheaf on Z is $\mathcal{O}_Z(\mathbf{1}_j)$. Moreover, if $g: Z \longrightarrow Y$ is the canonical projection, we have

$$g^*(\mathcal{O}_Y(\mathbf{m})) = \mathcal{O}_Z(m_1, \dots, m_{j-1}, 0, m_j, \dots, m_{r-1})$$

for $\mathbf{m} \in \mathbb{N}^{r-1}$.

In the following we are mostly interested in the case $S = R_A(I_1, \dots, I_r)$ where $I_1, \dots, I_r \subset A$ are ideals of positive height. Using [2, Exercise 4.4.12] it is easy to see that $\dim R_A(I_1, \dots, I_r) = \dim A + r$. Observe that by the above construction $Z = \operatorname{Proj} R_A(I_1, \dots, I_r)$ can be identified with the blow-up of $Y = R_A(I_1, \dots, I_{j-1}, I_{j+1}, \dots, I_r)$ along the sheaf of ideals $I_j\mathcal{O}_Y$ $(j = 1, \dots, r)$.

3. A Criterion for Cohen–Macaulayness

The main result of this section is Theorem 3.1 which characterizes the Cohen–Macaulay property of a multigraded ring S in terms of the sheaf cohomology of

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the corresponding multiprojective scheme $\operatorname{Proj} S$. We first deal with the sheaf cohomology of the blow-up $\operatorname{Proj} R_S(S^+)$. Our arguments will be based on the observation that $\operatorname{Proj} R_S(S^+)$ can be considered as a vector bundle over $\operatorname{Proj} S$:

LEMMA 3.1. *Let S be a standard r-graded ring. Set $Z = \operatorname{Proj} S$. Then*

$$\operatorname{Proj} R_S(S^+) = \mathbb{V}(\mathcal{O}_Z(\mathbf{1}_1) \oplus \cdots \oplus \mathcal{O}_Z(\mathbf{1}_r)).$$

Proof. Set $T = R_S(S^+)$ and $W = \operatorname{Proj} T$. Write $T = S[S^+t]$ where t is a variable. Cover W with open affine sets $D_+(st) = \operatorname{Spec} T_{(st)}$ where $s \in S_{\mathbf{1}}$. The elements of $T_{(st)}$ are quotients $f/(st)^k$ where $k \in \mathbb{N}$ and $f \in T_k$. As $T_k = \bigoplus_{\mathbf{n} \geqslant \mathbf{0}} S_{n_1+k,\ldots,n_r+k} t^k$, this implies that $T_{(st)} = \bigoplus_{\mathbf{n} \geqslant \mathbf{0}} (S(\mathbf{n}))_{(s)}$. But then

$$W = \mathit{Spec} \bigoplus_{\mathbf{n} \geqslant \mathbf{0}} \mathcal{O}_Z(\mathbf{n}) = \mathit{Spec}\, \operatorname{Sym}(\mathcal{O}_Z(\mathbf{1}_1) \oplus \cdots \oplus \mathcal{O}_Z(\mathbf{1}_r)).$$

Let S be a standard r-graded ring defined over a local ring $(A, \mathfrak{m})$. Set $Z = \operatorname{Proj} S$. We begin with some generalities about the sheaf cohomology of $W = \operatorname{Proj} T$ where $T = R_S(S^+)$. In particular, we want to indicate that sheaf cohomology is in fact r-graded. Observe first that $T = \bigoplus_{\mathbf{n} \geqslant \mathbf{0}, k \geqslant 0} S_{n_1+k,\ldots,n_r+k}$ can be considered as a standard $(r+1)$-graded ring. Note that the ring S^Δ coincides with the subring $\bigoplus_{k \geqslant 0} T_{\mathbf{0};k}$. Let M be any $(r+1)$-graded T-module. By writing $M = \bigoplus_{k \in \mathbb{Z}} M_{\bullet;k}$ where $M_{\bullet;k} = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n};k}$, we can consider M as a graded T-module. We then have the associated sheaf $\widetilde{M}$ on W. Let $\pi\colon W \longrightarrow Z$ be the canonical morphism. One checks that $\pi_*\widetilde{M} = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} \widetilde{M_{\mathbf{n};\bullet}}$ where $M_{\mathbf{n};\bullet}$ is the graded S^Δ-module $\bigoplus_{k \in \mathbb{Z}} M_{\mathbf{n};k}$. The module $\Gamma(W, \widetilde{M}) = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} \Gamma(Z, \widetilde{M_{\mathbf{n};\bullet}})$ has a natural structure of an r-graded S-module. We then see that $\Gamma(W, \tilde{-})$ defines a functor from the category of $(r+1)$-graded T-modules to the category of r-graded S-modules. Let $\mathfrak{A} \subset S$ be a homogeneous ideal. Set $F = W \times_S S/\mathfrak{A}$. Let $0 \longrightarrow M \longrightarrow I^\bullet$ be an $(r+1)$-graded injective resolution. Take $j \geqslant 0$. Let T^+ denote the usual irrelevant ideal of T. Now look at the Sancho de Salas sequence ([20, p. 150], see also [12, Theorem 1.4])

$$\cdots \longrightarrow H^i_{\mathfrak{A}}(I^j_{\bullet;0}) \longrightarrow H^i_F(W, \widetilde{I^j}) \longrightarrow [H^{i+1}_{(\mathfrak{A},T^+)}(I^j)]_0 \longrightarrow \cdots \quad .$$

Noting that $0 = H^i_{\mathfrak{A}T}(I^j) = \bigoplus_{k \in \mathbb{Z}} H^i_{\mathfrak{A}}(I^j_{\bullet;k})$, this shows that $H^i_F(W, \widetilde{I^j}) = 0$ when $i > 0$. We thus have a $\Gamma_F(W, -)$-acyclic resolution $0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{I^\bullet}$ which we can use to calculate the modules $H^i_F(W, \widetilde{M})$. Since $\Gamma_F(W, -) = H^0_{\mathfrak{A}}\Gamma(W, -)$, these are indeed r-graded S-modules.

Take in particular $\mathfrak{A} = \mathfrak{a}S$ where $\mathfrak{a} \subset A$ is an ideal. Set $E = Z \times_A A/\mathfrak{a}$. The morphism π being affine, we obtain $H^i_E(Z, \pi_*\widetilde{I^j}) = H^i_F(W, \widetilde{I^j}) = 0$ for all $j \geqslant 0$ when $i > 0$. This implies that $0 \longrightarrow \pi_*\widetilde{M} \longrightarrow \pi_*\widetilde{I^\bullet}$ is a $\Gamma_E(Z, -)$-acyclic resolution of $\pi_*\widetilde{M}$ in the category of graded $\mathcal{O}_Z$-modules. By means of this resolution one now verifies that as a graded S-module $H^i_F(W, \widetilde{M}) = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} H^i_E(Z, \widetilde{M_{\mathbf{n}}})$ for all $i \geqslant 0$.

Our main interest concerns the sheaf cohomology with supports in $F = W \times_S S/\mathfrak{M}$ where $\mathfrak{M}$ denotes the homogeneous maximal ideal of S. In calculating

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the modules $H^i_F(W, \mathcal{O}_W)$ our strategy is to determine first the local cohomology sheaves $\mathcal{H}^i_G(\mathcal{O}_W)$ with supports in $G = W \times_S S/S^*$ where $S^* = \bigoplus_{\mathbf{n}\neq\mathbf{0}} S_{\mathbf{n}}$ (for basic facts about local cohomology sheaves we refer to [5, §1.]).

LEMMA 3.2. *Let S be a standard r-graded ring. Set $T = R_S(S^+)$ and $W = \operatorname{Proj} T$. Also set $G = W \times_S S/S^*$ where $S^* = \bigoplus_{\mathbf{n}\neq\mathbf{0}} S_{\mathbf{n}}$. Then $\mathcal{H}^i_G(\mathcal{O}_W) = \big(H^i_{S^*T}(T)\big)^{\sim}$. Moreover, if $Z = \operatorname{Proj} S$ and $\pi\colon W \longrightarrow Z$ is the canonical morphism, we have*

$$\pi_*(\mathcal{H}^i_G(\mathcal{O}_W)) = \begin{cases} 0 & \text{if } i \neq r, \\ \bigoplus_{\mathbf{n}<\mathbf{0}} \mathcal{O}_Z(\mathbf{n}) & \text{if } i = r \end{cases}$$

as graded $\mathcal{O}_Z$-modules.

Proof. For any affine open set $D_+(f) \subset W$ where $f \in T$ is a homogeneous element, we have by definition

$$\mathcal{H}^i_G(\mathcal{O}_W)_{|D_+(f)} = \big(H^i_{S^*T_{(f)}}(T_{(f)})\big)^{\sim} = \big(\big(H^i_{S^*T}(T)\big)_{(f)}\big)^{\sim}.$$

This proves the first claim. In order to prove the second one, we cover Z with open affine sets $U = \operatorname{Spec} S_{(s_1\cdots s_r)}$ where $s_j \in S_{\mathbf{1}_j}$ $(j = 1, \dots, r)$. By the construction given in the proof of Lemma 3.1 $\pi^{-1}(U) = \operatorname{Spec} T_{(s_1\cdots s_r t)}$. Set $B = S_{(s_1\cdots s_r)}$ and $t_j = s_j/1 \in T_{(s_1\cdots s_r t)}$ $(j = 1, \dots, r)$. We observe that $T_{(s_1\cdots s_r t)} = B[t_1, \dots, t_r]$ is a polynomial ring. We also get $G \cap \pi^{-1}(U) = V(t_1, \dots, t_r)$. Therefore

$$\mathcal{H}^i_G(\mathcal{O}_W)_{|\pi^{-1}(U)} = H^i_{(t_1,\dots,t_r)}(B[t_1, \dots, t_r])^{\sim}.$$

The claim is then a consequence of [7, Proposition 2.1.12] according to which

$$H^i_{(t_1,\dots,t_r)}(B[t_1, \dots, t_r]) = \begin{cases} 0 & \text{if } i \neq r, \\ \bigoplus_{\mathbf{n}<\mathbf{0}} Bt_1^{n_1} \cdots t_r^{n_r} & \text{if } i = r. \end{cases}$$

PROPOSITION 3.1. *Let S be a standard r-graded ring defined over a local ring $(A, \mathfrak{m})$. Set $Z = \operatorname{Proj} S$ and $W = \operatorname{Proj} R_S(S^+)$. Also set $E = Z \times_A A/\mathfrak{m}$ and $F = W \times_S S/\mathfrak{M}$ where $\mathfrak{M}$ is the homogeneous maximal ideal of S. Then as a graded S-module $H^i_F(W, \mathcal{O}_W) = \bigoplus_{\mathbf{n}<\mathbf{0}} H^{i-r}_E(Z, \mathcal{O}_Z(\mathbf{n}))$ for all $i \geqslant 0$.*

Proof. Set $G = W \times_S S/S^*$ where $S^* = \bigoplus_{\mathbf{n}\neq\mathbf{0}} S_{\mathbf{n}}$. We consider the functor $\Gamma_F(W, \simeq)$ from the category of $(r+1)$-graded $R_S(S^+)$-modules to the category of r-graded S-modules. Since $\mathfrak{M} = \mathfrak{m} \oplus S^*$, it equals to the composite $\Gamma_{\pi^{-1}(E)}(W, \mathcal{H}^0_G(\simeq))$. It follows that there is a spectral sequence

$$E_2^{p,q} = H^p_{\pi^{-1}(E)}(W, \mathcal{H}^q_G(\mathcal{O}_W)) \Rightarrow H^{p+q}_F(W, \mathcal{O}_W).$$

On the other hand, by the remarks we have made earlier

$$H^p_{\pi^{-1}(E)}(W, \mathcal{H}^q_G(\mathcal{O}_W)) = H^p_E(Z, \pi_*(\mathcal{H}^q_G(\mathcal{O}_W)))$$

as graded S-modules. It now follows from Lemma 3.2 that the above spectral sequence degenerates to give the claim.

