

DIFFERENTIAL SIMPLICITY AND CYCLIC MAXIMAL IDEALS OF THE WEYL ALGEBRA $A_2(K)$

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Abstract. Let K be a field of characteristic zero and $A_2 := A_2(K)$ the 2nd–Weyl algebra over K . We establish a close connection between the maximal left ideals of A_2 and the simple derivations of $K[X_1, X_2]$.

MAIN THEOREM. Let $d = \partial_1 + \beta\partial_2$ be a simple derivation of $K[X_1, X_2]$ with $\beta \in K[X_1, X_2]$. Then, there exists $\gamma \in K[X_1, X_2]$ such that $d + \gamma$ generates a maximal left ideal of A_2 . More precisely, the following is true:

- (1) $\deg_{X_2}(\beta) \geq 2$ or $\deg_{X_2}(\beta) = 1$ and $\deg_{X_1}(\partial_2(\beta)) \geq 1$;
- (2) $d + gX_2$ generates a maximal left ideal of A_2 if $g \in K[X_1] \setminus \{0\}$ is such that
 - (a) $g \notin -(\partial_2^2(\beta)/2)\mathbb{N}$ when $\deg_{X_2}(\beta) \geq 2$,
 - (b) $\deg_{X_1}(g) < \deg_{X_1}(\partial_2(\beta))$ when $\deg_{X_2}(\beta) = 1$.

As applications, we obtain large families of concrete examples of cyclic maximal left ideals of A_2 ; such examples have been rather scarce so far.

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1. Introduction. Let K be a field of characteristic zero and $A_2 := A_2(K)$ the 2nd Weyl algebra over K .

In 1988, Bernstein and Lunts showed in [1] that the “generic” differential operator of A_2 with total degree $m \geq 4$ generates a maximal left ideal of A_2 . This result however

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is not effective and the only concrete known examples of cyclic maximal left ideals of A_2 are the following:

$$A_2.(\partial_1 + (\lambda X_1 X_2 + 1)\partial_2 + 2X_2) \text{ with } \lambda \in K \setminus \{0\},$$

$$A_2.(\partial_1 + (\lambda X_2^2 + X_1 X_2 + 1)\partial_2 + \nu X_2) \text{ with } \lambda \in K \setminus \mathbb{Q}, \nu \in K \text{ and } \nu/\lambda \notin \mathbb{Z},$$

that were given by Stafford in [10] for a field K of infinite degree over \mathbb{Q} , and

$$A_2.(\partial_1 + (aX_2 + b)\partial_2 + hX_2) \text{ with } a, b, h \in K[X_1] \setminus \{0\}, \text{ and } \deg(a) > \max\{\deg(b), \deg(h)\},$$

that was given by Coutinho in [3], and variations of them.

In this paper, we shall establish a close relationship between the cyclic maximal left ideals of A_2 and the simple derivations of $K[X_1, X_2]$.

MAIN THEOREM. *Let $d = \partial_1 + \beta\partial_2$ be a simple derivation of $K[X_1, X_2]$ with $\beta \in K[X_1, X_2]$. Then, there exists $\gamma \in K[X_1, X_2]$ such that $d + \gamma$ generates a maximal left ideal of A_2 .⁴ More precisely, the following is true:*

- (1) $\deg_{X_2}(\beta) \geq 2$ or $\deg_{X_2}(\beta) = 1$ and $\deg_{X_1}(\partial_2(\beta)) \geq 1$;
- (2) $d + gX_2$ generates a maximal left ideal of A_2 if $g \in K[X_1] \setminus \{0\}$ is such that
 - (a) $g \notin -(\partial_2^2(\beta)/2)\mathbb{N}$ when $\deg_{X_2}(\beta) \geq 2$,
 - (b) $\deg_{X_1}(g) < \deg_{X_1}(\partial_2(\beta))$ when $\deg_{X_2}(\beta) = 1$.

Through this connection, we obtain some new insight into maximal left ideals of A_2 . In particular, we obtain a rich source of concrete examples of cyclic maximal left ideals of A_2 (hence also of concrete irreducible non-holonomic modules over A_2). Besides recovering all the above concrete examples, we easily produce examples of the type $A_2.(\partial_1 + \beta\partial_2 + \gamma)$ with K an arbitrary field of characteristic zero, $\beta, \gamma \in K[X_1, X_2]$, $\deg_{X_2}(\beta)$ equal to an arbitrary integer ≥ 1 (and not only $\deg_{X_2}(\beta)$ equal to 1 or 2 as in the known cases).

Throughout this paper, K will be an arbitrary field of characteristic zero, X_1, X_2 two indeterminates over K and $A_2 := A_2(K) := K[X_1, X_2]\langle\partial_1, \partial_2\rangle$ the 2nd–Weyl algebra over K , where ∂_i denotes the derivation $\partial/\partial X_i$ of $K[X_1, X_2]$. For an element $f \in K[X_1]$, we shall often use f' instead of $\partial_1(f)$.

