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SINGULAR RATIONALLY CONNECTED THREEFOLDS WITH NON-ZERO PLURI-FORMS

WENHAO OU

Abstract. This paper is concerned with singular projective rationally connected threefolds X which carry non-zero pluri-forms, that is the reflexive hull of $(\Omega_X^1)^{\otimes m}$ has a non-zero global section for some positive integer m. If X has $\mathbb{Q}$-factorial terminal singularities, then we show that there is a fibration p from X to $\mathbb{P}^1$. Moreover, we give a formula for the numbers of m-pluri-forms as a function of the ramification of the fibration p.

§1. Introduction

Recall that a complex projective variety X is said to be rationally connected if for any two general points in X, there exists a rational curve passing through them (see [Kol96, Definition IV.3.2 and Proposition IV.3.6]). It is known that if X is a smooth rationally connected variety, then X does not carry any non-zero pluri-form, that is $H^0(X, (\Omega_X^1)^{\otimes m}) = \{0\}$ for $m > 0$ (see [Kol96, Corollary IV.3.8]). It is a conjecture that the converse is also true (see [Kol96, Conjecture IV.3.8.1]). For singular varieties, there are also some analog results. The following theorems can be found in [GKP14] and [GKKP11].

THEOREM 1.1. [GKP14, Theorem 3.3] *If X is a rationally connected variety with factorial canonical singularities, then $H^0(X, (\Omega_X^1)^{[\otimes m]}) = \{0\}$ for $m > 0$, where $(\Omega_X^1)^{[\otimes m]}$ is the reflexive hull of $(\Omega_X^1)^{\otimes m}$.*

The reader is referred to Definition 2.6 for the notion of a klt pair.

THEOREM 1.2. [GKKP11, Theorem 5.1] *If (X, D) is a klt pair such that X is rationally connected, then $H^0(X, \Omega_X^{[\wedge m]}) = \{0\}$ for $m > 0$, where $\Omega_X^{[\wedge m]}$ is the reflexive hull of Ω_X^m.*

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Note that Ω_X^m is a direct summand of $(\Omega_X^1)^{\otimes m}$, hence $H^0(X,(\Omega_X^1)^{[\otimes m]}) = \{0\}$ implies $H^0(X,\Omega_X^{[\wedge m]}) = \{0\}$. However, there are examples of klt rationally connected varieties whose canonical divisors are $\mathbb{Q}$-effective (see [Tot12, Example 10] or [Kol08, Example 43]). Moreover, Theorem 1.1 is not true if the variety is not factorial. A counterexample is given in [GKP14, Example 3.7]. In [Ou14], we classify all rationally connected surfaces X with canonical singularities such that $H^0(X,(\Omega_X^1)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$. We obtain the following theorem.

THEOREM 1.3. [Ou14, Theorem 1.4] *Let X be a rationally connected surface with canonical singularities. Then $H^0(X,(\Omega_X^1)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$ if and only if there is a fibration $p : X \to \mathbb{P}^1$ whose general fibers are isomorphic to $\mathbb{P}^1$ such that $\sum_{z\in\mathbb{P}^1}((m(p,z)-1)/m(p,z)) \geqslant 2$, where $m(p,z)$ is the smallest positive coefficient in the divisor p^*z.*

In this paper, we study the case of threefolds. We are interested in the structure of mildly singular rationally connected threefolds X which carry non-zero pluri-forms, and we try to find out the source of these forms. If $f : \bar{X} \to X$ is the $\mathbb{Q}$-factorial model, then f is an isomorphism in codimension 1 and any pluri-forms of X lift to $\bar{X}$. Hence, we can reduce to the case where the singularities of X are $\mathbb{Q}$-factorial. If the threefold has terminal singularities, we prove the following result which is similar to Theorem 1.3.

THEOREM 1.4. *Let X be a projective rationally connected threefold with $\mathbb{Q}$-factorial terminal singularities. Then $H^0(X,(\Omega_X^1)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$ if and only if there is a fibration $p : X \to \mathbb{P}^1$ whose general fibers are smooth rationally connected surfaces such that $\sum_{z\in\mathbb{P}^1}((m(p,z)-1)/m(p,z)) \geqslant 2$, where $m(p,z)$ is the smallest positive coefficient in the divisor p^*z. Moreover, if this is the case, then for all $m > 0$, we have*

$$H^0(X,(\Omega_X^1)^{[\otimes m]}) \cong H^0\left(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}\left(-2m + \sum_{z\in\mathbb{P}^1}\left\lfloor\frac{(m(p,z)-1)m}{m(p,z)}\right\rfloor\right)\right).$$

In particular, the number of m-pluri-forms depends only on the ramification of p.