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It is important to observe that certain homogeneous components of the local cohomology of $R_S(S^+)$ always vanish.

LEMMA 3.3. *Let S be a standard r-graded ring defined over a local ring. Set $T = R_S(S^+)$. Let $\mathfrak{N}$ denote the homogeneous maximal ideal of T. Considering T as an $(r+1)$-graded ring, we have $[H^i_{\mathfrak{N}}(T)]_{\mathbf{n};k} = 0$ for all $i \geqslant 0$ and $\mathbf{n} < \mathbf{0}$, $k \geqslant 0$.*

Proof. Let Q denote the irrelevant ideal of T when T is considered as an $(r+1)$-graded ring. We have the exact sequences

$$0 \longrightarrow Q \longrightarrow T \longrightarrow T/Q \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathrm{T}^+ \longrightarrow \mathrm{T} \longrightarrow \mathrm{T}/\mathrm{T}^+ \longrightarrow 0.$$

The corresponding long exact sequences of cohomology give the exact sequences

$$H^{i-1}_{\mathfrak{N}}(T/Q) \longrightarrow H^i_{\mathfrak{N}}(Q) \longrightarrow H^i_{\mathfrak{N}}(T) \longrightarrow H^i_{\mathfrak{N}}(T/Q)$$

and

$$H^{i-1}_{\mathfrak{N}}(T/T^+) \longrightarrow H^i_{\mathfrak{N}}(T^+) \longrightarrow H^i_{\mathfrak{N}}(T) \longrightarrow H^i_{\mathfrak{N}}(T/T^+).$$

Note that $[H^i_{\mathfrak{N}}(T/Q)]_{\mathbf{n};k} = 0$ if $n_j \neq 0$ for all $j = 1, \ldots, r$. We also have $[H^i_{\mathfrak{N}}(T/T^+)]_{\mathbf{n};k} = 0$ if $k \neq 0$. As $T = \bigoplus_{\mathbf{n} \geqslant \mathbf{0}, k \geqslant 0} S_{n_1+k,\ldots,n_r+k}$, we observe that there is an obvious isomorphism $Q \longrightarrow T^+(-\mathbf{1}; 1)$ which maps an element in $T_{\mathbf{n};k}$ to the corresponding element of $T_{\mathbf{n}-\mathbf{1};k+1}$. For any $k \geqslant 0$ and $\mathbf{n} < \mathbf{0}$, we thus obtain the isomorphisms

$$[H^i_{\mathfrak{N}}(T)]_{\mathbf{n};k} = [H^i_{\mathfrak{N}}(Q)]_{\mathbf{n};k} = [H^i_{\mathfrak{N}}(T^+)]_{\mathbf{n}-\mathbf{1};k+1} = [H^i_{\mathfrak{N}}(T)]_{\mathbf{n}-\mathbf{1};k+1}.$$

Since $[H^i_{\mathfrak{N}}(T)]_{\mathbf{n};k} = 0$ for $k \gg 0$, the claim follows.

In the sequel we shall frequently utilize the interplay between the vanishing of local cohomology, sheaf cohomology and sheaf cohomology with supports. The following proposition ([13, Lemma 1.1] which is a version of [20, Lemma 4.2]) is therefore crucial for our arguments.

PROPOSITION 3.2. *Let R be a standard graded ring defined over a local ring $(A, \mathfrak{m})$. Set $X = \operatorname{Proj} R$ and $E = X \times_A A/\mathfrak{m}$. Let $\mathfrak{M}$ be the homogeneous maximal ideal of R. Let M be a graded R-module. Let $n \in \mathbb{Z}$. Then the following conditions are equivalent:*

(1) $[H^i_{\mathfrak{M}}(M)]_n = 0$ *for all* $i \geqslant 0$;
(2) *The canonical homomorphism* $H^i_{\mathfrak{m}}(M_n) \longrightarrow H^i_E(X, \widetilde{M}(n))$ *is an isomorphism for all* $i \geqslant 0$;
(3) *The canonical homomorphism* $M_n \longrightarrow \Gamma(X, \widetilde{M}(n))$ *is an isomorphism and* $H^i(X, \widetilde{M}(n)) = 0$ *for* $i > 0$;
(4) $[H^i_{S^+}(M)]_n = 0$ *for all* $i \geqslant 0$.

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Let S be an $\mathbb{N}^r$-graded ring defined over a local ring, and let $\mathfrak{M}$ be the homogeneous maximal ideal of S. Define for all $j = 1, \ldots, r$ the *a-invariants*

$$a^j(S) = \sup\{k \in \mathbb{Z} \mid [H^{\dim S}_{\mathfrak{M}}(S)]_{\mathbf{n}} \neq 0 \text{ for some } \mathbf{n} \in \mathbb{Z}^r \text{ with } n_j = k\}.$$

Moreover, we set $\mathbf{a}(S) = (a^1(S), \ldots, a^r(S))$.

We are now ready to prove the following theorem:

THEOREM 3.1. *Let S be a standard r-graded ring defined over a local ring $(A, \mathfrak{m})$ such that S^+ has positive height. Set $Z = \operatorname{Proj} S$ and $E = Z \times_A A/\mathfrak{m}$. Then S is Cohen–Macaulay with $\mathbf{a}(S) < \mathbf{0}$ if and only if the following conditions are satisfied*

(1) $\Gamma(Z, \mathcal{O}_Z(\mathbf{n})) = S_{\mathbf{n}}$ *for all* $\mathbf{n} \geqslant \mathbf{0}$*;*
(2) $H^i(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ *for all* $i > 0$ *and* $\mathbf{n} \geqslant \mathbf{0}$*;*
(3) $H^i_E(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ *for all* $i < \dim Z$ *and* $\mathbf{n} < \mathbf{0}$.

If this is the case, then also S^Δ is Cohen–Macaulay with $a(S^\Delta) < 0$.

Proof. Set $T = R_S(S^+)$ and $W = \operatorname{Proj} T$. Let $\mathfrak{N}$ denote the homogeneous maximal ideal of T. Also set $F = W \times S/\mathfrak{M}$ where $\mathfrak{M}$ is the homogeneous maximal ideal of S. One now checks that the Sancho de Salas sequence

$$\cdots \longrightarrow [H^i_{\mathfrak{N}}(T)]_0 \longrightarrow H^i_{\mathfrak{M}}(S) \longrightarrow H^i_F(W, \mathcal{O}_W) \longrightarrow \cdots$$

is r-graded (cf. [20, p. 150] or [12, the proof of Theorem 1.4]). We saw in Lemma 3.3 that $[H^i_{\mathfrak{N}}(T)]_{\mathbf{n};0} = 0$ for $\mathbf{n} < \mathbf{0}$. Using Proposition 3.1, it therefore follows that there is an isomorphism $[H^i_{\mathfrak{M}}(S)]_{\mathbf{n}} = H^{i-r}_E(Z, \mathcal{O}_Z(\mathbf{n}))$. Noting that $\dim S = \dim Z + r$, we thus see that $[H^i_{\mathfrak{M}}(S)]_{\mathbf{n}} = 0$ for $i < \dim S$ and $\mathbf{n} < \mathbf{0}$ if and only if (3) holds. On the other hand, the Sancho de Salas sequence also implies that

$$[H^i_{\mathfrak{N}}(T)]_0 = \bigoplus_{\text{some } n_j \geqslant 0} [H^i_{\mathfrak{M}}(S)]_{\mathbf{n}}.$$

Therefore $[H^i_{\mathfrak{M}}(S)]_{\mathbf{n}} = 0$ for all $i \geqslant 0$ and $\mathbf{n} \in \mathbb{Z}^r$ such that $n_j \geqslant 0$ for some j if and only if $[H^i_{\mathfrak{N}}(T)]_0 = 0$ for all $i \geqslant 0$. But according to Proposition 3.2 this is equivalent to having $\Gamma(W, \mathcal{O}_W)_{\mathfrak{M}} = S_{\mathfrak{M}}$ and $H^i(W, \mathcal{O}_W)_{\mathfrak{M}} = 0$ for $i > 0$. Because $\Gamma(W, \mathcal{O}_W)$ and $H^i(W, \mathcal{O}_W)$ are r-graded S-modules, this is the same as $\Gamma(W, \mathcal{O}_W) = S$ and $H^i(W, \mathcal{O}_W) = 0$ for $i > 0$. Since $\Gamma(W, \mathcal{O}_W) = \bigoplus_{\mathbf{n} \geqslant 0} \Gamma(Z, O_Z(\mathbf{n}))$ and $H^i(W, \mathcal{O}_W) = \bigoplus_{\mathbf{n} \geqslant 0} H^i(Z, O_Z(\mathbf{n}))$, the proof is now complete.

The last statement is proved by utilizing the isomorphism $\operatorname{Proj} S^\Delta \cong Z$ and noting that $\mathcal{O}_{Z^\Delta}(n)$ then corresponds to $\mathcal{O}_Z(n, \ldots, n)$ for all $n \in \mathbb{Z}$.

Remark 3.1. Let A be a local ring, and let $I_1, \ldots, I_r \subset A$ be ideals of positive height. Recall from [9, Lemma 2.1] that $\mathbf{a}(R_A(I_1, \ldots, I_r)) = -\mathbf{1}$. We can therefore apply Theorem 3.1 in the case $S = R_A(I_1, \ldots, I_r)$. In particular, the Cohen–Macaulayness of $R_A(I_1, \ldots, I_r)$ implies that of $R_A(I_1 \cdots I_r)$. This recovers [12, Corollary 2.10].

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Let S be a standard r-graded ring. Let $j \in \{1, \ldots, r\}$. Let S_0 denote the $(r-1)$-graded subring $\bigoplus_{\mathbf{m}\in\mathbb{N}^{r-1}} S_{m_1,\ldots,m_{j-1},0,m_j,\ldots,m_{r-1}}$. For every $k \in \mathbb{N}$, we then get an $(r-1)$-graded S_0-module $S_k = \bigoplus_{\mathbf{m}\in\mathbb{N}^{r-1}} S_{m_1,\ldots,m_{j-1},k,m_j,\ldots,m_{r-1}}$.

The Cohen–Macaulay property of S does not in general imply that of S_0. We will return to this question later in Theorem 4.5 when S is a multi-Rees algebra (for a counterexample in this case, see, e.g., [8, Example 3.11]). We mention here only the following general fact:

PROPOSITION 3.3. *Set* $Z = \operatorname{Proj} S$ *and* $Y = \operatorname{Proj} S_0$. *For every* $k \in \mathbb{N}$, *let* $\mathcal{S}_k$ *denote the* $\mathcal{O}_Y$*-module associated to* S_k. *Suppose that*

(1) $\Gamma(Z, \mathcal{O}_Z(\mathbf{n})) = S_{\mathbf{n}}$ *for all* $\mathbf{n} \geqslant \mathbf{0}$;
(2) $H^i(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ *for all* $i > 0$ *and* $\mathbf{n} \geqslant \mathbf{0}$.

Then

(a) $\Gamma(Y, \mathcal{S}_k(\mathbf{m})) = S_{m_1,\ldots,m_{j-1},k,m_j,\ldots,m_{r-1}}$ for all $k \in \mathbb{N}$, $\mathbf{m} \geqslant \mathbf{0}$;
(b) $H^i(Y, \mathcal{S}_k(\mathbf{m})) = 0$ for all $i > 0$ and $k \in \mathbb{N}$, $\mathbf{m} \geqslant \mathbf{0}$.

In particular, this means that

(a′) $\Gamma(Y, \mathcal{O}_Y(\mathbf{m})) = S_{m_1,\ldots,m_{j-1},0,m_j,\ldots,m_{r-1}}$ for all $k \in \mathbb{N}$, $\mathbf{m} \geqslant \mathbf{0}$;
(b′) $H^i(Y, \mathcal{O}_Y(\mathbf{m})) = 0$ for all $i > 0$ and $\mathbf{m} \geqslant \mathbf{0}$.

