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FRACTIONAL CONVOLUTION

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Abstract

A continuous one-parameter set of binary operators on $L^2(\mathbb{R})$ called fractional convolution operators and which includes those of multiplication and convolution as particular cases is constructed by means of the Condon-Bargmann fractional Fourier transform. A fractional convolution theorem generalizes the standard Fourier convolution theorems and a fractional unit distribution generalizes the unit and delta distributions. Some explicit double-integral formulas for the fractional convolution between two functions are given and the induced operation between their corresponding Wigner distributions is found.

1. Introduction

The operation $*$, of convolution, on the space $L^2(\mathbb{R})$ is the dual under the Fourier-Plancherel operator $\mathscr{F}$ of the operation $\times$, of multiplication [6]; that is, defining

$$\mathscr{F} f(y) = \widehat{f}(y) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-iyx} f(x)\, dx \tag{1.1}$$

and

$$f * g(y) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} f(y-x) g(x)\, dx \tag{1.2}$$

one gets for the convolution theorem the dual pair

$$\widehat{f \times g} = \widehat{f} * \widehat{g} \quad \text{and} \quad \widehat{f * g} = \widehat{f} \times \widehat{g}. \tag{1.3}$$

In the context of signal processing the multiplication of signal f by signal g corresponds to a modulation of f by g whereas the convolution of f with g corresponds to a filtering of f by the filter with spectral response $\widehat{g}$ [10]. It is of interest to see whether these two operations can be extended to a one-parameter family of operations $\{*_\theta\}$ in

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which $*$ and $\times$ appear just as particular cases. The operations $*_\theta$ between multiplication and convolution would then correspond to an influence between modulation and filtering.

In this paper I construct such a set of "fractional convolution" operators $\{*_\theta\}_{\theta\in\mathbb{T}}$ (where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$), in which $*_0 = \times$ and $*_{\pi/2} = *$, by means of the Condon-Bargmann fractional Fourier transform [1, 4, 8, 9] and investigate some of its formal properties. The operator $*_\theta$ is commutative, associative and bilinear and obeys a fractional convolution theorem that includes both results (1.3) as particular cases. I find a "fractional unit" for $*_\theta$ that generalizes the units under $\times$ and $*$ and a deconvolution formula.

From the initial triple-integral construction of $f *_\theta g$ I get some other representations. Two are double-integral formulas and one shows $f *_\theta g$ as a product of certain elementary operations and Fourier transform operations.

In another paper [9] I have shown the Radon-transform relationship between the Condon-Bargmann fractional Fourier transform $\mathscr{F}_\theta f$ and the Wigner distribution [2, 7, 11] W_f of f. The Wigner distributions W_{fg} and W_{f*g} are equal to one-dimensional convolutions of W_f and W_g in the directions of the two axes in the Wigner plane. I define a θ-angled one-dimensional convolution, also denoted by $*_\theta$, between W_f and W_g, generalizing the two axial ones, and show that $W_{f*_\theta g} = W_f *_\theta W_g$, which generalizes and extends the earlier results.

2. The fractional convolution operator $*_\theta$

The integer powers of $\mathscr{F}$ form a cyclic group of order 4 of unitary operators on $L^2(\mathbb{R})$ [6] in which the inner product and associated 2-norm are defined by

$$\langle f, g\rangle = (2\pi)^{-1/2}\int_{\mathbb{R}} \bar{f}(x)g(x)\,dx \quad \text{and} \quad \|f\| = \langle f, f\rangle^{1/2}.$$

This finite discrete group can be imbedded in a continuous one-parameter group of unitary operators, $\{\mathscr{F}_\theta\}_{\theta\in\mathbb{T}}$, the Condon-Bargmann group of fractional Fourier transforms [1, 4, 8, 9], obeying

$$\forall\,\theta_1, \theta_2 \in \mathbb{T} \qquad \mathscr{F}_{\theta_1}\mathscr{F}_{\theta_2} = \mathscr{F}_{\theta_1+\theta_2} \text{ and } \forall\, k \in \mathbb{Z} \qquad \mathscr{F}^k = \mathscr{F}_{k\pi/2}. \tag{2.1}$$

