ON THE ABSOLUTE SUMMABILITY BY BOREL'S INTEGRAL METHOD OF THE DERIVED FOURIER SERIES AND ITS CONJUGATE SERIES

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Summary. Mohanty (1) and (3) considered the absolute summability of conjugate series and Fourier series by Borel's integral method by proving the following theorems.

Theorem A. *If* $\psi(t)\log\frac{k}{t}$ *is of bounded variation in* $(0, \pi)$, *then* $\sum_{n=1}^{\infty} B_n(\theta)$ *is summable* $|B'|$.

Theorem B. *If* $g(t)$ *is of bounded variation in* $(0, \pi)$, *then the series* $\sum_{n=1}^{\infty} A_n(\theta)$ *is summable* $|B'|$.

The present author considered the absolute summability of derived Fourier series and its conjugate series by Borel's integral method. Theorems proved by the present author are

Theorem 1. *If*

(i) $\psi(+0) = 0$

and

(ii) $\int_0^\delta t^{-2}\,|d\psi(t)| < \infty;\ 0<\delta<1.$

then the series $\sum_1^{\infty} nB_n(\theta)$ *is summable* $|B'|$.

Theorem 2. *If*

(i) $\phi(+0) = O(1)$

and

(ii) $\int_0^\delta t^{-2}\,|d\phi(t)| < \infty;\ 0<\delta<1.$

then the series $\sum_1^{\infty} nA_n(\theta)$ *is summable* $|B'|$.

1. Definition

A series $\sum_0^\infty a_n$ is said to be summable (B') to sum A if

$$\int_0^\infty e^{-x}.\sum_0^\infty \frac{a_n x^n}{n!}\,dx = \lim_{x\to\infty}\int_0^x e^{-x}.\sum_0^\infty \frac{a_n x^n}{n!}\,dx = A.$$

If the above integral is absolutely convergent, we say that the series $\sum_0^\infty a_n$ is absolutely summable by Borel's integral method (2) or summable $|B'|$.

2. Let $f(t)$ be Lebesgue integrable in $(-\pi, \pi)$ and periodic with period 2π and let

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt) = \tfrac{1}{2}a_0 + \sum_1^\infty A_n(t). \tag{2.1}$$

The allied series of (2.1) at $t = \theta$ is

$$\sum_1^\infty (b_n \cos n\theta - a_n \sin n\theta) = \sum_1^\infty B_n(\theta)$$

and the derived Fourier series is

$$\sum_1^\infty nB_n(\theta). \tag{2.2}$$

The conjugate series of (2.2) is

$$\sum_1^\infty nA_n(\theta). \tag{2.3}$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(\theta+t)+f(\theta-t)\} \tag{2.4}$$

$$\psi(t) = \tfrac{1}{2}\{f(\theta+t)-f(\theta-t)\} \tag{2.5}$$

and

$$g(t) = \phi(t)\log\frac{k}{t}. \tag{2.6}$$

Mohanty (1) and (3) proved

Theorem A. *If* $\psi(t)\log\frac{k}{t}$ *is of bounded variation in* $(0, \pi)$, *then* $\sum_{n=1}^\infty B_n(\theta)$ *is summable* $|B'|$.

Theorem B. *If* $g(t)$ *is of bounded variation in* $(0, \pi)$ *then the series* $\sum_{n=1}^\infty A_n(\theta)$ *is summable* $|B'|$.

3. The object of the present paper is to prove the following two theorems.

Theorem 1. *If* (i) $\psi(+0) = 0$

and (ii) $\int_0^\delta t^{-2}\,|d\psi(t)| < \infty;\ 0<\delta<1,$

then the series $\sum_{n=1}^\infty nB_n(\theta)$ *is summable* $|B'|$.

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Theorem 2. *If* (i) $\phi(+0) = O(1)$

and (ii) $\int_0^\delta t^{-2} \, |d\phi(t)| < \infty; \; 0<\delta<1$

then the series $\sum_{n=1}^{\infty} nA_n(\theta)$ *is summable* $|B'|$.