Proof. The claim is an immediate consequence of Lemma 3.4 below.

LEMMA 3.4. *Use the preceding notation. Let* $\mathbf{n} \in \mathbb{N}^r$. *Suppose that there exists an* $N \geqslant 0$ *such that* $\Gamma(Z, \mathcal{O}_Z(\mathbf{n}')) = S_{\mathbf{n}'}$ *and* $H^i(Z, \mathcal{O}_Z(\mathbf{n}')) = 0$ *for all* $i > 0$ *when* $n'_j = n_j$ *and* $n'_1, \ldots, n'_{j-1}, n'_{j+1}, \ldots, n'_r \geqslant N$. *Then*

$$H^i(Z, \mathcal{O}_Z(\mathbf{n})) = H^i(Y, \mathcal{S}_{n_j}(n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_r))$$

for all $i \geqslant 0$.

Proof. Let $g\colon Z \longrightarrow Y$ denote the canonical projection. The claim will follow from the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q g_*(\mathcal{O}_Z(\mathbf{n}))) \Rightarrow H^{p+q}(Z, \mathcal{O}_Z(\mathbf{n}))$$

as soon as we show that

$$g_*(\mathcal{O}_Z(\mathbf{n})) = \mathcal{S}_{n_j}(n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_r))$$

and $R^q g_*(\mathcal{O}_Z(\mathbf{n})) = 0$ for all $q > 0$. For the first statement, it is enough to prove that

$$\Gamma(Y, (g_*(\mathcal{O}_Z(\mathbf{n})))(\mathbf{m})) = \Gamma(Y, (\mathcal{S}_{n_j}(n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_r))(\mathbf{m}))$$

for all $\mathbf{m} \in \mathbb{N}^{r-1}$ with $\mathbf{m} \gg \mathbf{0}$. Indeed, using the assumption and the r-graded version

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of the theorem of Serre (see [16, Lemma 4.2]), we obtain for $\mathbf{m} \gg \mathbf{0}$

$$\begin{aligned}
&\Gamma(Y, g_*(\mathcal{O}_Z(\mathbf{n})) \otimes \mathcal{O}_Y(\mathbf{m}))\\
&\quad = \Gamma(Z, \mathcal{O}_Z(\mathbf{n}) \otimes g^*(\mathcal{O}_Y(\mathbf{m})))\\
&\quad = \Gamma(Z, \mathcal{O}_Z(\mathbf{n}) \otimes \mathcal{O}_Z(m_1, \dots, m_{j-1}, 0, m_j, \dots, m_{r-1}))\\
&\quad = \Gamma(Z, \mathcal{O}_Z(n_1 + m_1, \dots, n_{j-1} + m_{j-1}, n_j, n_{j+1} + m_j, \dots, n_r + m_{r-1}))\\
&\quad = S_{n_1+m_1, \dots, n_{j-1}+m_{j-1}, n_j, n_{j+1}+m_j, \dots, n_r+m_{r-1}}\\
&\quad = \Gamma(Y, (\mathcal{S}_{n_j}(n_1, \dots, n_{j-1}, n_{j+1}, \dots, n_r))(\mathbf{m})).
\end{aligned}$$

The second statement is a consequence of [12, Lemma 2.1].

4. Cohen–Macaulayness and Small Analytic Spread

Let A be a local ring, and let $I_1, \dots, I_r \subset A$ be ideals of positive height. We saw in Remark 3.1 that the Cohen–Macaulayness of $S = R_A(I_1, \dots, I_r)$ implies that of $S^\Delta = R_A(I_1 \cdots I_r)$. The converse implication does not hold in general (see [2, Example 2.11]). Our purpose is to find out what additional conditions are needed for the Cohen–Macaulay property of $R_A(I_1, \dots, I_r)$ in this situation. Set $Z = \operatorname{Proj} S$ and $Z^\Delta = \operatorname{Proj} S^\Delta$. In light of Theorem 3.1, one key point is to understand how the vanishing of the cohomology modules $H^i(Z^\Delta, \mathcal{O}_{Z^\Delta}(n)) = H^i(Z, \mathcal{O}_Z(n, \dots, n))$ $(n \in \mathbb{N})$ affects the vanishing of the modules $H^i(Z, \mathcal{O}_Z(\mathbf{n}))$ $(\mathbf{n} \in \mathbb{N}^r)$. Our main Lemma 4.2 is an r-graded variant of the Castelnuovo–Mumford lemma. It is a based on the following.

LEMMA 4.1. *Let Y be a scheme, and let $\mathcal{L}$ be an invertible sheaf on Y generated by finitely many global sections. Let $\mathcal{F}$ be a coherent sheaf on Y. Let $m \in \mathbb{Z}$ and $p \in \mathbb{N}$. Then the following holds:*

(a) *If $H^i(Y, \mathcal{F} \otimes \mathcal{L}^{m-i}) = 0$ for all $i > p$, then $H^i(Y, \mathcal{F} \otimes \mathcal{L}^{n-i}) = 0$ for all $i > p$ and $n \geqslant m$. Moreover, if $p = 0$ and $s_1, \dots, s_\ell \in \Gamma(Y, \mathcal{L})$ generate $\mathcal{L}$, then the induced homomorphism $\Gamma(Y, \mathcal{F} \otimes \mathcal{L}^n)^{\oplus \ell} \longrightarrow \Gamma(Y, \mathcal{F} \otimes \mathcal{L}^{n+1})$ is surjective for $n \geqslant m$.*
(b) *Let $E \subset Y$ be a closed subset. If $H^i_E(Y, \mathcal{F} \otimes \mathcal{L}^{m-i}) = 0$ for all $i < p$, then $H^i_E(Y, \mathcal{F} \otimes \mathcal{L}^{n-i}) = 0$ for all $i < p$ and $n \leqslant m$.*

The proof of (a) is similar to that of [11, Lemma 2.6] and [21, Lemma 5.1]. Therefore we prove here only (b). Let us use descending induction on n. If $n = m$, then there is nothing to prove. Suppose $n < m$. Since $\mathcal{L}$ is generated by global sections, there is for some $\ell > 0$ an epimorphism $\mathcal{O}_Y^{\oplus \ell} \longrightarrow \mathcal{L}$ which further gives an epimorphism $\sigma\colon (\mathcal{L}^{-1})^{\oplus \ell} \longrightarrow \mathcal{O}_Y$. The Koszul cocomplex $\mathcal{F} \otimes K^\bullet(\sigma^\vee)$ corresponding to the dual morphism $\sigma^\vee\colon \mathcal{O}_Y \longrightarrow \mathcal{L}^{\oplus \ell}$ is then exact. Note that

$$\mathcal{F} \otimes K^j(\sigma^\vee) = \mathcal{F} \otimes \textstyle\bigwedge^j \mathcal{L}^{\oplus \ell} = \mathcal{F} \otimes (\mathcal{L}^j)^{\oplus \binom{\ell}{j}} \quad (j = 0, \dots, \ell).$$

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By tensoring $\mathcal{F} \otimes K^\bullet(\sigma^\vee)$ with $\mathcal{L}^{n-i}$ we obtain an exact sequence

$$0 \longrightarrow \mathcal{L}_0 \longrightarrow \cdots \longrightarrow \mathcal{L}_j \longrightarrow \mathcal{L}_{j+1} \longrightarrow \cdots \longrightarrow \mathcal{L}_\ell \longrightarrow 0$$

where

$$\mathcal{L}_j = \mathcal{F} \otimes (\mathcal{L}^{n-i+j})^{\oplus\binom{\ell}{j}} \quad (j = 0, \ldots, \ell).$$

This sequence gives rise to exact sequences

$$0 \longrightarrow \mathcal{K}_{j-1} \longrightarrow \mathcal{L}_j \longrightarrow \mathcal{K}_j \longrightarrow 0 \quad (j = 1, \ldots, \ell - 1)$$

where $\mathcal{K}_0 = \mathcal{F} \otimes \mathcal{L}^{n-i}$ and $\mathcal{K}_{\ell-1} = \mathcal{F} \otimes \mathcal{L}^{n-i+\ell}$. By the induction hypothesis we now have $H_E^{i-j}(Y, \mathcal{F} \otimes \mathcal{L}^{n+1-i+j}) = 0$ for all $j \geqslant 0$. By looking at the corresponding long exact sequences of cohomology, we obtain for every $j = 0, \ldots, \ell - 2$ an epimorphism $H_E^{i-j-1}(Y, \mathcal{K}_{j+1}) \longrightarrow H_E^{i-j}(Y, \mathcal{K}_j)$. But using the induction hypothesis again, we get $H_E^{i-\ell+1}(Y, \mathcal{K}_{\ell-1}) = 0$. Therefore $H_E^{i-j}(Y, \mathcal{K}_j) = 0$ for all $j = 0, \ldots, \ell - 1$. In particular, $H_E^i(Y, \mathcal{F} \otimes \mathcal{L}^{n-i}) = 0$ as wanted.

LEMMA 4.2. *Let S be a standard graded ring defined over the ring $A = S_{\mathbf{0}}$. Let $\mathfrak{a} \subset A$ be an ideal. Set $Z = \operatorname{Proj} S$ and $E = Z \times_A A/\mathfrak{a}$. Let M be an r-graded S-module. Set $\mathcal{F} = \widetilde{M}$. Let $\mathbf{m} \in \mathbb{Z}^r$ and $p \in \mathbb{N}$.*

(a) *Suppose that $H^i(Z, \mathcal{F}(m_1 - i, \ldots, m_r - i)) = 0$ for all $i > p$. Then*

$$H^i(Z, \mathcal{F}(n_1 - i, \ldots, n_r - i)) = 0$$

for all $i > p$ and $\mathbf{n} \geqslant \mathbf{m}$. Moreover, if $p = 0$ and $s_1, \ldots, s_\ell \in \Gamma(Z, \mathcal{O}_Z(\mathbf{1}_j))$ generate $\mathcal{O}_Z(\mathbf{1}_j)$, then the induced homomorphism

$$\Gamma(Z, \mathcal{F}(\mathbf{n}))^{\oplus\ell} \longrightarrow \Gamma(Z, \mathcal{F}(\mathbf{n} + \mathbf{1}_j))$$

is surjective for $\mathbf{n} \geqslant (m_1 - 1, \ldots, m_{j-1} - 1, m_j, m_{j+1} - 1, \ldots, m_r - 1)$ and $j = 1, \ldots, r$.

(b) *Suppose that $H_E^i(Z, \mathcal{F}(m_1 - i, \ldots, m_r - i)) = 0$ for all $i < p$. Then*

$$H_E^i(Z, \mathcal{F}(n_1 - i, \ldots, n_r - i)) = 0$$

for all $i < p$ and $\mathbf{n} \leqslant \mathbf{m}$.

Proof. Let us prove the first statement of (a) by descending induction on i. Everything being clear for large i, suppose $H^j(Z, \mathcal{F}(n_1 - j, \ldots, n_r - j)) = 0$ for $j > i$ and $\mathbf{n} \geqslant \mathbf{m}$. Let us show by induction on $k = 0, \ldots, r$ that

$$H^i(Z, \mathcal{F}(n_1 - i, \ldots, n_k - i, m_{k+1} - i, \ldots, m_r - i)) = 0$$

when $n_j \geqslant m_j$ for $j = 1, \ldots, k$. Since $H^i(Z, \mathcal{F}(m_1 - i, \ldots, m_r - i)) = 0$, the case $k = 0$ is clear. Let $k > 0$. By the original induction hypothesis we know that

$$H^j(Z, \mathcal{F}(n_1 - i, \ldots, n_{k-1} - i, m_k - j, m_{k+1} - i, \ldots, m_r - i)) = 0.$$

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for $j > i$. As this now also holds for $j = i$, the first statement of Lemma 4.1(a) gives

$$H^i(Z, \mathcal{F}(n_1 - i, \dots, n_{k-1} - i, n_k - i, m_{k+1} - i, \dots, m_r - i)) = 0$$

for all $n_k \geqslant m_k$ as wanted. In particular, given $j \in \{1, \dots, r\}$, we have

$$H^i(Z, \mathcal{F}(n_1, \dots, n_{j-1}, m_j - i, n_{j+1}, \dots, n_r)) = 0$$

for all $i > 0$ and $n_k \geqslant m_k - 1$ ($k \neq j$). By the second statement of Lemma 4.1(a) this implies the remaining statement of (a). The proof of (b) is done by induction on i. As it is analogous to the proof of (a), we omit it.