This one-dimensional fractional Fourier operator is defined by

$$\mathscr{F}_\theta f(y) = f_\theta(y) = (2\pi)^{-1/2}\int_{\mathbb{R}} K_\theta(x, y) f(x)\,dx \qquad (\theta \in \mathbb{T}), \tag{2.2a}$$

where

$$K_0(x, y) = \sqrt{2\pi}\,\delta(x - y) \quad \text{and} \quad K_\pi(x, y) = \sqrt{2\pi}\,\delta(x + y) \tag{2.2b}$$

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(giving $\mathscr{F}_0 = \mathscr{F}^0 = \mathscr{I}$, the identity operator, and $\mathscr{F}_\pi = \mathscr{F}^2 = \mathscr{P}$, the reversal operator, defined by $\mathscr{P}f(x) = f(-x)$) and where for $0 < |\theta| < \pi$

$$K_\theta(x, y) = A_\theta \exp\left[\frac{-i}{2\sin\theta}\left\{-\left(x^2 + y^2\right)\cos\theta + 2xy\right\}\right], \tag{2.2c}$$

where

$$A_\theta = |\sin\theta|^{-1/2} \exp\left[-\frac{i}{2}\left(\frac{\pi}{2}\operatorname{sgn}\theta - \theta\right)\right]. \tag{2.2d}$$

If one rewrites (1.3) replacing $*$ by "$*_{\pi/2}$" and $\times$ by "$*_0$" and using the notation of (2.2a) they suggest a generalization to a theorem involving a fractional convolution operator and a definition of $*_\theta$ *provided* that $*_\theta$ satisfies $*_{\pi/2} = *_{-\pi/2}$ and $*_0 = *_\pi$.

DEFINITION 2.1. Let $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then the fractional convolution $f *_\theta g$ is defined by

$$f *_\theta g = (f_{-\theta} g_{-\theta})_\theta \qquad (\theta \in \mathbb{T}). \tag{2.3}$$

PROPOSITION 2.1.

$$\forall \phi \in \mathbb{T} \qquad *_\phi = *_{\phi+\pi}. \tag{2.4}$$

PROOF. From Definition 2.1 and (2.1)

$$f_0 *_{\phi+\pi} g_0 = \left(f_{-(\phi+\pi)} g_{-(\phi+\pi)}\right)_{\phi+\pi} = \left[\mathscr{F}_\pi\left(f_{-(\phi+\pi)} g_{-(\phi+\pi)}\right)\right]_\phi. \tag{2.5}$$

But $\mathscr{F}_\pi = \mathscr{P}$, the reversal operator, for which $\mathscr{P}(fg) = (\mathscr{P}f)(\mathscr{P}g) = f_\pi g_\pi$, so

$$\mathscr{F}_\pi\left(f_{-(\phi+\pi)} g_{-(\phi+\pi)}\right) = f_{-(\phi+\pi)+\pi} g_{-(\phi+\pi)+\pi} = f_{-\phi} g_{-\phi}. \tag{2.6}$$

Using (2.6) in (2.5):

$$f_0 *_{\phi+\pi} g_0 = \left(f_{-\phi} g_{-\phi}\right)_\phi;$$

that is, by (2.3),

$$f_0 *_{\phi+\pi} g_0 = f_0 *_\phi g_0,$$

which is just what the proposition claims.

COROLLARY.

$$*_0 = *_\pi \quad \textit{and} \quad *_{\pi/2} = *_{-\pi/2}; \tag{2.7}$$

so the definition of $*_\theta$ *does satisfy the proviso made earlier.*

We are now ready for the fractional convolution theorem.