Construction 1.5. Thanks to Theorem 1.4, every rationally connected threefold X with $\mathbb{Q}$-factorial terminal singularities such that $H^0(X,(\Omega_X^1)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$ can be constructed as follows. There is a fibration $q : T \to B$ from a normal threefold T to a smooth curve B which

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has positive genus such that $m(q,b)=1$ for all $b \in B$. There is a finite group G which acts on T and B such that q is G-equivariant. By taking the quotient by G, we have $B/G \cong \mathbb{P}^1$ and $T/G \cong X$. For more details, see Section 3.

If the threefold X has canonical singularities, we also obtain some necessary conditions for the existence of non-zero pluri-forms. As before, we reduce to the case where X is $\mathbb{Q}$-factorial. We have the following theorem.

THEOREM 1.6. *Let X be a projective rationally connected threefold with $\mathbb{Q}$-factorial canonical singularities which carries non-zero pluri-forms. Let $X \dashrightarrow X'$ be the result of a minimal model program. Then there is always an equidimensional fibration $q : X' \to Z$ with* $\dim Z > 0$ *such that $K_Z + \Delta$ is $\mathbb{Q}$-effective. The divisor Δ is defined by $\Delta = \sum_i ((m(q,D_i)-1)/m(q,D_i))D_i$, where the D_i are all the prime divisors in Z such that q^*D_i is not reduced and $m(q,D_i)$ is the smallest positive coefficient in q^*D_i. More precisely, we have the following two possibilities.*

(1) *The variety Z is a surface and $q : X' \to Z$ is a Mori fibration such that there is a positive integer l such that $l(K_Z+\Delta)$ is an effective divisor. Moreover, the sections of $\mathscr{O}_Z(l(K_Z+\Delta))$ lift to sections of $\Omega_{X'}^{[\otimes 2l]}$.*
(2) *The variety Z is isomorphic to $\mathbb{P}^1$ and for any $m>0$, we have*

$$H^0(X', (\Omega^1_{X'})^{[\otimes m]}) \cong H^0\left(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}\left(-2m + \sum_{z \in \mathbb{P}^1} \left\lfloor \frac{(m(q,z)-1)m}{m(q,z)} \right\rfloor\right)\right).$$

The difficulty in the case of canonical singularities is that the rational map $X \dashrightarrow Z$ is not always regular. Moreover, even if it is, it may not be equidimensional. If klt singularities are permitted for our threefolds, then there exist other structures. We give an example of a rationally connected threefold of general type in Example 2.15.

Outline of the paper. The main objective of this paper is to prove Theorem 1.4. We do this in several steps. First, we consider a projective rationally connected threefold X with $\mathbb{Q}$-factorial canonical singularities which carries non-zero pluri-forms. Since it is rationally connected and has canonical singularities, its canonical divisor K_X is not pseudo-effective. Hence, if $f : X \dashrightarrow X^*$ is the result of a minimal model program for X, then K_{X^*} is not pseudo-effective either. Thus there is a Mori fibration $p : X^* \to Z$ where Z is a normal variety of dimension less than 3. However, since X carries non-zero pluri-forms, we have $h^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) \neq 0$ for some $m>0$. This implies that $\dim Z > 0$ by [Ou14, Theorem 3.1]. Hence, either

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$\dim Z = 1$ or $\dim Z = 2$. We treat these two cases separately in Sections 3 and 4. If $\dim Z = 1$, then $Z \cong \mathbb{P}^1$, and we show that

$$K_{\mathbb{P}^1} + \sum_{z \in \mathbb{P}^1} \frac{m(p, z) - 1}{m(p, z)} z$$

is $\mathbb{Q}$-effective, which is the same condition as in Theorem 1.4. If $\dim Z = 2$, then we define the $\mathbb{Q}$-divisor Δ on Z as in Theorem 1.6. We prove that either $K_Z + \Delta$ is $\mathbb{Q}$-effective or there is a fibration $X^* \to \mathbb{P}^1$ such that we can reduce to the situation of Section 3. In the last section, we assume that the variety X has terminal singularities. In this case, there is always a fibration from X^* to $\mathbb{P}^1$ and we are always in the situation of Section 3. This fibration induces a dominant rational map from X to $\mathbb{P}^1$. In the end, we prove that this rational map is regular and complete the proof of Theorem 1.4.