Remark 4.1. Set $Z^\Delta = \operatorname{Proj} S^\Delta$ and $\mathcal{F}^\Delta = \widetilde{M^\Delta}$ where $M^\Delta = \bigoplus_{n \in \mathbb{Z}} M_{n,\dots,n}$. Recall that the Castelnuovo–Mumford regularity of $\mathcal{F}^\Delta$ is by definition the smallest integer $m \in \mathbb{Z}$ such that

$$H^i(Z^\Delta, \mathcal{F}^\Delta(m - i)) = H^i(Z, \mathcal{F}(m - i, \dots, m - i)) = 0$$

for all $i > 0$.

We now return to consider the case where S is a multi-Rees algebra. We would like to find an integer $m \in \mathbb{Z}$ such that $\mathbf{m} = (m, \dots, m) \in \mathbb{Z}^r$ in Lemma 4.2 is optimal for our purposes. Therefore, we are next going to investigate the vanishing of the sheaf cohomology of the projective scheme associated to a Cohen–Macaulay Rees algebra of a single ideal.

Let $(A, \mathfrak{m})$ be a local ring, and let $I \subset A$ be an ideal of positive height. Set $X = \operatorname{Proj} R_A(I)$. As the closed fiber of the canonical projection $X \longrightarrow \operatorname{Spec} A$ has dimension $\ell(I) - 1$, it is well known from [7, Corollaire (4.2.2)] that $H^i(X, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on X if $i \geqslant \ell$. When the ideal has small analytic spread, we thus see that most of the sheaf cohomology vanishes. The following lemma shows that the same also holds for sheaf cohomology with supports in the closed fiber.

LEMMA 4.3. *Let $(A, \mathfrak{m})$ be a local ring of dimension d, and let $I \subset A$ be an ideal of positive height. Set $X = \operatorname{Proj} R_A(I)$ and $E = X \times_A A/\mathfrak{m}$. Let $\mathcal{L}$ be an invertible sheaf on X. If X is Cohen–Macaulay, then $H^i_E(X, \mathcal{L}) = 0$ for all $i \leqslant d - \ell(I)$.*

Proof. We may assume that A is complete. By the local–global duality of Lipman ([18, p. 188]) we then have

$$H^i_E(X, \mathcal{L}) = \operatorname{Hom}_A(H^{d-i}(X, \omega_X \otimes \mathcal{L}^{-1}), E_A(k)).$$

Using [7, Corollaire (4.2.2)] we get $H^{d-i}(X, \omega_X \otimes \mathcal{L}^{-1}) = 0$ for $i \leqslant d - \ell(I)$ so that $H^i_E(X, \mathcal{L}) = 0$ for $i \leqslant d - \ell(I)$ as wanted.

LEMMA 4.4. *Let $(A, \mathfrak{m})$ be a local ring of dimension d, and let $I \subset A$ be an ideal of positive height. Set $X = \operatorname{Proj} R_A(I)$ and $E = X \times_A A/\mathfrak{m}$. Also set $\ell = \ell(I)$ and $a = a(G_A(I))$ where $G_A(I)$ denotes the form ring $R_A(I)/IR_A(I)$. If $R_A(I)$ is Cohen–Macaulay, then*

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(a) $H^i(X, \mathcal{O}_X(\ell - 1 - i)) = 0$ *for all* $i > 0$*;*
(b) $H^i_E(X, \mathcal{O}_X(d - \ell - i)) = 0$ *for all* $i < d$.

Suppose, moreover that A is Cohen–Macaulay. Then

(a′) $H^i(X, \mathcal{O}_X(\ell + a - i)) = 0$ *for all* $i > 0$*;*
(b′) $H^i_E(X, \mathcal{O}_X(d - \ell + 1 - i)) = 0$ *for all* $i < d$.

Proof. If $R_A(I)$ is Cohen–Macaulay, then $H^i(X, \mathcal{O}_X(n)) = 0$ for all $i > 0$ and $n \geqslant 0$ by the case $r = 1$ of Theorem 3.1. This clearly implies (a). We also obtain $H^i_E(X, \mathcal{O}_X(n)) = 0$ for all $i < d$ and $n < 0$. Together with Lemma 4.3 this implies (b). To prove (a′) recall first that if A and $R_A(I)$ are Cohen–Macaulay, then so is $G_A(I)$ (see, e.g., [27, Theorem 5.1.23]). Let $\mathfrak{M}$ denote the homogeneous maximal ideal of $R_A(I)$. We now see that $[H^i_{\mathfrak{M}}(G)]_n = 0$ for all $i \geqslant 0$ and $n > a$. Set $Y = \operatorname{Proj} G_A(I)$. By Proposition 3.2 we obtain $H^i(Y, \mathcal{O}_Y(n)) = 0$ for all $i > 0$ and $n > a$. Using the long exact sequences of cohomology corresponding to the exact sequences

$$O \longrightarrow \mathcal{O}_X(n+1) \longrightarrow \mathcal{O}_X(n) \longrightarrow j_*(\mathcal{O}_Y(n)) \longrightarrow 0$$

where $j\colon Y \longrightarrow X$ is the inclusion, this gives $H^i(X, \mathcal{O}_X(n)) = 0$ for all $i > 0$ and $n > a$. This implies (a′). Finally, to prove (b′), we need to show that if A is Cohen–Macaulay, then $H^i_E(X, \mathcal{O}_X) = 0$ for $i < d$. But this follows from the Sancho de Salas sequence

$$\cdots \longrightarrow H^i_{\mathfrak{m}}(A) \longrightarrow H^i_E(X, \mathcal{O}_X) \longrightarrow [H^{i+1}_{\mathfrak{M}}(R_A(I))]_0 \longrightarrow \cdots$$

(see [20, p. 150]).

We still need two lemmas about the global section modules.

LEMMA 4.5. *Let A be a local ring, and let* $I_1, \ldots, I_r \subset A$ *be ideals of positive grade. Set* $Z = \operatorname{Proj} R_A(I_1, \ldots, I_r)$*. Then*

$$\Gamma(Z, \mathcal{O}_Z(\mathbf{n} - \mathbf{m})) = \operatorname{Hom}_A(I_1^{m_1} \cdots I_r^{m_r}, \Gamma(Z, \mathcal{O}_Z(\mathbf{n})))$$

for all $\mathbf{n}, \mathbf{m} \geqslant \mathbf{0}$*. Moreover,*

$$\Gamma(Z, \mathcal{O}_Z(\mathbf{n} - \mathbf{m})) = \Gamma(Z, \mathcal{O}(\mathbf{n})) :_{\Gamma(Z, \mathcal{O}_Z)} (I_1^{m_1} \cdots I_r^{m_r})$$

for all $\mathbf{n} \geqslant \mathbf{m} \geqslant \mathbf{0}$.

Proof. Observe first that

$$\begin{aligned}\Gamma(Z, \mathcal{O}_Z(\mathbf{n} - \mathbf{m})) &= \operatorname{Hom}_Z(\mathcal{O}_Z, \mathcal{O}_Z(\mathbf{n} - \mathbf{m}))\\ &= \operatorname{Hom}_Z(\mathcal{O}_Z(\mathbf{m}), \mathcal{O}_Z(\mathbf{n}))\\ &= \operatorname{Hom}_Z(I_1^{m_1} \cdots I_r^{m_r} \mathcal{O}_Z, \mathcal{O}_Z(\mathbf{n})).\end{aligned}$$

Because $I_1^{n'_1} \cdots I_r^{n'_r} \Gamma(Z, \mathcal{O}_Z(\mathbf{n})) \subset \Gamma(Z, \mathcal{O}_Z(\mathbf{n})(\mathbf{n}'))$ for all $\mathbf{n}' \geqslant \mathbf{0}$, it is easily checked

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that

$$\operatorname{Hom}_Z(I_1^{m_1}\cdots I_r^{m_r}\mathcal{O}_Z,\mathcal{O}_Z(\mathbf{n}))=\operatorname{Hom}_A(I_1^{m_1}\cdots I_r^{m_r},\Gamma(Z,\mathcal{O}(\mathbf{n}))).$$

This proves the first claim. Let us then prove the second claim by showing that the canonical homomorphism

$$\Gamma(Z,\mathcal{O}(\mathbf{n})):_{\Gamma(Z,\mathcal{O}_Z)}(I_1^{m_1}\cdots I_r^{m_r})\longrightarrow\operatorname{Hom}_A(I_1^{m_1}\cdots I_r^{m_r},\Gamma(Z,\mathcal{O}_Z(\mathbf{n})))$$

is an isomorphism. It is clearly injective, because every regular element of A is also $\Gamma(Z,\mathcal{O}_Z)$-regular. To prove the surjectivity, take

$$u\in\operatorname{Hom}_A(I_1^{m_1}\cdots I_r^{m_r},\Gamma(Z,\mathcal{O}_Z(\mathbf{n}))).$$

Choose a regular element $a\in A$ such that $aI_1^{m_1}\cdots I_r^{m_r}\in I_1^{n_1}\cdots I_r^{n_r}$. Since

$$\operatorname{Hom}_A(I_1^{m_1}\cdots I_r^{m_r}/I_1^{n_1}\cdots I_r^{n_r},\Gamma(Z,\mathcal{O}_Z(\mathbf{n})))=0,$$

there is a monomorphism

$$\operatorname{Hom}_A(I_1^{m_1}\cdots I_r^{m_r},\Gamma(Z,\mathcal{O}_Z(\mathbf{n})))\longrightarrow\operatorname{Hom}_A(I_1^{n_1}\cdots I_r^{n_r},\Gamma(Z,\mathcal{O}_Z(\mathbf{n}))).$$

Because

$$\operatorname{Hom}_A(I_1^{n_1}\cdots I_r^{n_r},\Gamma(Z,\mathcal{O}_Z(\mathbf{n})))=\Gamma(Z,\mathcal{O}_Z),$$

we can find an element $s\in\Gamma(Z,\mathcal{O}_Z)$ such that $u(x)=sx$ for all $x\in I_1^{n_1}\cdots I_r^{n_r}$. In particular, $u(ay)=say$ for all $y\in I_1^{m_1}\cdots I_r^{m_r}$. But then $u(y)=sy$, which proves the claim.

LEMMA 4.6. *Let A be a local ring. Let $I_1,\dots,I_r\subset A$ be ideals of positive grade such that*

$$(I_{j_1}^m\cdots I_{j_k}^m):I_{j_l}=I_{j_1}^m\cdots I_{j_{l-1}}^m I_{j_l}^{m-1}I_{j_{l+1}}^m\cdots I_{j_k}^m$$

for all $1\leqslant j_1<\cdots<j_k\leqslant r$ and $1\leqslant l\leqslant k$. Set $Z=\operatorname{Proj}R_A(I_1,\dots,I_r)$. Suppose that $H^i(Z,\mathcal{O}_Z(\mathbf{n}))=0$ for all $i>0$ and $\mathbf{n}\geqslant(m-i,\dots,m-i)$. If $\Gamma(Z,\mathcal{O}_Z)=A$ and $\Gamma(Z,\mathcal{O}_Z(m,\dots,m))=I_1^m\cdots I_r^m$, then $\Gamma(Z,\mathcal{O}_Z(\mathbf{n}))=I_1^{n_1}\cdots I_r^{n_r}$ for all $\mathbf{n}\geqslant(m-1,\dots,m-1)$.