THEOREM 2.1. *Let* $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *then*

$$\forall\, \theta, \phi \in \mathbb{T} \qquad \left(f_0 *_\phi g_0\right)_\theta = f_\theta *_{\phi+\theta} g_\theta. \tag{2.8}$$

PROOF. From (2.3) and using (2.1)

$$\left(f_0 *_\phi g_0\right)_\theta = \left(\left(f_{-\phi} g_{-\phi}\right)_\phi\right)_\theta = \left(f_{-\phi} g_{-\phi}\right)_{\phi+\theta} \tag{2.9}$$

and

$$f_\theta *_{\phi+\theta} g_\theta = \left(f_{\theta-(\phi+\theta)} g_{\theta-(\phi+\theta)}\right)_{\phi+\theta} = \left(f_{-\phi} g_{-\phi}\right)_{\phi+\theta};$$

that is, using (2.9), the required result, (2.8).

COROLLARY. *Taking* $\phi = 0$ *and* $\theta = \pi/2$ (2.8) *yields* $\widehat{fg} = \widehat{f} * \widehat{g}$; *then taking* $\phi = \pi/2$ *and* $\theta = \pi/2$ *and using* (2.7) (2.8) *yields* $\widehat{f * g} = \widehat{f}\widehat{g}$ *so both the standard convolution results appear as particular cases of the fractional convolution theorem.*

PROPOSITION 2.2. $*_\theta$ *is associative; that is,*

$$\forall\, \theta \in \mathbb{T} \quad \forall\, f, g, h \qquad f *_\theta (g *_\theta h) = (f *_\theta g) *_\theta h. \tag{2.10}$$

PROOF. From the definition (2.3) and (2.1)

$$f *_\theta (g *_\theta h) = f *_\theta (g_{-\theta} h_{-\theta})_\theta = \left(f_{-\theta}\, (g_{-\theta} h_{-\theta})_{\theta-\theta}\right)_\theta = (f_{-\theta} g_{-\theta} h_{-\theta})_\theta$$

from which the result is obvious.

REMARK. Generally for $(\theta_1 - \theta_2)/\pi \notin \mathbb{Z} \qquad f *_{\theta_1} \left(g *_{\theta_2} h\right) \neq (f *_{\theta_1} g) *_{\theta_2} h.$

3. The fractional convolution unit 1_θ and deconvolution

DEFINITION 3.1. Define the distribution 1_θ by

$$1_\theta = \mathscr{F}_\theta 1 \qquad (\theta \in \mathbb{T}). \tag{3.1}$$

The relations between the unit for $\times$, 1, and the unit for $*$ (as in (1.2)), $\sqrt{2\pi}\,\delta$, are now expressed $1_{\pi/2} = \sqrt{2\pi}\,\delta$ and $\sqrt{2\pi}\,\delta_{\pi/2} = 1$ and so

$$1_\theta = \sqrt{2\pi}\,\delta_{\pi/2+\theta}. \tag{3.2}$$

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From (2.2) one gets immediately for $0 < |\theta| < \pi$

$$\sqrt{2\pi}\,\delta_\theta(y) = |\sin\theta|^{-1/2}\exp\left[-\frac{i}{2}\left(\frac{\pi}{2}\operatorname{sgn}\theta - \theta - y^2\cot\theta\right)\right]. \tag{3.3}$$

Replacing θ by $\pi/2 + \theta$ in (3.3) and using (3.2) one gets for $|\theta| < \pi/2$ the explicit function

$$1_\theta(y) = |\cos\theta|^{-1/2}\exp\left[-\frac{i}{2}\left(-\theta + y^2\tan\theta\right)\right]. \tag{3.4}$$

PROPOSITION 3.1. *1_θ is the unit under fractional convolution $*_\theta$; that is,*

$$\forall\, f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \quad 1_\theta *_\theta f = f. \tag{3.5}$$

PROOF. From the definitions of $*_\theta$ and 1_θ in (2.3) and (3.1) and (2.1)

$$1_\theta *_\theta f = (1_{\theta-\theta} f_{-\theta})_\theta = (1_0 f_{-\theta})_\theta = f_{-\theta+\theta} = f.$$

Given $f *_\theta g$, where f is known, then to *deconvolve* it by a further fractional convolution with some x, so as to recover g, means x must satisfy

$$\forall g \quad x *_\theta (f *_\theta g) = g. \tag{3.6}$$

By Propositions 2.2 and 3.1 this means x must satisfy

$$x *_\theta f = 1_\theta; \tag{3.7}$$

that is, x is a convolutional inverse of f.