Every non-zero element R of $K[X_1, X_2]\langle\partial_2\rangle$ may uniquely be written in the form $R = \sum_{i=0}^r g_i \partial_2^i$, with $g_i \in K[X_1, X_2]$ for every i , $g_r \neq 0$; r is then said to be the order of R . If d is a derivation of $K[X_1, X_2]$, if $\gamma \in K[X_1, X_2]$ and if $S \in K[X_1, X_2]\langle\partial_2\rangle \setminus K$, we say that S is a Darboux operator of $d + \gamma$ if $[d + \gamma, S] \in K[X_1, X_2]S$.

If d is a derivation of a ring B , an ideal I of B is said to be a d –ideal if $d(I) \subseteq I$. If the only d –ideals of B are (0) and (1) , d is said to be a simple derivation; we shall also say that B is d –simple.

2. Proof of the Main Theorem. One important ingredient in the proof of this theorem will be the following result.

THEOREM 2.1. ([5, Theorem 2.7], [2, Theorem 2.2]) *Let $d = \partial_1 + \beta\partial_2$, with $\beta \in K[X_1, X_2]$, be a derivation of $K[X_1, X_2]$. Let $\gamma \in K[X_1, X_2]$. Then the following statements are equivalent:*

- (i) $A_2 \cdot (d + \gamma)$ is a maximal left ideal of A_2 .
- (ii) $d + \gamma$ does not admit any Darboux operator in $K[X_1, X_2]\langle\partial_2\rangle$.

We recall a technical lemma.

⁴Actually, the converse of this statement is also true; this is proved in [5].

LEMMA 2.2. Let $d = \partial_1 + \beta\partial_2$, with $\beta \in K[X_1, X_2]$, be a derivation of $K[X_1, X_2]$ and $\gamma \in K[X_1, X_2]$. Let $j \geq 0$ be an integer and $f_j \in K[X_1, X_2]$. Then

$$[d + \gamma, f_j \partial_2^j] = (d(f_j) - j\partial_2(\beta)f_j) \partial_2^j - f_j \sum_{i=1}^j (C_j^{i+1} \partial_2^{i+1}(\beta) + C_j^i \partial_2^i(\gamma)) \partial_2^{j-i}.$$

Proof. This is an easy computation. □

DEFINITION. Let $d = \partial_1 + \beta\partial_2$, with $\beta \in K[X_1, X_2]$, be a simple derivation of $K[X_1, X_2]$. Let $m \geq 1$ be an integer and $\gamma \in K[X_1, X_2]$. We define $P_m(Z, \gamma)$ in the following way:

$$P_m(Z, \gamma) := \partial_1(Z) + \beta\partial_2(Z) + \partial_2(\beta)Z - (C_m^2 \partial_2^2(\beta) + C_m^1 \partial_2(\gamma)). \tag{1}$$

Proof of the Main Theorem. (1) Since d is a simple derivation of $K[X_1, X_2]$, $\deg_{X_2}(\beta) \geq 1$ because, otherwise, letting $f \beta$ be an element of $K[X_1]$ whose derivative is β , the ideal $(X_2 - f \beta)K[X_1, X_2]$ would be a non-trivial d -ideal of $K[X_1, X_2]$.

Now, if $\deg_{X_2}(\beta) = 1$, say $\beta = aX_2 + b$ with $a, b \in K[X_1]$, $a \neq 0$, then $b \neq 0$ and $\deg(a) \geq 1$. Indeed, if b were equal to zero, then $X_2K[X_1, X_2]$ would be a non-trivial d -ideal of $K[X_1, X_2]$. If a belonged to $K \setminus \{0\}$, then taking $h := -\sum_{i=0}^r \partial_1^i(b)/a^{i+1}$ where $r := \deg_{X_1}(b) \geq 0$, h would be an element of $K[X_1]$ such that $h' = ah + b$, and hence such that $d(X_2 - h) = (aX_2 + b) - (ah + b) = a(X_2 - h)$ and $(X_2 - h)K[X_1, X_2]$ would be a non-trivial d -ideal of $K[X_1, X_2]$.

(2) First we establish the following result.

Claim. Let $m \geq 1$ be an integer and $\gamma \in K[X_1, X_2]$. If $d + \gamma$ admits a Darboux operator of order m in $K[X_1, X_2]\langle \partial_2 \rangle$, then the differential equation $P_m(Z, \gamma) = 0$ has a solution in $K[X_1, X_2]$.