Proof. Using Lemma 4.2(a), we see that $\Gamma(Z,\mathcal{O}_Z(\mathbf{n}))=I_1^{n_1}\cdots I_r^{n_r}$ if $\mathbf{n}\geqslant\mathbf{m}$. Let us show that the same holds for all $\mathbf{n}\geqslant(m-1,\dots,m-1)$. It suffices to consider the case $\mathbf{n}\leqslant(m,\dots,m)$. Suppose, for simplicity, that $n_1=\ldots=n_k=m$ and $n_{k+1}=\cdots=n_r=m-1$. The assumption now implies that

$$(I_1^m\cdots I_r^m):(I_{k+1}\cdots I_r)=I_1^m\cdots I_k^m I_{k+1}^{m-1}\cdots I_r^{m-1}.$$

By Lemma 4.5 we then get

$$\Gamma(Z,\mathcal{O}_Z(\mathbf{n}))=\Gamma(Z,\mathcal{O}_Z(m,\dots,m)):(I_{k+1}\cdots I_r)=I_1^m\cdots I_k^m I_{k+1}^{m-1}\cdots I_r^{m-1}.$$

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We are now ready to prove the first main result of this section, which concerns the Cohen–Macaulay property of a multi-Rees algebra in the case the product of the ideals has analytic spread at most two.

THEOREM 4.1. *Let A be a local ring. Let $I_1, \ldots, I_r \subset A$ be ideals of positive grade with $\ell(I_1 \cdots I_r) \leqslant 2$. Then $R_A(I_1, \ldots, I_r)$ is Cohen–Macaulay if and only if $R_A(I_1 \cdots I_r)$ is Cohen–Macaulay and the condition $(I_{j_1} \cdots I_{j_k}) : I_{j_l} = I_{j_1} \cdots I_{j_{l-1}} I_{j_{l+1}} \cdots I_{j_k}$ holds for all $1 \leqslant j_1 < \cdots < j_k \leqslant r$ and $1 \leqslant l \leqslant k$.*

Proof. Using Theorem 3.1 and Lemma 4.5, we easily see that the conditions of the theorem are necessary.

Let us prove that they are also sufficient. Set $Z = \operatorname{Proj} R_A(I_1, \ldots, I_r)$ and $E = Z \times_A A/\mathfrak{m}$. Since $Z \cong \operatorname{Proj} R_A(I_1 \cdots I_r)$, Lemmas 4.2(a) and 4.4(a) imply that $H^i(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i > 0$ and $\mathbf{n} \geqslant \mathbf{0}$. Because $\Gamma(Z, \mathcal{O}_Z) = A$ and $\Gamma(Z, \mathcal{O}_Z(1, \ldots, 1)) = I_1 \cdots I_r$, it follows from Lemma 4.6 that $\Gamma(Z, \mathcal{O}_Z(\mathbf{n})) = I_1^{n_1} \cdots I_r^{n_r}$ for all $\mathbf{n} \geqslant \mathbf{0}$. Using Lemmas 4.2(b) and 4.4(b), we get $H^i_E(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i < \dim A$ and $\mathbf{n} < \mathbf{0}$. The claim is now a consequence of Theorem 3.1.

J. Verma showed in [29, Theorem 3.4] that in a two-dimensional regular local ring integrally closed ideals primary to the maximal ideal always have Cohen–Macaulay multi-Rees algebras. In the following Corollary 4.1 we are going to generalize this to equimultiple ideals of height two in a local ring of arbitrary dimension. However, we first need two lemmas.

LEMMA 4.7. *Let A be a local ring, and let $I, J \subset A$ be ideals of positive height. Then* $\max\{\ell(I), \ell(J)\} \leqslant \ell(IJ)$.

Proof. It is enough to show that $\ell(I) \leqslant \ell(IJ)$. Set $X = \operatorname{Proj} R_A(I)$ and $Z = \operatorname{Proj} R_A(IJ)$. Also set $E = X \times_A A/\mathfrak{m}$ and $F = Z \times_A A/\mathfrak{m}$ where $\mathfrak{m}$ denotes the maximal ideal of A. We have $\ell(I) = \dim E + 1$ and $\ell(IJ) = \dim F + 1$. If $f: Z \longrightarrow X$ is the canonical projection, then clearly $F = f^{-1}(E)$ so that there is a proper surjection $F \longrightarrow E$. Therefore $\dim E \leqslant \dim F$ (see, e.g., [16, Lemma 3.2]).

In lack of a suitable reference we state the following doubtlessly well-known corollary of a result of Lipman and Teissier ([21, Corollary 5.4]).

LEMMA 4.8. *Let A be a Cohen–Macaulay local ring which satisfies the condition (R_h), and let $I \subset A$ be an integrally closed equimultiple ideal of height h such that A/I is unmixed. Then $I^2 = JI$ for any reduction $J \subset I$.*

Proof. We may assume that J is a minimal reduction of I. Then J is generated by a regular sequence of length h. Hence, $J/J^2 \cong (A/J)^h$ so that $J/JI = J/J^2 \otimes A/I \cong (A/I)^h$. The exact sequence

$$0 \longrightarrow J/JI \longrightarrow A/JI \longrightarrow A/J \longrightarrow 0$$

then implies that $\mathit{Ass}\ A/JI \subset \mathit{Min}\ A/I$. It is therefore enough to check the equality $I^2 = JI$ at minimal primes of I. But then one can use [21, Corollary 5.4].

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COROLLARY 4.1. *Let A be a Cohen–Macaulay local ring which satisfies the condition (R_2). Let $I_1, \ldots, I_r \subset A$ be integrally closed equimultiple ideals of height two. Suppose that also $I_1 \cdots I_r$ is equimultiple of height two. Then $R_A(I_1, \ldots, I_r)$ is Cohen–Macaulay if and only if $A/I_1 \cdots I_r$ is Cohen–Macaulay.*

Proof. If $R_A(I_1, \ldots, I_r)$ is Cohen–Macaulay, then so is $R_A(I_1 \cdots I_r)$ by Theorem 3.1. According to [10, Proposition 45.5] this implies that $A/I_1 \cdots I_r$ is Cohen–Macaulay.

Conversely, suppose that $A/I_1 \cdots I_r$ is Cohen–Macaulay. By Lemma 4.8 and [27, Proposition 5.1.12] $R_A(I_1 \cdots I_r)$ is Cohen–Macaulay. We want to apply Theorem 4.1. By the 'determinant trick' it suffices to show that the ideal $I_{j_1} \cdots I_{j_{l-1}} I_{j_{l+1}} \cdots I_{j_k}$ is integrally closed for $1 \leqslant j_1 < \cdots < j_k \leqslant r$ and $1 \leqslant l \leqslant k$. Because $\ell(I_{j_1} \cdots I_{j_{l-1}} I_{j_{l+1}} \cdots I_{j_k}) \leqslant 2$ by Lemma 4.7, we necessarily have $\ell(I_{j_1} \cdots I_{j_{l-1}} I_{j_{l+1}} \cdots I_{j_k}) = 2$. The above claim then follows from [28, Theorem 4.1].

EXAMPLE 4.1. In general the Cohen–Macaulayness of $R_A(IJ)$ does not imply that of $R_A(I, J)$ even if $R_A(I, J)$ is normal.

Let k be a field of characteristic zero. Take $A = k[x, y, z]_{(x,y,z)}$ and $I_0 = \mathfrak{p}A$ where $\mathfrak{p} \subset k[x, y, z]$ is the prime ideal defining the monomial curve $x = t^{11}$, $y = t^{14}$, $z = t^{15}$. It has been proven in [11, Example 3.6] that $R_A(I_0)$ does not have rational singularities although it is normal and Cohen–Macaulay. It is possible to find an ideal $J_0 \subset A$ that $Y = \operatorname{Proj} R_A(J_0)$ is regular and that $I_0\mathcal{O}_Y$ is invertible. Then $\operatorname{Proj} R_A(I_0, J_0) \cong \operatorname{Proj} R_A(I_0J_0) \cong Y$ is normal. For $p, q \gg 0$, we have $\Gamma(Y, I_0^p J_0^q \mathcal{O}_Y) = I_0^p J_0^q$ by the bigraded version of the theorem of Serre (see [16, Lemma 4.2]). On the other hand, $\Gamma(Y, I_0^p J_0^q \mathcal{O}_Y) = \overline{I_0^p J_0^q}$ for all $p, q \geqslant 0$ (see [21, p. 100]). Hence, $I_0^p J_0^q = \overline{I_0^p J_0^q}$ for $p, q \gg 0$. We also know that $J_0^q = \overline{J_0^q}$ for $q \gg 0$. Take $I = I_0^N$ and $J = J_0^N$ where $N \gg 0$. Then $R_A(I, J)$ is normal (cp. [17, p. 126]). Moreover, it still holds that $R_A(I)$ does not have rational singularities (use, e.g., [11, Proposition 2.1]). Now $R_A(I, J)$ is not Cohen–Macaulay. To see this, consider the blow-up $Z = \operatorname{Proj} R_{R_A(I)}(JR_A(I))$. By looking at the affine open sets which cover Z, one easily checks that $Z = \mathbb{V}(I\mathcal{O}_Y)$. In particular, Z is regular. The Cohen–Macaulayness of $R_A(I, J)$ would then by [20, Theorem 4.1] imply that $R_A(I)$ has rational singularities. Finally, since A has rational singularities, using [20, Theorem 4.1] again, we may choose $N \gg 0$ in such a way that $R_A(IJ)$ is Cohen–Macaulay.

We now move to the analytic spread three case. Then an additional cohomological condition comes in. For simplicity, we assume $r = 2$.

THEOREM 4.2. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, and let $I, J \subset A$ be ideals of positive height such that $\ell(IJ) = 3$. Let $\mathfrak{M}$ and $\mathfrak{N}$ denote the homogeneous maximal ideals of $R_A(I)$ and $R_A(J)$ respectively. Then $R_A(I, J)$ is Cohen–Macaulay if and only if the following conditions are satisfied*

(1) *$R_A(IJ)$ is Cohen–Macaulay;*
(2) *$I^2J^2 : I = IJ^2$ and $I^2J^2 : J = I^2J$;*

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(3) $[H^i_{\mathfrak{M}}(JR_A(I))]_0 = 0$ *and* $[H^i_{\mathfrak{N}}(IR_A(J))]_0 = 0$ *for all* $i < d+1$.

Proof. If $R_A(I,J)$ is Cohen–Macaulay, then Theorem 3.1 implies that also $R_A(IJ)$ is Cohen–Macaulay. So (1) holds. Moreover, using Lemma 4.5 we get (2). Finally, to prove (3), we first verify that always

$$[H^{d+1}_{\mathfrak{M}}(JR_A(I))]_0 = 0 \quad \text{and} \quad [H^{d+1}_{\mathfrak{N}}(IR_A(J))]_0 = 0.$$

Let us show, for example, that $[H^{d+1}_{\mathfrak{M}}(JR_A(I))]_0 = 0$. Indeed, there is for some $N > 0$ an exact sequence

$$0 \longrightarrow K \longrightarrow (R_A(I))^{\oplus N} \longrightarrow JR_A(I) \longrightarrow 0$$

of graded $R_A(I)$-modules. As $[H^{d+1}_{\mathfrak{M}}(R_A(I))]_0 = 0$ (see [4, Part I, 6.3]), the corresponding long exact sequence of cohomology yields $[H^{d+1}_{\mathfrak{M}}(JR_A(I))]_0 = 0$ as wanted. Set $X = \operatorname{Proj} R_A(I)$ and $Y = \operatorname{Proj} R_A(J)$. It now follows from Proposition 3.2 that condition (3) is equivalent to having $\Gamma(X, J\mathcal{O}_X) = J$, $\Gamma(Y, I\mathcal{O}_Y) = I$ and $H^i(X, J\mathcal{O}_X) = 0$, $H^i(Y, I\mathcal{O}_Y) = 0$ for all $i > 0$. Condition (3) is thus a consequence of Theorem 3.1 and Proposition 3.3.