4. In order to simplify the proof we require the following estimates for the function

$$g_1(x, t) = \int_t^\delta e^{x \cos u} . \sin (u + x \sin u) du; \; 0<t<\delta<1, \; x>0.$$

$$= O(e^x) \tag{4.1}$$

$$= O(x^{-1} e^{x \cos t}) \tag{4.2}$$

$$= O(x^{-1} e^x) \tag{4.3}$$

$$g_2(x, t) = \int_t^\delta e^{x \cos u} . \cos (u + x \sin u) du; \; 0<t<\delta<1, \; x>0.$$

$$= O(e^x) \tag{4.4}$$

$$= O(x^{-1} e^{x \cos t}) \tag{4.5}$$

$$= O(x^{-1} e^x) \tag{4.6}$$

Proof. Let $0<t<\delta<1$, $x>0$ and ε_1, ε_2 be either 0 or 1. Define

$$h(x, t) = \int_t^\delta e^{x \cos u} . \sin (u + \tfrac{1}{2}\pi\varepsilon_1) \sin (x \sin u + \tfrac{1}{2}\pi\varepsilon_2) du$$

using the second mean value theorem for integrals twice

$$h(x, t) = e^{x \cos t} \int_t^s \sin (u + \tfrac{1}{2}\pi\varepsilon_1) \sin (x \sin u + \tfrac{1}{2}\pi\varepsilon_2) du; \; (t<s<\delta)$$

$$= x^{-1} . e^{x \cos t} \int_t^s x \cos u \sin (x \sin u + \tfrac{1}{2}\pi\varepsilon_2) \sec u \sin (u + \tfrac{1}{2}\pi\varepsilon_1) du$$

$$= x^{-1} . e^{x \cos t} [\varepsilon_1 + (1-\varepsilon_1) \tan s] \int_r^s x \cos u \sin (x \sin u + \tfrac{1}{2}\pi\varepsilon_2) du \qquad (t \leqq r < s)$$

$$= 0(x^{-1} e^{x \cos t})$$

from which properties (4.2), (4.3), (4.5) and (4.6) follow at once.

5. Proof of Theorem 1

$\sum_{n=1}^{\infty} nB_n(\theta)$ is summable $|B'|$ if

$$I = \int_0^\infty e^{-x} \left| \sum_1^\infty \frac{nB_n(\theta)}{n!} x^n \right| dx < \infty.$$

Now

$$I = 2\pi^{-1}\int_0^\infty e^{-x}\left|\sum_1^\infty \frac{\int_0^\pi \psi(t)\sin nt}{(n-1)!}\, x^n dt\right| dx$$

$$= 2\pi^{-1}\int_0^\infty e^{-x}\left|\int_0^\pi \psi(t)\,.\sum_1^\infty \frac{x^n \sin nt}{(n-1)!}\, dt\right| dx$$

$$= 2\pi^{-1}\int_0^\infty e^{-x}\left|\int_0^\pi \psi(t)\,.\, xe^{x\cos t}\,.\sin(t+x\sin t)dt\right| dx$$

$$\leqq 2\pi^{-1}\int_0^\infty xe^{-x}\left|\int_0^\delta \psi(t)e^{x\cos t}\,.\sin(t+x\sin t)dt\right| dx$$

$$+2\pi^{-1}\int_0^\infty xe^{-x}\left|\int_\delta^\pi \psi(t)\,.\,e^{x\cos t}\,.\sin(t+x\sin t)dt\right| dx;\ (0<\delta<1)$$

$$\leqq I_1+I_2,\ \text{say} \tag{5.1}$$

We have

$$I_2 \leqq 2\pi^{-1}\int_0^\infty xe^{-x}dx\int_\delta^\pi |\psi(t)|\,.\,e^{x\cos t}dt$$

$$\leqq 2\pi^{-1}\int_0^\infty xe^{-x}\,.\,e^{x\cos\delta}dx\int_\delta^\zeta |\psi(t)|\,dt,\quad (\delta<\zeta<\pi)$$

$$\leqq 2\pi^{-1}\left\{\left[\frac{-xe^{-2x\sin^2\frac{1}{2}\delta}}{2\sin^2\frac{1}{2}\delta}\right]_0^\infty+\int_0^\infty \frac{e^{-2x\sin^2\frac{1}{2}\delta}}{2\sin^2\frac{1}{2}\delta}\,dx\right\}\int_\delta^\zeta |\psi(t)|\,dt$$

$$\leqq \tfrac{1}{2}\pi^{-1}\operatorname{cosec}^4\tfrac{1}{2}\delta\,.\int_0^\pi |\psi(t)|\,dt<\infty. \tag{5.2}$$