Indeed, by hypothesis, there exists an operator $R = \sum_{i=0}^m f_i \partial_2^i$ with $f_i \in K[X_1, X_2]$ for every i , $f_m \neq 0$ and $g \in K[X_1, X_2]$, such that

$$[d + \gamma, R] = gR. \tag{2}$$

By Lemma 2.2, we have the following equalities:

$$(\text{coefficient of } \partial_2^m \text{ in } [d + \gamma, R]) = d(f_m) - m\partial_2(\beta)f_m, \tag{3}$$

$$(\text{coefficient of } \partial_2^{m-1} \text{ in } [d + \gamma, R]) = (d(f_{m-1}) - (m-1)\partial_2(\beta)f_{m-1}) - (C_m^2 \partial_2^2(\beta) + C_m^1 \partial_2(\gamma))f_m. \tag{4}$$

Evidently, we also have

$$gR = \sum_{i=0}^m g f_i \partial_2^i,$$

Hence, by (2) and (3), we get

$$d(f_m) = (g + m\partial_2(\beta))f_m. \tag{5}$$

Thus $f_mK[X_1, X_2]$ is a non-zero d -ideal of $K[X_1, X_2]$, and therefore

$$f_m \in K \setminus \{0\},$$

since $K[X_1, X_2]$ is d -simple. Then $d(f_m) = 0$,

$$g = -m\partial_2(\beta) \tag{6}$$

by (5), and

$$(\text{coefficient of } \partial_2^{m-1} \text{ in } gR) = -m\partial_2(\beta)f_{m-1}. \tag{7}$$

Now, looking at the terms in ∂_2^{m-1} in (2), we have, by (4) and (7), $d(f_{m-1}) + \partial_2(\beta)f_{m-1} - (C_m^2\partial_2^2(\beta) + C_m^1\partial_2^1(\gamma))f_m = 0$, that is

$$\partial_1(f_{m-1}) + \beta\partial_2(f_{m-1}) + \partial_2(\beta)f_{m-1} - (C_m^2\partial_2^2(\beta) + C_m^1\partial_2^1(\gamma))f_m = 0.$$

Evidently, $(f_{m-1}/f_m) \in K[X_1, X_2]$ since $f_m \in K \setminus \{0\}$ and, dividing the last equality by f_m , we obtain $P_m(f_{m-1}/f_m, \gamma) = 0$. Our claim is then proved.

Now, if $\deg_{X_2}(\beta) = 1$, write $\beta = aX_2 + b$ with $a, b \in K[X_1]$, $\deg a \geq 1$, and take $g \in K[X_1] \setminus \{0\}$ such that $\deg_{X_1}(g) < \deg_{X_1}(a)$. Let $m \geq 1$ be an integer and suppose that the equation $P_m(Z, gX_2) = 0$ has a solution $f \in K[X_1, X_2]$:

$$\partial_1(f) + (aX_2 + b)\partial_2(f) + af - mg = 0. \tag{8}$$

Clearly, we have $f \neq 0$. Write

$$f = \sum_{i=0}^r f_i X_2^i, \text{ with } f_i \in K[X_1] \text{ for every } i = 0, \dots, r, f_r \neq 0.$$

If $r \geq 1$, then looking at the coefficient of X_2^r in equality (8), we obtain $f'_r + raf_r + af_r = 0$. Hence $a(r + 1)f_r = -f'_r$, which is absurd, since $a \neq 0$ and $r + 1 \neq 0$. If $r = 0$, then by (8) we have $f'_0 + af_0 - mg = 0$, which is also absurd since $\deg(g) < \deg(a)$ by hypothesis. Thus, for every integer $m \geq 1$, the equation $P_m(Z, gX_2) = 0$ does not have any solution in $K[X_1, X_2]$. Hence the operator $d + gX_2$ does not admit any Darboux operator in $K[X_1, X_2][\partial_2]$ of order m by the above claim. Then $A_2 \cdot (d + gX_2)$ is a maximal left ideal of A_2 by Theorem 2.1.

If $\deg_{X_2}(\beta) \geq 2$, take $g \in K[X_1]$ such that $g \notin -(\partial_2^2(\beta)/2)\mathbb{N}$. Let $m \geq 1$ be an integer and suppose that the equation $P_m(Z, gX_2) = 0$ has a solution $f \in K[X_1, X_2]$

$$\partial_1(f) + \beta\partial_2(f) + \partial_2(\beta)f - (m(m - 1)/2)\partial_2^2(\beta) - mg = 0.$$

This equality can also be written as

$$\partial_1(f) + \partial_2(\beta f) - (m(m - 1)/2)\partial_2^2(\beta) - mg = 0.$$

By the choice of g , we have $f \neq 0$. Then, the above equality is absurd: indeed, since $\deg_{X_2}(\beta) \geq 2$, the degree in X_2 of its left side is equal to $\deg_{X_2}(\beta) + \deg_{X_2}(f) - 1 \geq 1$. Thus, for every integer $m \geq 1$, the equation $P_m(Z, gX_2) = 0$ does not have any solution in $K[X_1, X_2]$. Hence the operator $d + gX_2$ does not admit any Darboux operator in $K[X_1, X_2][\partial_2]$ of order m by the above claim. Then $A_2 \cdot (d + gX_2)$ is a maximal left ideal of A_2 by Theorem 2.1. □

EXAMPLE 2.7. Cyclic maximal left ideals of A_2 .