Suppose then that the conditions (1)–(3) hold. Set $E = Z \times_A A/\mathfrak{m}$. It is now clear from Lemma 4.2(b) and Lemma 4.4(b′) that $H^i_E(Z, \mathcal{O}_Z(p,q)) = 0$ for all $i < d$ and $p, q \leqslant -1$. Using Lemmas 4.2(a) and 4.4(a), we get $H^i(Z, \mathcal{O}_Z(p,q)) = 0$ for all $i > 0$ and $p, q \geqslant 2 - i$. Since $R_A(IJ)$ is Cohen–Macaulay, we have $\Gamma(Z, \mathcal{O}_Z(n,n)) = I^nJ^n$ for all $n \geqslant 0$. By 2) it then follows from Lemma 4.6 that $\Gamma(Z, \mathcal{O}_Z(p,q)) = I^pJ^q$ for all $p, q \geqslant 1$. By Lemma 3.4 the conditions $\Gamma(X, J\mathcal{O}_X) = J$ and $\Gamma(Y, I\mathcal{O}_Y) = I$ mean that $\Gamma(Z, J\mathcal{O}_Z) = J$ and $\Gamma(Z, I\mathcal{O}_Z) = I$. By Lemma 4.5 this is the same as $IJ : J = I$ and $IJ : I = J$. Therefore, as soon as we know that $H^1(Z, \mathcal{O}_Z(p,q)) = 0$ for all $p, q \geqslant 0$ Lemma 4.6 will tell us that $\Gamma(Z, \mathcal{O}_Z(p,q)) = I^pJ^q$ for all $p, q \geqslant 0$. As $H^2(Z, \mathcal{O}_Z(0,0)) = 0$, it follows from Lemma 4.1(a) that if $H^1(Z, \mathcal{O}_Z(1,0)) = 0$ and $H^1(Z, \mathcal{O}_Z(0,1)) = 0$, then $H^1(Z, \mathcal{O}_Z(p,0)) = 0$ and $H^1(Z, \mathcal{O}_Z(0,q)) = 0$ for all $p, q \geqslant 1$. But using Lemma 3.4 again, we have $H^1(Z, \mathcal{O}_Z(0,1)) = H^1(X, J\mathcal{O}_X)$ and $H^1(Z, \mathcal{O}_Z(1,0)) = H^1(Y, I\mathcal{O}_Y)$. This completes the proof.

EXAMPLE 4.2. Let k be a field. Let $A = k[x,y,z]_{(x,y,z)}$ where x, y, z are variables. Take $I = (x^2, y, z)A$ and $J = \mathfrak{m}$, where $\mathfrak{m} = (x, y, z)$ is the maximal ideal of A. We now have $I^2J^2 : I = IJ^2$ and $I^2J^2 : J = I^2J$. Moreover, one checks that the ideal $L = (x^3 + y^2 + z^2, xy, xz + yz)$ is a minimal reduction of $IJ = (x^3, y^2, z^2, xy, xz, yz)$ with $(IJ)^2 = L(IJ)$. Therefore $R_A(IJ)$ is Cohen–Macaulay (see, e.g., [27, Corollary 5.1.13]). Let U, V and W be variables. Then $R_A(I)/JR_A(I) = k[U, V, W]$. The long exact sequence of cohomology corresponding to the exact sequence

$$0 \longrightarrow JR_A(I) \longrightarrow R_A(I) \longrightarrow R_A(I)/JR_A(I) \longrightarrow 0$$

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now gives $[H^i_{\mathfrak{M}}(JR_A(I))]_0 = 0$ for all $i < 4$. On the other hand,

$$R_A(J) = A[U, V, W]/(xV - yU, xW - zU, yW - zV)$$

so that $R_A(J)/IR_A(J) = k[x][U, V, W]/(x^2, xV, xW)$. Using the long exact sequences of cohomology corresponding to the exact sequences

$$0 \longrightarrow (x, V, W) \xrightarrow{x} k[x][U, V, W] \longrightarrow R_A(J)/IR_A(J) \longrightarrow 0$$

and

$$0 \longrightarrow (x, V, W) \longrightarrow k[x][U, V, W] \longrightarrow k[U] \longrightarrow 0,$$

one easily verifies that $[H^i_{\mathfrak{R}}(R_A(J)/IR_A(J))]_0 = 0$ for all $i \geqslant 0$. Hence, also $[H^i_{\mathfrak{R}}(IR_A(J))]_0 = 0$ for all $i \geqslant 0$. It thus follows from Theorem 4.2 that $R_A(I, J)$ is Cohen–Macaulay. However, one checks that $[H^2_{\mathfrak{R}}(IR_A(J))]_n \neq 0$ for all $n < 0$. It follows that neither $IR_A(J)$ nor even $I\mathcal{O}_Y$ is Cohen–Macaulay.

We now want to investigate rational singularities of $R_A(I, J)$ in this context. For basic facts about rational singularities, we refer to [1]. The following lemma is a multigraded variant of a theorem of Flenner ([3, Satz 3.1]).

LEMMA 4.9. *Let S be a normal r-graded ring defined over an excellent local ring of equicharacteristic zero. Set $Z = \operatorname{Proj} S$. Then S has rational singularities if and only if Z has rational singularities and $H^i(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i > 0$ and $\mathbf{n} \geqslant \mathbf{0}$.*

Proof. Suppose first that S has rational singularities. Cover Z with open affine sets $D_+(s_1 \cdots s_r) = \operatorname{Spec} S_{(s_1 \cdots s_r)}$ where $s_j \in S_{\mathbf{1}_j}$ $(j = 1, \ldots, r)$. Because $S_{s_1 \cdots s_r} = S_{(s_1 \cdots s_r)}[t_1, t_1{}^{-1}, \ldots, t_r, t_r{}^{-1}]$ where $t_1, \ldots, t_r$ are variables, it is easy to see that if $S_{s_1 \cdots s_r}$ has rational singularities, then so has $S_{(s_1 \cdots s_r)}$. Therefore Z has rational singularities.

From now on we can thus assume that Z has rational singularities. Set $W = \operatorname{Proj} R_S(S^+)$. Let $f: W \longrightarrow \operatorname{Spec} S$ be the canonical projection. Let $g: W' \longrightarrow W$ be a desingularization. Then $fg: W' \longrightarrow \operatorname{Spec} S$ is a desingularization of $\operatorname{Spec} S$. By Lemma 3.1 we know that $W = \mathbb{V}(\mathcal{O}_Z(\mathbf{1}_1) \oplus \cdots \oplus \mathcal{O}_Z(\mathbf{1}_r))$. Since Z has rational singularities, then so does W. Hence, $R^i g_* \mathcal{O}_{W'} = 0$ for all $i > 0$. Because W is normal, we also have $g_* \mathcal{O}_{W'} = \mathcal{O}_W$. An easy application of the Leray spectral sequence now shows that $R^i(fg)_* \mathcal{O}_{W'} = R^i f_* \mathcal{O}_W$ for all $i > 0$. Therefore $R^i(fg)_* \mathcal{O}_{W'} = 0$ for all $i > 0$ if and only if $H^i(W, \mathcal{O}_W) = 0$ for all $i > 0$. Because $H^i(W, \mathcal{O}_W) = \bigoplus_{\mathbf{n} \geqslant \mathbf{0}} H^i(Z, \mathcal{O}_Z(\mathbf{n}))$, we thus see that S has rational singularities if and only if $H^i(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i > 0$ and $\mathbf{n} \geqslant \mathbf{0}$. The claim has so been proven.

THEOREM 4.3. *Let A be an excellent local ring of equicharacteristic zero. Let $I, J \subset A$ be ideals of positive height such that $\ell(IJ) = 3$. Then $R_A(I, J)$ has rational singularities if and only if $R_A(I)$, $R_A(J)$ and $R_A(IJ)$ have rational singularities. When A is essentially of finite type over a field of characteristic zero, this implies in particular that $R_A(I, J)$ is Cohen–Macaulay.*

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Proof. Suppose first that $R_A(I)$, $R_A(J)$ and $R_A(IJ)$ have rational singularities. Set $X = \operatorname{Proj} R_A(I)$, $Y = \operatorname{Proj} R_A(J)$ and $Z = \operatorname{Proj} R_A(I, J)$. Because $Z \cong \operatorname{Proj} R_A(IJ)$ has rational singularities, it is by Lemma 4.9 enough to show that $R_A(I, J)$ is normal and that $H^i(Z, \mathcal{O}_Z(p, q)) = 0$ for all $i > 0$ and $p, q \geqslant 0$. Consider first the latter claim. Since $R_A(IJ)$ has rational singularities, we have $H^i(Z, \mathcal{O}_Z(n, n)) = 0$ for all $i > 0$ and $n \geqslant 0$. By Lemma 4.2(a) we then get $H^i(Z, \mathcal{O}_Z(p, q)) = 0$ for all $i > 0$ and $p, q \geqslant 2 - i$. It remains to prove that $H^1(Z, \mathcal{O}_Z(p, 0)) = 0$ and $H^1(Z, \mathcal{O}_Z(0, q)) = 0$ for all $p, q \geqslant 0$. But because $R_A(I)$ and $R_A(J)$ have rational singularities, this again follows from Lemma 4.9 by an easy Leray spectral sequence argument. We still need to show that $R_A(I, J)$ is normal. Note first that A is normal. Because Z is normal, we know by [21, p. 100] that $\Gamma(Z, I^pJ^q\mathcal{O}_Z) = \overline{I^pJ^q}$ for all $p, q \geqslant 0$. This means in particular that $\Gamma(Z, I\mathcal{O}_Z) = I$, $\Gamma(Z, J\mathcal{O}_Z) = J$ and $\Gamma(Z, IJ\mathcal{O}_Z) = IJ$. Lemma 4.2(a) then yields $\Gamma(Z, I^pJ^q\mathcal{O}_Z) = I^pJ^q$ for all $p, q \geqslant 0$. So $I^pJ^q = \overline{I^pJ^q}$ for all $p, q \geqslant 0$. Hence, $R_A(I, J)$ is normal (cf. [17, p. 126]).

Conversely, if $R_A(I, J)$ has rational singularities, then so have $R_A(I)$ and $R_A(J)$ (use e.g. [11, Proposition 2.3]). By utilizing again the isomorphism $\operatorname{Proj} R_A(IJ) \cong Z$, it is also clear from Lemma 4.9 that $R_A(IJ)$ must have rational singularities.

Let A be a ring and let $\mathfrak{a} \subset A$ be an ideal. Recall that the reduction number of $\mathfrak{a}$ with respect to a reduction $\mathfrak{b} \subset \mathfrak{a}$ is the least integer r such that $\mathfrak{a}^{r+1} = \mathfrak{b}\mathfrak{a}^r$. It is denoted by $r_{\mathfrak{b}}(\mathfrak{a})$. The reduction number of $\mathfrak{a}$ is then defined as

$$r(\mathfrak{a}) = \min\{r_{\mathfrak{b}}(\mathfrak{a}) \mid \mathfrak{b} \subset \mathfrak{a} \text{ is a reduction of } \mathfrak{a}\}.$$

The next theorem deals with the Cohen–Macaulay property of a multi-Rees algebra when the product of the ideals has a small reduction number. In this case we need not to assume anything about the analytic spread.