PROPOSITION 3.2. *If $(1/f_{-\theta})_\theta$ exists then it is a convolutional inverse of f and solves the deconvolution problem* (3.6).

PROOF. Formally solve (3.7) for x using (2.3), (3.1) and (2.1).

(If the given f is in $L^2(\mathbb{R})$ then $1/f_{-\theta}$ is not and so $(1/f_{-\theta})_\theta$ exists only in a distributional sense.)

4. Explicit formulas for $f *_\theta g$

The calculation of a fractional convolution directly from Definition 2.1 involves three integrations however a little manipulation yields double-integral formulas.

One gets

$$f *_\theta g(t) = (2\pi)^{-1} \int_{\mathbb{R}^2} C_\theta(u, v; t) f(u) g(v)\, du\, dv, \tag{4.1a}$$

where

$$C_\theta(u, v; t) = |\sin\theta|^{-1} \exp[i(u-t)(v-t)\cot\theta] 1_{-\theta}(u+v-t), \tag{4.1b}$$

a double-integral formula symmetric in f and g.

Changing the variables in (4.1) by putting $z = u + v - t$ and $w = u - v$ leads to

$$f *_\theta g(t) = (2\pi)^{-1/2} \int_{\mathbb{R}} \frac{\exp[i(\frac{z-t}{2})^2 \cot\theta]}{|\sin\theta|^{1/2}} 1_{-\theta}(z) Q_\theta(f, g; z+t)\, dz, \tag{4.2a}$$

where

$$Q_\theta(f, g; z) = (2\pi)^{-1/2} \int_{\mathbb{R}} \frac{\exp[-i\, w^2 \cot\theta]}{|\sin\theta|^{1/2}} f\left(\frac{z}{2} + w\right) g\left(\frac{z}{2} - w\right) dw, \tag{4.2b}$$

a repeated integral, again symmetric in f and g.

To develop numerical algorithms to approximate $f *_\theta g$ it may be useful to represent it as the result of elementary operations for which efficient algorithms are already known.

Define the chirp and scaling groups of unitary operators $\{C_a\}_{a\in\mathbb{R}}$ and $\{S_a\}_{a\in\mathbb{R}^*}$ (where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$) by $C_a f(t) = \exp[iat^2/2] f(t)$ and $S_a f(t) = |a|^{1/2} f(at)$ then the fractional Fourier transform $\mathscr{F}_\theta$ of (2.2) can be written

$$\mathscr{F}_\theta = \alpha_\theta C_{\cot\theta} S_{csc\theta} \mathscr{F} C_{\cot\theta} \quad \text{where} \quad \alpha_\theta = \exp\left[-i\frac{1}{2}\left(\frac{\pi}{2}\operatorname{sgn}\theta - \theta\right)\right].$$

Applying this representation of $\mathscr{F}_\theta$ to the definition of $f *_\theta g$ yields

$$f *_\theta g = A_{-\theta} C_{\cot\theta} \mathscr{F}^{-1} C_{-\cos\theta\sin\theta} \mathscr{F}\left[(C_{-\cot\theta} f) * (C_{-\cot\theta} g)\right], \tag{4.3}$$

where A_θ is defined in (2.2d).

5. Fractional convolution and the Wigner distribution

One member of the Cohen class $\{C_f\}$ of generalised phase-space distributions [3] associated with a function $f \in L^2(\mathbb{R})$ is the Wigner distribution [11] $W : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ where $f \mapsto W_f$ and

$$W_f(\mathbf{x}) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix_2 p} f\left(x_1 + \frac{p}{2}\right) \overline{f}\left(x_1 - \frac{p}{2}\right) dp. \tag{5.1}$$