The Main Theorem provides a powerful tool to produce explicit examples of cyclic maximal left ideals of A_2 : one considers a simple derivation of $K[X_1, X_2]$ of the type $d = \partial_1 + \beta\partial_2$ and then takes $A_2 \cdot (d + gX_2)$ where $g \in K[X_1] \setminus \{0\}$, $g \notin -(\partial_2^2(\beta)/2)\mathbb{N}$ when $\deg_{X_2}(\beta) \geq 2$ and $\deg_{X_1}(g) < \deg_{X_1}(\partial_2(\beta))$ when $\deg_{X_2}(\beta) = 1$.

Simple derivations of the type $d = \partial_1 + \beta\partial_2$ with $\beta \in K[X_1, X_2]$ can be found in the literature. We give a few examples.

Case $\deg_{X_2}(\beta) = 1$.

The best known example in this case is obtained by taking

$$\beta = aX_2 + b \text{ with } a, b \in K[X_1] \setminus \{0\}, \deg(a) > \deg(b).$$

The simplicity of d is an immediate consequence of a theorem of Shamsuddin [7, Theorem 3.2.1, p. 151] or [3, Proposition 3.2, p. 410].

Other interesting “linear” examples can be obtained by using an algorithm that has been established in [7, p. 152–155]; for example, we get one of them by taking

$$\beta = X_1X_2 + (X_1^3 + 1).$$

Case $\deg_{X_2}(\beta) = 2$.

Two important examples in this case are obtained by taking

(i) $\beta = X_2^2 - p(X_1)$ with $p(X_1) \in K[X_1]$, $\deg(p(X_1))$ odd,

(ii) $\beta = X_2^2 - (X_1^2 - e)$ with $e \in K \setminus \{2k + 1; k \in \mathbb{Z}\}$.

The simplicity of such d 's is proved in [6].

Other interesting “quadratic” examples are given in the same paper.

Case $\deg_{X_2}(\beta) = n$, $n \geq 2$ arbitrary.

One important example in this case is obtained by taking

$$\beta = X_2^n + pX_1 \text{ with } p \in K \setminus \{0\}.$$

The simplicity of d is proved in [8].

Another example is given in [4].

if $\beta \in \mathbb{Q}[X_1, X_2]$ is an homogeneous polynomial of degree $n \geq 2$ that is irreducible over \mathbb{Q} , then $d := \partial_1 + \beta\partial_2$ is a simple derivation of $\mathbb{C}[X_1, X_2]$.

REMARK 2.8. (a) The family of examples of Coutinho (hence also the first family of examples of Stafford), given in the introduction can easily be obtained using our Main Theorem. Indeed, if $\beta = aX_2 + b$ with $a, b \in K[X_1] \setminus \{0\}$, $\deg(a) > \deg(b)$, then $d := \partial_1 + \beta\partial_2$ is a simple derivation of $K[X_1, X_2]$ as was mentioned above. Then, if we take $g \in K[X_1] \setminus \{0\}$ with $\deg(g) < \deg(a)$, we can conclude that $A_2 \cdot (d + gX_2)$ is a maximal left ideal of A_2 .

(b) Similarly, the second family of examples of Stafford given in the introduction can also be obtained with our Main Theorem. Indeed, if $\beta = \lambda X_2^2 + X_1X_2 + 1$ with $\lambda \in K \setminus \mathbb{Z}$, then by [6, p. 5106–5107 and Theorem 8.3 p. 5108], $d := \partial_1 + \beta\partial_2$ is a simple derivation of $K[X_1, X_2]$. Now, if we take $\gamma = \nu X_2$ with $\nu \in K[X_1] \setminus \{0\}$, $\nu/\lambda \notin -\mathbb{N}$, then $\nu \notin -(\partial_2^2(\beta)/2)\mathbb{N}$ and we can conclude that $A_2 \cdot (\partial_1 + (\lambda X_2^2 + X_1X_2 + 1)\partial_2 + \nu X_2)$ is a maximal left ideal of A_2 . Note that with our method, and contrary to what was done by Stafford, we need not suppose that K is a field of infinite degree over \mathbb{Q} (we only need to know that K contains \mathbb{Q}), nor do we have to suppose that $\lambda \in K \setminus \mathbb{Q}$ (we only need to know that $\lambda \in K \setminus \mathbb{Z}$).

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