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NOTES ON ERDÖS-TURÁN INEQUALITY

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Abstract

A new version of Erdös-Turán's inequality is described. The purpose of the present paper is to show that the inequality provides better upper bounds for the discrepancies of some sequences than usual Erdös-Turán's inequality.

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1. Introduction

Let $\{x\} = x - [x]$ be the fractional part of $x \in \mathbb{R}$. For a subinterval E of the unit interval $U = [0, 1)$, the characteristic function of E is defined by $\chi_E(x) = 1$ for $\{x\} \in E$ and $\chi_E(x) = 0$ otherwise. Let $e(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$. Let $\omega = (x_n)$, $n = 1, 2, \ldots$, be an infinite sequence in $\mathbb{R}$. For a positive integer N, the discrepancy of the sequence ω is defined by

$$(1) \qquad D_N(\omega) = \sup_{0<y\leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} \chi_{[0,y)}(x_n) - y \right| .$$

Erdös-Turán's inequality is very useful to obtain an upper bound of the discrepancy ([2, 3]). Baker and Harman gave a new version of Erdös-Turán's inequality for the logarithmic discrepancy, which is defined by adapting the logarithmic mean instead of the arithmetic mean to (1) [1, Lemma 1]. By using the techniques developed by Baker and Harman [1, Lemma 1], we can analogously obtain Theorem 1, so we omit the proof of Theorem 1.

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THEOREM 1. *Let* $\omega = (x_n)$ *be a real sequence and let* $0 < \delta \le 1$. *Then there exists a constant* $C(\delta) > 0$ *such that*

$$D_N(\omega) \le F(N) + \frac{C(\delta)}{N} \sum_{1 \le h \le N^\delta} \frac{1}{h} \sup_{h^{1/\delta} < B \le N} \left| \sum_{h^{1/\delta} \le n \le B} e(hx_n) \right|, \tag{2}$$

where

$$F(N) = \begin{cases} \left(\dfrac{1}{2^{1-\delta} - 1} + 1\right) \dfrac{1}{N^\delta} & \text{for } 0 < \delta < 1, \\ \left(\dfrac{1}{\log 2} + 1\right) \dfrac{1 + \log N}{N} & \text{for } \delta = 1. \end{cases}$$

2. Main results

The purpose of the present paper is to show that Theorem 1 provides better upper bounds for the discrepancies of some sequences than usual Erdös-Turán's inequality. The following is an analogue of Baker and Harman's theorem [1, Theorem 2].

THEOREM 2. *Let* $f(x)$, $x \ge 1$, *be a real-valued twice differentiable function such that* $f''(x) \ll x^{-2+\epsilon}$ *for some* $0 < \epsilon < 1/2$. *Suppose that there are real numbers* $1 = x_0 < x_1 < \cdots < x_H < \infty$ *such that* $f''(x)$ *is of constant sign and monotone in each of the intervals* $[x_{j-1}, x_j]$ $(j = 1, \ldots, H)$ *and* $[x_H, \infty)$. *Then the sequence* $\omega = (f(n))$ *satisfies*

$$D_N(\omega) \ll \frac{1}{N|f''(N)|^{1/2}}. \tag{3}$$

PROOF. We use van der Corput's method for bounding exponential sums. Let $1 \le A < x_H < B$. There exists an integer $1 \le k \le H$ such that $x_{k-1} \le A < x_k$. Let $j \in \mathbb{Z}$ with $k \le j < H$ and let $h \in \mathbb{Z}$ with $h \ge 1$. Suppose that $f''(x) < 0$ in the interval $[x_j, x_{j+1}]$. Let $hf'(x_{j+1}) = \alpha$, $hf'(x_j) = \beta$. Applying [5, Lemma 4.7], we have

$$\sum_{x_j < n \le x_{j+1}} e(hf(n)) = \sum_{\alpha - 1/2 < \nu < \beta + 1/2} \int_{x_j}^{x_{j+1}} e(hf(x) - \nu x)dx + O(\log(\beta - \alpha + 2)). \tag{4}$$

By [5, Lemma 4.4] we get

$$\int_{x_j}^{x_{j+1}} e(hf(x) - \nu x)dx \ll \frac{1}{\min\{|f''(x_j)|, |f''(x_{j+1})|\}^{1/2} h^{1/2}}. \tag{5}$$

Since $f''(x) \ll x^{-2+\epsilon}$, we have

$$\beta - \alpha \ll A^{\epsilon-1}h. \tag{6}$$

From (4), (5) and (6) we get

$$\sum_{x_j < n \le x_{j+1}} e(hf(n)) \ll \frac{(A^{\epsilon-1}h+1)h^{-1/2}}{\min_{j=0,\dots,H} |f''(x_j)|^{1/2}} + \log h. \tag{7}$$

The inequality (7) holds also for the interval $[A, x_k]$. Similarly, we can obtain the same estimate even if $f''(x) > 0$ in the interval $[x_j, x_{j+1}]$.