THEOREM 4.4. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of positive dimension, and let $I, J \subset A$ be $\mathfrak{m}$-primary ideals such that $r(IJ) \leqslant 1$. Then $R_A(I, J)$ is Cohen–Macaulay if and only if $IJ : I = J$ and $IJ : J = I$.*

Proof. Since $r(IJ) \leqslant 1$, we know that $R_A(IJ)$ is Cohen–Macaulay (see [27, Corollary 5.13]). By [26, Proposition 3.2] we obtain $a(G_A(IJ)) \leqslant -d + 1$ where $d = \dim A$. Lemmas 4.2(a) and 4.4(a′) therefore imply that $H^i(Z, \mathcal{O}_Z(p, q)) = 0$ for all $i > 0$ and $p, q \geqslant 1 - i$. Let us now show that $H^i_E(Z, \mathcal{O}_Z(p, q)) = 0$ for all $i < d$ and $p, q \leqslant -1$. We use induction on i. Note first that by Lemmas 4.2(b) and 4.4(b′) we have $H^i_E(Z, \mathcal{O}_Z(p, q)) = 0$ for all $i < d$ and $p, q \leqslant 1 - i$. For $i \leqslant 2$, there is thus nothing to prove. Let $i > 2$. Set $U = \operatorname{Spec} A \setminus \{\mathfrak{m}\}$ and $V = f^{-1}(U)$ where $f: Z \longrightarrow \operatorname{Spec} A$ is the canonical projection. We then have the exact sequence

$$\cdots \longrightarrow H^{i-1}(V, \mathcal{O}_V) \longrightarrow H^i_E(Z, \mathcal{O}_Z(p, q)) \longrightarrow H^i(Z, \mathcal{O}_Z(p, q)) \longrightarrow \cdots \quad .$$

Here $H^{i-1}(V, \mathcal{O}_V) \cong H^{i-1}(U, \mathcal{O}_U) \cong H^i_{\mathfrak{m}}(A)$. By the Cohen–Macaulayness of A, this implies that $H^i_E(Z, \mathcal{O}_Z(p, q)) = 0$ for $i < d$ and $1 - i \leqslant p, q \leqslant -1$. Then suppose, for example, that $1 - i \leqslant q \leqslant -1$, but $p < 1 - i$. By the case $j \leqslant 2$ and the induction hypothesis we now have $H^j_E(Z, \mathcal{O}_Z(1 - j, q)) = 0$ for all $j < i$. We just saw that

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also $H^i_E(Z, \mathcal{O}_Z(1-i, q)) = 0$. It thus follows from Lemma 4.2(b) that $H^i_E(Z, \mathcal{O}_Z(p, q)) = 0$ also for all $p < 1 - i$ and $1 - i \leqslant q \leqslant -1$. It remains to show that $\Gamma(Z, \mathcal{O}_Z(p, q)) = I^p J^q$ for all $p, q \geqslant 0$ if and only if $IJ : I = J$ and $IJ : J = I$. But this is now a consequence of Lemmas 4.5 and 4.6.

Finally, we want to investigate the effect of the Cohen–Macaulayness of a multi-Rees algebra on the depths of the corresponding single Rees algebras.

THEOREM 4.5. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, and let $I \subset A$ be an ideal of positive height satisfying for some $k \in \mathbb{N}$ the conditions $A/I^n \geqslant d + 1 - k - n$ $(n = 1, \ldots, d - k)$. Suppose that there exist ideals $I_1, \ldots, I_r \subset A$ of positive height such that $R_A(I, I_1, \ldots, I_r)$ is Cohen–Macaulay. Then $R_A(I) \geqslant d + 3 - k$. In particular, we always have depth $R_A(I) \geqslant 3$. When $d = 2$, this means that $R_A(I)$ is Cohen–Macaulay.*

Proof. Set $R = R_A(I)$, and let $\mathfrak{M}$ denote the homogeneous maximal ideal of R. Also set $X = \operatorname{Proj} R$ and $E = X \times_A A/\mathfrak{m}$. Using Proposition 3.3 it follows from Theorem 3.1 that $\Gamma(X, \mathcal{O}_X(n)) = I^n$ and $H^i(X, \mathcal{O}_X(n)) = 0$ for all $i > 0$ and $n \geqslant 0$. By Proposition 3.2 this gives $[H^i_{\mathfrak{M}}(R)]_n = 0$ for all $i \geqslant 0$ and $n \geqslant 0$. We also obtain $H^i_E(X, \mathcal{O}_X(n)) = H^i_{\mathfrak{m}}(I^n)$ for all $n \geqslant 0$. Because A is Cohen–Macaulay, we have $H^i_{\mathfrak{m}}(I^n) = H^{i-1}_{\mathfrak{m}}(A/I^n)$ for all $n \geqslant 0$. It therefore follows that $H^i_E(X, \mathcal{O}_X(d + 1 - k - i)) = 0$ for all $i \leqslant d + 1 - k$. By Lemma 4.2(b) this implies that $H^i_E(X, \mathcal{O}_X(n)) = 0$ for all $i \leqslant d + 1 - k$ and $n < 0$. By the Sancho de Salas sequence ([20, p. 150]) we then obtain $[H^i_{\mathfrak{M}}(R)]_n = 0$ for all $n < 0$ and $i \leqslant d + 2 - k$. This proves the claim.

5. An Application to Joint Reductions

We begin by recalling from [15, p. 218] the definition of a joint reduction of a standard graded ring $S = \bigoplus_{n \in \mathbb{N}^r} S_{\mathbf{n}}$. Let $\mathbf{q} \in \mathbb{N}^r$. A set $\{z_{i,j} \in S_{\mathbf{1}_i} \mid i = 1, \ldots, r; j = 1, \ldots, q_i\}$ is called a joint reduction of S of type $\mathbf{q}$ if

$$S_{\mathbf{n}} = \sum_{i=1}^{r} (z_{i,1}, \ldots, z_{i,q_i}) S_{\mathbf{n}-\mathbf{1}_i} \tag{\dagger}$$

for all $\mathbf{n} \gg \mathbf{0}$. Suppose that $S_{\mathbf{0}}$ is local with the maximal ideal $\mathfrak{m}$. If the residue field of $S_{\mathbf{0}}$ is infinite, then joint reductions always exist for $|\mathbf{q}| \geqslant \dim S^{\Delta}/\mathfrak{m}S^{\Delta}$. Finally, when $I_1, \ldots, I_r \subset A$ are ideals, one says that a set $\{a_{i,j} \in I_i \mid i = 1, \ldots, r; j = 1, \ldots, q_i\}$ is a joint reduction of $I_1, \ldots, I_r$ of type $\mathbf{q}$ if the set $\{a_{i,j} t_i \mid i = 1, \ldots, r; j = 1, \ldots, q_i\}$ is a joint reduction of $R_A(I_1, \ldots, I_r) = A[I_1 t_1, \ldots, I_r t_r]$ of type $\mathbf{q}$.

The main result of this section is Theorem 5.1, which says that in the Cohen–Macaulay case the formula (†) already holds for all $\mathbf{n} \geqslant \mathbf{q}$. Its proof is based on the following lemma.

LEMMA 5.1. *Let S be a standard r-graded ring. Set $Z = \operatorname{Proj} S$. Let $\{z_{i,j} \in S_{\mathbf{1}_i} \mid i = 1, \ldots, r; j = 1, \ldots, q_i\}$ be a joint reduction of S of type $\mathbf{q}$. Let $\mathbf{n} \in \mathbb{N}^r$. Suppose*

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that $H^i(Z, \mathcal{O}(\mathbf{n}-\mathbf{p})) = 0$ *for all* $0 < i < |\mathbf{q}|$ *and* $\mathbf{0} \leqslant \mathbf{p} \leqslant \mathbf{q}$ *with* $|\mathbf{p}| = i+1$. *Then the elements* $z_{i,j}$ *define a surjective homomorphism*

$$\sigma_{\mathbf{n}}\colon \bigoplus_{i=1}^{r} \Gamma(Z, \mathcal{O}_Z(\mathbf{n}-\mathbf{1}_i))^{\oplus q_i} \longrightarrow \Gamma(Z, \mathcal{O}_Z(\mathbf{n})).$$

Proof. In any case the elements $z_{i,j}$ induce an epimorphism

$$\sigma\colon \bigoplus_{i=1}^{r} \mathcal{O}_Z(-\mathbf{1}_i)^{\oplus q_i} \longrightarrow \mathcal{O}_Z.$$

Set $\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{O}_Z(-\mathbf{1}_i)^{\oplus q_i}$. It follows that the corresponding Koszul complex

$$K_\bullet(\mathcal{F}, \sigma)\colon\ \wedge^{|\mathbf{q}|}\mathcal{F} \longrightarrow \cdots \longrightarrow \wedge^{j}\mathcal{F} \longrightarrow \wedge^{j-1}\mathcal{F} \longrightarrow \cdots \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z$$

is exact. We now have

$$\wedge^{j}\mathcal{F} = \bigoplus_{|\mathbf{p}|=j} \bigotimes_{i=1}^{r} \left(\mathcal{O}_Z(-p_i\mathbf{1}_i)^{\oplus\binom{q_i}{p_i}} \right).$$

Note that

$$\bigotimes_{i=1}^{r} \left(\mathcal{O}_Z(-p_i\mathbf{1}_i)^{\oplus\binom{q_i}{p_i}} \right) = 0$$

if $p_i > q_i$ for some i. By tensoring $K_\bullet(\mathcal{F}, \sigma)$ with $\mathcal{O}_Z(\mathbf{n})$ we obtain an exact sequence

$$0 \longrightarrow \mathcal{L}_{|\mathbf{q}|} \longrightarrow \cdots \longrightarrow \mathcal{L}_j \longrightarrow \mathcal{L}_{j-1} \longrightarrow \cdots \longrightarrow \mathcal{L}_0 \longrightarrow 0$$

where $\mathcal{L}_j = \wedge^j \mathcal{F} \otimes \mathcal{O}_Z(\mathbf{n})$. This sequence gives raise to exact sequences

$$0 \longrightarrow \mathcal{K}_j \longrightarrow \mathcal{L}_j \longrightarrow \mathcal{K}_{j-1} \longrightarrow 0 \qquad (j = 1, \dots, |\mathbf{q}|-1)$$

where $\mathcal{K}_0 = \mathcal{L}_0$ and $\mathcal{K}_{|\mathbf{q}|-1} = \mathcal{L}_{|\mathbf{q}|}$. The assumption now guarantees that we have $H^{j-1}(Z, \mathcal{L}_j) = 0$ for all $1 < j \leqslant |\mathbf{q}|$. So there is for every $1 \leqslant j \leqslant |\mathbf{q}|-1$ a monomorphism $H^{j-1}(Z, \mathcal{K}_{j-1}) \longrightarrow H^j(Z, \mathcal{K}_j)$. As $H^{|\mathbf{q}|-1}(Z, \mathcal{K}_{|\mathbf{q}|-1}) = 0$, this implies that $H^1(Z, \mathcal{K}_1) = 0$. But then the homomorphism $\Gamma(Z, \mathcal{F}(\mathbf{n})) \longrightarrow \Gamma(Z, \mathcal{O}(\mathbf{n}))$ is an epimorphism, which proves the claim.

THEOREM 5.1. *Let* S *be a standard* r*-graded ring defined over a local ring such that* S^+ *has positive height. If* S *is Cohen–Macaulay with* $\mathbf{a}(S) < \mathbf{0}$, *then*

$$S_{\mathbf{n}} = \sum_{i=1}^{r} (z_{i,1}, \dots, z_{i,q_i}) S_{\mathbf{n}-\mathbf{1}_i}$$

when $\mathbf{n} \geqslant \mathbf{q}$ *for all joint reductions* $\{z_{i,j} \in S_{\mathbf{1}_i} \mid i = 1, \dots, r; j = 1, \dots, q_i\}$ *of type* $\mathbf{q}$.

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Proof. Set $Z = \operatorname{Proj} S$. Theorem 3.1 implies that $H^i(Z, \mathcal{O}_Z(\mathbf{n}-\mathbf{p})) = 0$ for all $i > 0$ and $\mathbf{n} \geqslant \mathbf{p}$. Moreover, we have $\Gamma(Z, \mathcal{O}_Z(\mathbf{n})) = S_{\mathbf{n}}$ for all $\mathbf{n} \geqslant \mathbf{0}$. The claim therefore follows from Lemma 5.1.