Now

$$\int_0^\delta \psi(t)e^{x\cos t}\,.\sin(t+x\sin t)dt$$

$$= -\left[\psi(t)\int_t^\delta e^{x\cos u}\,.\sin(u+x\sin u)du\right]_0^\delta+\int_0^\delta d\psi(t)\int_t^\delta e^{x\cos u}\,.\sin(u+x\sin u)du$$

$$= \int_0^\delta d\psi(t)g_1(x,t), \tag{5.3}$$

by condition (i) of the theorem. Therefore

$$I_1 = 2\pi^{-1}\int_0^\infty xe^{-x}\left|\int_0^\delta d\psi(t)g_1(x,t)\right| dx$$

$$\leqq 2\pi^{-1}\int_0^\delta |d\psi(t)|\int_0^\infty xe^{-x}|g_1(x,t)|\,dx$$

$$\leqq 2\pi^{-1}\int_0^\delta |d\psi(t)|\,.\,J,\ \text{say} \tag{5.4}$$

Now

$$J=\int_0^1 xe^{-x}\,|\,g_1(x,t)|\,dx+\int_1^{t^{-1}} xe^{-x}\,|\,g_1(x,t)|\,dx+\int_{t^{-1}}^{\infty} xe^{-x}\,|\,g_1(x,t)|\,dx$$

$$=J_1+J_2+J_3,\text{ say} \tag{5.5}$$

By (4.1) we have

$$J_1=\int_0^1 xe^{-x}\,.\,O(e^x)dx$$

$$=O(1) \tag{5.6}$$

By (4.3) we have

$$J_2=\int_1^{t^{-1}} xe^{-x}\,.\,O(x^{-1}e^x)dx$$

$$=O(t^{-1}). \tag{5.7}$$

By (4.2) we have

$$J_3=\int_{t^{-1}}^{\infty} xe^{-x}\,.\,O(x^{-1}e^{x\cos t})dx$$

$$=O\left(\int_{t^{-1}}^{\infty} e^{-2x\sin^2\frac{1}{2}t}\,.\,dx\right)$$

$$=O(t^{-2}). \tag{5.8}$$

There is therefore an A with

$$I\leqq I_1+I_2\leqq A+A\int_0^{\delta} t^{-2}\,|\,d\psi(t)|<\infty.$$

Thus the theorem is proved

6. Proof of Theorem 2

$$\sum_{n=1}^{\infty} nA_n(\theta)\text{ is summable }|\,B'\,|\text{ if}$$

$$I'=\int_0^{\infty} e^{-x}\left|\sum_1^{\infty}\frac{nA_n(\theta)}{n!}x^n\right|dx<\infty.$$

Now

$$I'=2\pi^{-1}\int_0^{\infty} e^{-x}\left|\sum_1^{\infty}\frac{\int_0^{\pi}\phi(t)\cos ntdt}{(n-1)!}x^n\right|dx$$

$$\leqq 2\pi^{-1}\int_0^{\infty} xe^{-x}\left|\int_0^{\delta}\phi(t)e^{x\cos t}\,.\cos(t+x\sin t)dt\right|dx$$

$$+2\pi^{-1}\int_0^{\infty} xe^{-x}\left|\int_{\delta}^{\pi}\phi(t)e^{x\cos t}\,.\cos(t+x\sin t)dt\right|dx$$

$$\leqq I_1'+I_2'; \tag{6.1}$$

say, where $0<\delta<1$ and

$$I_2' \leqq 2\pi^{-1}\int_0^\infty xe^{-x}.e^{x\cos\delta}dx\int_\delta^\xi |\phi(t)|\,dt;\ (\delta<\zeta<\pi)<\infty \tag{6.2}$$

Now

$$\int_0^\delta \phi(t)e^{x\cos t}.\cos(t+x\sin t)dt = O(1)+\int_0^\delta d\phi(t)g_2(x,t)$$

by condition (i) of the theorem. Therefore

$$I_1' \leqq 2\pi^{-1}\int_0^\delta |d\phi(t)|.\int_0^\infty xe^{-x}.|g_2(x,t)|\,dx+A$$

$$\leqq 2\pi^{-1}\int_0^\delta |d\phi(t)|.J'+A, \text{ say} \tag{6.3}$$

Proceeding as in the proof of Theorem 1 and using (4.4), (4.5) and (4.6) we have

$$J' = O(t^{-2}) \tag{6.4}$$

Therefore

$$I' \leqq A+A\int_0^\delta t^{-2}\,|d\phi(t)|<\infty.$$

Thus the theorem is proved.

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