On the other hand, since $|f''(x)|$ is monotone decreasing on $[x_H, B]$, in like manner we have also

$$\sum_{x_H < n \le B} e(hf(n)) \ll \frac{(A^{\epsilon-1}h+1)h^{-1/2}}{|f''(B)|^{1/2}} + \log h. \tag{8}$$

From (7) and (8) it follows that

$$\sum_{A < n \le B} e(hf(n)) \ll \left(\frac{1}{\min\limits_{j=0,\dots,H} |f''(x_j)|^{1/2}} + \frac{1}{|f''(B)|^{1/2}} \right)(A^{\epsilon-1}h+1)h^{-1/2} + \log h. \tag{9}$$

In the same way we obtain

$$\sum_{A < n \le B} e(hf(n)) \ll \begin{cases} \dfrac{(A^{\epsilon-1}h+1)h^{-1/2}}{\min\limits_{j=0,\dots,H} |f''(x_j)|^{1/2}} + \log h & \text{if } A < x_H \text{ and } B \le x_H, \\ \dfrac{(A^{\epsilon-1}h+1)h^{-1/2}}{|f''(B)|^{1/2}} + \log h & \text{if } x_H \le A \text{ and } x_H < B. \end{cases} \tag{10}$$

Now, we set

$$S_1 = \sum_{1 \le h < x_H^\delta} \frac{1}{h} \sup_{h^{1/\delta} < B \le N} \left| \sum_{h^{1/\delta} \le n \le B} e(hf(n)) \right|,$$

$$S_2 = \sum_{x_H^\delta \le h \le N^\delta} \frac{1}{h} \sup_{h^{1/\delta} < B \le N} \left| \sum_{h^{1/\delta} \le n \le B} e(hf(n)) \right|,$$

where it is assumed that $x_H \le N$.

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First we consider S_1. From (9) and (10) it follows that

$$S_1 = \sum_{1\le h<x_H^\delta} \frac{1}{h} \max\left\{ \sup_{h^{1/\delta}<B\le x_H} \left| \sum_{h^{1/\delta}\le n\le B} e(hf(n)) \right|, \sup_{x_H\le B\le N} \left| \sum_{h^{1/\delta}\le n\le B} e(hf(n)) \right| \right\}$$

$$\ll \sum_{1\le h<x_H^\delta} \frac{1}{h} \left\{ \left(\frac{1}{\min_{j=0,\dots,H} |f''(x_j)|^{1/2}} + \frac{1}{|f''(N)|^{1/2}} \right) \left(h^{(\epsilon-1)/\delta+1} + 1\right) h^{-1/2} + \log h \right\}$$

$$\ll \frac{1}{|f''(N)|^{1/2}} + 1. \tag{11}$$

Furthermore, from (10) we have

$$S_2 \ll \sum_{x_H^\delta\le h\le N^\delta} \frac{1}{h} \sup_{h^{1/\delta}<B\le N} \left\{ \frac{1}{|f''(B)|^{1/2}} \left(h^{(\epsilon-1)/\delta+1} + 1\right) h^{-1/2} + \log h \right\}$$

$$\ll \frac{1}{|f''(N)|^{1/2}} + (\log N)^2, \tag{12}$$

according to $0 < \epsilon < 1/2$. From (11) and (12) we infer

$$\sum_{1\le h\le N^\delta} \frac{1}{h} \sup_{h^{1/\delta}<B\le N} \left| \sum_{h^{1/\delta}\le n\le B} e(hf(n)) \right| \ll \frac{1}{|f''(N)|^{1/2}} + (\log N)^2; \tag{13}$$

and (13) holds even if $N < x_H$. By (13) and Theorem 1 we have

$$D_N(\omega) \ll F(N) + \frac{1}{N|f''(N)|^{1/2}} + \frac{(\log N)^2}{N}, \tag{14}$$

where

$$F(N) \ll \begin{cases} N^{-\delta} & \text{for } 0 < \delta < 1, \\ \dfrac{\log N}{N} & \text{for } \delta = 1. \end{cases}$$

Since $f''(N) \ll N^{-2+\epsilon}$, by choosing $\delta = \epsilon/2$, the desired result follows. □

REMARK 1. Suppose that $f(x)$ satisfies the conditions of Theorem 2. By (9) we have

$$\sum_{n=1}^{N} e(h(f(n))) \ll \left(\frac{1}{|f''(N)|^{1/2}} + 1 \right) h^{1/2}. \tag{15}$$

Applying usual Erdös-Turán's inequality together with (15), for any positive integer m we obtain

$$D_N(\omega) \ll \frac{1}{m} + \frac{m^{1/2}}{N|f''(N)|^{1/2}}. \tag{16}$$

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Choosing $m = [N^{2/3}|f''(N)|^{1/3}]$, from (16) we have

$$D_N(\omega) \ll \frac{1}{N^{2/3}|f''(N)|^{1/3}}. \tag{17}$$

If $N^{-2} \ll |f''(N)|$, then the upper bound of (3) is smaller than that of (17).