As an immediate consequence of this we obtain

COROLLARY 5.1. *Let A be a local ring, and let $I_1, \dots, I_r$ be ideals of positive height. Let $\{a_{i,j} \in I_i \mid i = 1, \dots, r; j = 1, \dots, q_i\}$ be a joint reduction of $I_1, \dots, I_r$ of type* $\mathbf{q}$. *If $R_A(I_1, \dots, I_r)$ is Cohen–Macaulay, then*

$$I_1^{n_1} \cdots I_r^{n_r} = \sum_{i=1}^{r} (a_{i,1}, \dots, a_{i,q_i}) I_1^{n_1} \cdots I_{i-1}^{n_{i-1}} I_i^{n_i - 1} I_{i+1}^{n_{i+1}} \cdots I_r^{n_r}$$

for all $\mathbf{n} \geqslant \mathbf{q}$.

6. A Formula for Mixed Multiplicities

Let $(A, \mathfrak{m})$ be a local ring of dimension d, and let $I_1, \dots, I_r$ be $\mathfrak{m}$-primary ideals. Mixed multiplicities were introduced by Teissier in [25]. Recall first that the function $l_A(A/I_1^{n_1} \cdots I_r^{n_r})$ is a numerical polynomial of degree d for $\mathbf{n} \gg \mathbf{0}$. Let $i_1, \dots, i_r \in \mathbb{N}$ with $i_1 + \cdots + i_r = d$. The mixed multiplicity of $I_1, \dots, I_r$ of type $(i_1, \dots, i_r)$ which is denoted by $[I_1^{[i_1]}, \dots, I_r^{[i_r]}]$, is then defined as the coefficient of $n_1^{i_1} \cdots n_r^{i_r}/i_1! \cdots i_r!$ in this polynomial.

It has been proven by J. Lipman ([19, Corollary 3.7]) (see also [28, Corollary 3.3]) that in a two-dimensional regular local ring $(A, \mathfrak{m})$ any two integrally closed $\mathfrak{m}$-primary ideals $I, J \subset A$ satisfy the 'the mixed multiplicity formula'

$$[I^{[1]}, J^{[1]}] = \ell(A/IJ) - \ell(A/I) - \ell(A/J).$$

A result of Verma ([29, Theorem 3.4]) says that in this case the multi-Rees algebra $R_A(I, J)$ is Cohen–Macaulay. We are now going to show that under the latter assumption an analogous formula holds in an arbitrary local ring.

THEOREM 6.1. *Let $(A, \mathfrak{m})$ be a local ring of dimension d, and let $I_1, \dots, I_r$ be $\mathfrak{m}$-primary ideals. If $R_A(I_1, \dots, I_r)$ is Cohen–Macaulay, then*

$$[I_1^{[i_1]}, \dots, I_r^{[i_r]}] = \sum_{n_1=0}^{i_1} \cdots \sum_{n_r=0}^{i_r} \binom{i_1}{n_1} \cdots \binom{i_r}{n_r} (-1)^{d - n_1 - \cdots - n_r} l_A(A/I_1^{n_1} \cdots I_r^{n_r}).$$

Proof. We first show that there is a numerical polynomial $H \in \mathbb{Q}[t_1, \dots, t_r]$ such that $H(\mathbf{n}) = \ell_A(A/I_1^{n_1} \cdots I_r^{n_r})$ when $\mathbf{n} \geqslant \mathbf{0}$. We proceed inductively by proving for all $j = 0, \dots, r$ the existence of numerical polynomials $H_j \in \mathbb{Q}[t_1, \dots, t_j]$ such that $H_j(n_1, \dots, n_j) = \ell_A(A/I_1^{n_1} \cdots I_j^{n_j})$ when $n_1, \dots, n_j \geqslant 0$. If $j = 0$, then there is nothing to prove. Suppose $j > 0$. Set

$$Y = \operatorname{Proj} R_A(I_1, \dots, I_r)/I_j R_A(I_1, \dots, I_r).$$

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Recall that the Euler–Poincaré characteristic

$$\chi(Y, \mathcal{O}_Y(\mathbf{n})) = \sum_{i \geqslant 0} (-1)^i l_A(H^i(Y, \mathcal{O}_Y(\mathbf{n}))) \quad (\mathbf{n} \in \mathbb{Z}^r)$$

is a numerical polynomial (see [16, Lemma 4.3]). This implies that also

$$\sum_{k=0}^{n_j-1} \chi(Y, \mathcal{O}_Y(n_1, \ldots, n_{j-1}, k, 0, \ldots, 0))$$

is a numerical polynomial. Consider the long exact sequence of cohomology corresponding to the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(\mathbf{n} + \mathbf{1}_j) \longrightarrow \mathcal{O}_Z(\mathbf{n}) \longrightarrow (j_* \mathcal{O}_Y)(\mathbf{n}) \longrightarrow 0$$

where $Z = \operatorname{Proj} R_A(I_1, \ldots, I_r)$ and $j: Y \longrightarrow Z$ is the inclusion. The Cohen–Macaulayness of $R_A(I_1, \ldots, I_r)$ implies by Theorem 3.1 that $\Gamma(Z, \mathcal{O}_Z(\mathbf{n})) = I_1^{n_1} \cdots I_r^{n_r}$ and $H^i(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i > 0$ and $\mathbf{n} \geqslant \mathbf{0}$. It follows that

$$\Gamma(Y, \mathcal{O}_Y(\mathbf{n})) = I_1^{n_1} \cdots I_r^{n_r} / I_1^{n_1} \cdots I_{j-1}^{n_{j-1}} I_j^{n_j+1} I_{j+1}^{n_{j+1}} \cdots I_r^{n_r}$$

and $H^i(Y, \mathcal{O}_Y(\mathbf{n})) = 0$ for all $i > 0$ and $\mathbf{n} \geqslant \mathbf{0}$. This means that

$$\chi(Y, \mathcal{O}_Y(\mathbf{n})) = l_A(I_1^{n_1} \cdots I_r^{n_r} / I_1^{n_1} \cdots I_{j-1}^{n_{j-1}} I_j^{n_j+1} I_{j+1}^{n_{j+1}} \cdots I_r^{n_r})$$

for all $\mathbf{n} \geqslant \mathbf{0}$. Therefore

$$\begin{aligned} & l_A(A/I_1^{n_1} \cdots I_j^{n_j}) \\ & \quad = l_A(A/I_1^{n_1} \cdots I_{j-1}^{n_{j-1}}) + \sum_{k=0}^{n_j-1} \chi(Y, \mathcal{O}_Y(n_1, \ldots, n_{j-1}, k, 0, \ldots, 0)) \end{aligned}$$

for all $n_1, \ldots, n_j \geqslant 0$. We can thus take

$$\begin{aligned} & H_j(n_1, \ldots, n_j) \\ & \quad = H_{j-1}(n_1, \ldots, n_{j-1}) + \sum_{k=0}^{n_j-1} \chi(Y, \mathcal{O}_Y(n_1, \ldots, n_{j-1}, k, 0, \ldots, 0)). \end{aligned}$$

For $H = H_r$, we then have $H(\mathbf{n}) = l_A(A/I_1^{n_1} \cdots I_r^{n_r})$ for all $\mathbf{n} \geqslant \mathbf{0}$. Let $\Delta_j = H(\mathbf{n} + \mathbf{1}_j) - H(\mathbf{n})$ be the jth difference function corresponding to H. According to [24, Proposition 1.1] we then have $[I_1^{[i_1]}, \ldots, I_r^{[i_r]}] = (\Delta_1^{i_1} \cdots \Delta_r^{i_r} H)(\mathbf{0})$ for all $i_1 + \cdots + i_r = d$. Using [24, Proposition 1.2] it is easy to see that

$$(\Delta_1^{i_1} \cdots \Delta_r^{i_r} H)(\mathbf{0}) = \sum_{n_1=0}^{i_1} \cdots \sum_{n_r=0}^{i_r} \binom{i_1}{n_1} \cdots \binom{i_r}{n_r} (-1)^{d-n_1-\cdots-n_r} H(\mathbf{n}).$$

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Hence

$$[I_1^{[i_1]}, \dots, I_r^{[i_r]}] = \sum_{n_1=0}^{i_1} \cdots \sum_{n_r=0}^{i_r} \binom{i_1}{n_1} \cdots \binom{i_r}{n_r} (-1)^{d-n_1-\dots-n_r} l_A(A/I_1^{n_1} \cdots I_r^{n_r}).$$

Let $(A, \mathfrak{m})$ be a regular local ring. Let $I \subset A$ be an ideal. Recall that the order of I, denoted by $\operatorname{ord}(I)$, is the largest integer k such that $I \subset \mathfrak{m}^k$. It has been proven in [19, Corollary 3.2] that if A is two-dimensional and I is an $\mathfrak{m}$-primary integrally closed ideal, then $\operatorname{ord}(I) = \mu(I) - 1$. In the following proposition we will show that an analogous formula holds in any dimension if the bi-Rees algebra $R(I, \mathfrak{m})$ is Cohen–Macaulay.

PROPOSITION 6.1. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension d, and let $I \subset A$ be an $\mathfrak{m}$-primary ideal. If $R_A(\mathfrak{m}, I)$ is Cohen–Macaulay, then*

$$\operatorname{ord}(I) = \sum_{k=0}^{d-2} \binom{d-2}{k} (-1)^{d-k} \mu(m^k I) - d + 1.$$

Proof. We first recall from [30, Lemma 1.1] that $\operatorname{ord}(I) = [\mathfrak{m}^{[d-1]}, I^{[1]}]$. According to Theorem 6.1 we now have

$$\begin{aligned}&[\mathfrak{m}^{[d-1]}, I^{[1]}] \\ &\quad = \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^{d-k} l_A(A/m^k) + \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^{d-1-k} l_A(A/m^k I).\end{aligned}$$

Clearly

$$\begin{aligned}&\sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^{d-k} l_A(A/m^k) \\ &\quad = \sum_{k=1}^{d-1} (-1)^{d-k} \binom{d-1}{k} \binom{d+k-1}{k-1} \\ &\quad = \sum_{k=0}^{d-2} (-1)^{d-1-k} \binom{d-1}{k+1} \binom{d+k}{k}.\end{aligned}$$

By considering the coefficient of t^{d-2} in

$$(1-t)^{d-1} \frac{1}{(1-t)^{d+1}} = \frac{1}{(1-t)^2},$$

we obtain

$$\sum_{k=0}^{d-2} (-1)^{d-1-k} \binom{d-1}{k+1} \binom{d+k}{k} = -d + 1.$$

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Since

$$\binom{d-1}{k} = \binom{d-2}{k-1} + \binom{d-2}{k},$$

we observe that

$$\begin{aligned}
&\sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^{d-1-k} l_A(A/m^k I) \\
&\quad = \sum_{k=1}^{d-1} \binom{d-2}{k-1} (-1)^{d-1-k} l_A(A/m^k I) + \sum_{k=0}^{d-2} \binom{d-2}{k} (-1)^{d-1-k} l_A(A/m^k I) \\
&\quad = \sum_{k=0}^{d-2} \binom{d-2}{k} (-1)^{d-k} l_A(A/m^{k+1} I) + \sum_{k=0}^{d-2} \binom{d-2}{k} (-1)^{d-1-k} l_A(A/m^k I) \\
&\quad = \sum_{k=0}^{d-2} \binom{d-2}{k} (-1)^{d-k} (l_A(A/m^{k+1} I) - l_A(A/m^k I)) \\
&\quad = \sum_{k=0}^{d-2} \binom{d-2}{k} (-1)^{d-k} l_A(m^k I/m^{k+1} I) \\
&\quad = \sum_{k=0}^{d-2} \binom{d-2}{k} (-1)^{d-k} \mu(m^k I).
\end{aligned}$$

We have thus proven the claim.

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