Many members of the Cohen class have marginal distributions along the two axes given by

$$(2\pi)^{-1/2} \int_{\mathbb{R}} C_f(\mathbf{x})\, dx_1 = |\widehat{f}(x_2)|^2 \quad \text{and} \quad (2\pi)^{-1/2} \int_{\mathbb{R}} C_f(\mathbf{x})\, dx_2 = |f(x_1)|^2$$

and this has the natural generalization that

$$\forall\, \theta \in \mathbb{T} \qquad (2\pi)^{-1/2} \int_{\ell(r,\theta)} C_f(x)\, d\ell = |f_\theta(r)|^2\,, \tag{5.2}$$

where $d\ell$ is the element of Euclidean arc length along the line $\ell(r, \theta)$ whose equation is $x_1 \cos\theta + x_2 \sin\theta = r$. This generalization states that the Radon transform [5] of C_f is the energy-density function $|f_\theta(r)|^2$ of the fractional Fourier transform regarded as a function on $\mathbb{R}^2$ in polar coordinates r and θ. In another paper [9] I have shown that this Radon-transform relationship (2) with the fractional Fourier transform holds *only* for the Wigner distribution W_f.

One now naturally asks what is the operation between W_f and W_g that is induced by fractional convolution under the map $f \mapsto W_f$. First define $*^1$ and $*^2$ as the one-dimensional convolutions in the Wigner plane with respect to the first and second arguments.

DEFINITION 5.1.

$$W_f *^1 W_g(\mathbf{x}) = (2\pi)^{-1/2} \int_{\mathbb{R}} W_f(x_1 - u, x_2) W_g(u, x_2)\, du \tag{5.3a}$$

and

$$W_f *^2 W_g(\mathbf{x}) = (2\pi)^{-1/2} \int_{\mathbb{R}} W_f(x_1, x_2 - u) W_g(x_1, u)\, du. \tag{5.3b}$$

It is easy to show the following relationships linking multiplication and convolution between f and g to $*^1$ and $*^2$ between W_f and W_g.

PROPOSITION 5.1.

$$W_{f*g} = W_f *^1 W_g \quad \textit{and} \quad W_{fg} = W_f *^2 W_g. \tag{5.4}$$

I now define "convolution in direction θ" on the Wigner plane and denote it also by "$*_\theta$".

DEFINITION 5.2. Let

$$P_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad W_{\theta,f}(\mathbf{x}) = W_f(P_\theta \mathbf{x}) \tag{5.5}$$

then define $W_f *_\theta W_g$ by

$$W_f *_\theta W_g(\mathbf{x}) = W_{-\theta,f} *^2 W_{-\theta,g}(P_\theta \mathbf{x}). \tag{5.6}$$

One can show (for example, from the more general result of Proposition 4.28 in [7] and using there $\mathscr{A} = P_\theta$ from above) that

$$W_{\theta,f} = W_{f_\theta}. \tag{5.7}$$

The generalization of the results of Proposition 5.1 and the answer to the question raised earlier are contained in the following theorem.

THEOREM 5.1.

$$\forall\theta \in \mathbb{T}, \quad \forall f, g \in L^2(\mathbb{R}) \qquad W_{f*_\theta g} = W_f *_\theta W_g. \tag{5.8}$$

PROOF.

$$\begin{aligned} W_{f*_\theta g}(\mathbf{x}) &= W_{(f_{-\theta}g_{-\theta})_\theta}(\mathbf{x}) \quad \text{(by Definition 2.1)} \\ &= W_{f_{-\theta}g_{-\theta}}(P_\theta \mathbf{x}) \quad \text{(by (5.7) and (5.5))} \\ &= W_{f_{-\theta}} *^2 W_{g_{-\theta}}(P_\theta \mathbf{x}) \quad \text{(by (5.4))} \\ &= W_{-\theta,f} *^2 W_{-\theta,g}(P_\theta \mathbf{x}) \quad \text{(by (5.7))} \\ &= W_f *_\theta W_g(\mathbf{x}) \quad \text{(by Definition 5.2).} \end{aligned}$$

COROLLARY. *Choosing $\theta = 0$ and $\theta = \pi/2$ one recovers as particular cases the results of Proposition* 5.1.

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