As the examples of functions $f(x)$ satisfying the conditions of Theorem 2, we consider $f(x) = \alpha x + \beta(\log x)^\sigma$, $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0, \sigma > 1$, and $f(x) = \alpha x + \beta x^\sigma$, $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, $0 < \sigma < 1/2$. Then we have the following.

COROLLARY 1. *The discrepancy of the sequence $\omega = (\alpha n + \beta(\log n)^\sigma)$ with real numbers α, β with $\beta \neq 0$ and $\sigma > 1$ satisfies*

$$D_N(\omega) \ll (\log N)^{-(\sigma-1)/2}.$$

COROLLARY 2. *The discrepancy of the sequence $\omega = (\alpha n + \beta n^\sigma)$ with real numbers α, β with $\beta \neq 0$ and $0 < \sigma < 1/2$ satisfies*

$$D_N(\omega) \ll N^{-\sigma/2}.$$

The following is another application of Theorem 1.

THEOREM 3. *If α is an irrational number of finite type $\eta \geq 1$ and β is a nonzero real number, then for any $\varepsilon > 0$ the discrepancy of $\omega = (\alpha n + \beta \log n)$ satisfies*

$$D_N(\omega) \ll N^{-1/(\eta+1/2)+\varepsilon}. \tag{18}$$

PROOF. We also use van der Corput's method for bounding exponential sums. Let $g(x) = \alpha x + \beta \log x$ and let $1 \leq A < B$. Applying integration by parts, for integers ν and $h \geq 1$ we have

$$\int_A^B e(hg(x) - \nu x)dx \ll \frac{1}{|h\alpha - \nu|}\left(1 + h\left|\int_A^B \frac{1}{x}e(hg(x) - \nu x)dx\right|\right). \tag{19}$$

Since [1, Lemma 2] implies

$$\int_A^B \frac{1}{x}e(hg(x) - \nu x)dx \ll h^{-1/2},$$

(19) yields

$$\int_A^B e(hg(x) - \nu x)dx \ll \frac{h^{1/2}}{\|h\alpha\|}, \tag{20}$$

where $\|x\| = \min\{|x-n| : n \in \mathbb{Z}\}$ for $x \in \mathbb{R}$. By using [5, Lemma 4.7], from (20) we obtain

$$\sum_{A \le n \le B} e(hg(n)) \ll \left(\frac{h}{A}+1\right)\frac{h^{1/2}}{\|h\alpha\|}. \tag{21}$$

Let $0 < \delta \le 1$. Then from (21) it follows that

$$\begin{aligned}\sum_{1\le h\le N^\delta} \frac{1}{h} \sup_{h^{1/\delta}<B\le N^\delta} \left|\sum_{h^{1/\delta}\le n\le B} e(hg(n))\right| &\ll \sum_{1\le h\le N^\delta}\left(\frac{1}{h^{1/\delta-1/2}\|h\alpha\|}+\frac{1}{h^{1/2}\|h\alpha\|}\right) \\ &\ll \sum_{1\le h\le N^\delta}\frac{1}{h^{1/2}\|h\alpha\|}.\end{aligned} \tag{22}$$

By using (22) with the following analogue of [3, Lemma 3.3]: for every $\vartheta > 0$

$$\sum_{h=1}^{m} \frac{1}{h^{1/2}\|h\alpha\|} \ll m^{\eta-1/2+\vartheta}, \tag{23}$$

Theorem 1 implies

$$D_N(\omega) \ll F(N) + N^{(\eta-1/2+\vartheta)\delta-1}. \tag{24}$$

By choosing $\delta = (\eta+1/2)^{-1}$, the desired result follows. □

REMARK 2. By applying usual Erdös-Turán's inequality, Tichy and Turnwald [4, p. 357] showed that $D_N(\omega) \ll N^{-1/(\eta+1)+\epsilon}$.

In the same way as in the proof of Theorem 3, we obtain the following.

THEOREM 4. *If α is an irrational number of constant type and β is a nonzero real number, then the discrepancy of $\omega = (\alpha n + \beta \log n)$ satisfies*

$$D_N(\omega) \ll N^{-2/3} \log N.$$

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