§2. Preliminaries

Throughout this paper, we work over $\mathbb{C}$, the field of complex numbers. Unless otherwise specified, every variety is an integral $\mathbb{C}$-scheme of finite type. A curve is a variety of dimension 1, a surface is a variety of dimension 2 and a threefold is a variety of dimension 3. A projective variety is called rationally chain connected if any two general points can be connected by a chain of rational curves. By [HM07, Corollary 1.8], a projective variety with klt singularities is rationally chain connected if and only if it is rationally connected. For a normal variety X, let K_X be a canonical divisor of X. We denote the sheaf of Kähler differentials by Ω^1_X and let $\Omega^{[1]}_X$ be its reflexive hull. Denote $\bigwedge^p \Omega^1_X$ by Ω^p_X for $p \in \mathbb{N}$. Let $\Omega^{[\wedge p]}_X$ (resp. $(\Omega^1_X)^{[\otimes p]}$) be the reflexive hull of Ω^p_X (resp. $(\Omega^1_X)^{\otimes p}$). We say that a normal variety X carries non-zero pluri-forms if $H^0(X, (\Omega^1_X)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$. Let $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{Pic}(X) \otimes \mathbb{Q}$, where $\mathrm{Pic}(X)$ is the Picard group of X. A $\mathbb{Q}$-divisor Δ on X is called $\mathbb{Q}$-effective if there is a positive integer k such that kD is a divisor and $\mathscr{O}_X(kD)$ has a non-zero global section.

A fibration $p : X \to Z$ between normal quasi-projective varieties is a dominant proper morphism such that every fiber is connected, that is $p_*\mathscr{O}_X \cong \mathscr{O}_Z$. If $X \to Z$ is just a proper morphism, then we have the Stein factorization $X \to V \to Z$ such that $X \to V$ is a fibration and $V \to Z$ is finite (see [Har77, Corollary III. 11.5]). A Fano fibration is a fibration $X \to Z$ such that $-K_X$ is relatively ample. A Mori fibration is a Fano fibration such that the relative Picard number is 1.

Let $p : X \to Z$ be a morphism and D be a prime $\mathbb{Q}$-Cartier Weil divisor in Z. Let k be a positive integer such that kD is Cartier. Denote the $\mathbb{Q}$-divisor $(1/k)p^*(kD)$ by p^*D. We define the multiplicity $m(p, D)$ by

$$m(p, D) = \min \{\text{coefficient of } E \text{ in } p^*D \mid E \text{ is an irreducible component of } p^*D \text{ which dominates } D\}.$$

By generic smoothness, there are only finitely many divisors D such that $m(p, D) > 1$. Moreover, if $p : X \to Z$ is a morphism from a normal variety to a smooth curve, then we define the ramification divisor R as $\sum_{z\in Z} p^*z - (p^*z)_{red}$, where $(p^*z)_{red}$ is the sum of the irreducible components of p^*z.

2.1 Reflexive sheaves on normal varieties

In this subsection, we gather some properties of reflexive sheaves. For a coherent sheaf $\mathscr{F}$ on a normal variety X, we denote by $\mathscr{F}^{**}$ the double dual of $\mathscr{F}$. The sheaf $\mathscr{F}$ is reflexive if and only if $\mathscr{F} \cong \mathscr{F}^{**}$. In particular, $\mathscr{F}^{**}$ is reflexive and we also call it the reflexive hull of $\mathscr{F}$. Denote $(\mathscr{F}^{\otimes m})^{**}$ by $\mathscr{F}^{[\otimes m]}$ and $(\bigwedge^m \mathscr{F})^{**}$ by $\mathscr{F}^{[\wedge m]}$ for any $m > 0$. For two coherent sheaves $\mathscr{F}$ and $\mathscr{G}$, let $\mathscr{F}[\otimes]\mathscr{G} = (\mathscr{F} \otimes \mathscr{G})^{**}$. The following proposition is an important criterion for reflexive sheaves on normal varieties.

PROPOSITION 2.1. [Har80, Proposition 1.6] *Let $\mathscr{F}$ be a coherent sheaf on a normal variety X. Then $\mathscr{F}$ is reflexive if and only if $\mathscr{F}$ is torsion-free and for each open $U \subseteq X$ and each closed subset $Y \subseteq U$ of codimension at least 2, $\mathscr{F}|_U \cong j_*(\mathscr{F}|_{U\setminus Y})$, where $j : U \setminus Y \to U$ is the inclusion map.*

As a corollary of this proposition, we prove the following lemma.

LEMMA 2.2. *If $\varphi : X \dashrightarrow X'$ is a birational map between normal projective varieties such that φ^{-1} does not contract any divisor, then we have a natural injection $H^0(X, (\Omega^1_X)^{[\otimes m]}) \hookrightarrow H^0(X', (\Omega^1_{X'})^{[\otimes m]})$ for any integer $m > 0$.*

Proof. Since the birational map $\varphi^{-1} : X' \dashrightarrow X$ does not contract any divisor, it induces an isomorphism from an open subset W' of X' such that codim $X'\setminus W \geqslant 2$ onto an open subset W of X. By Proposition 2.1, we have $H^0(X', (\Omega^1_{X'})^{[\otimes m]}) \cong H^0(W', (\Omega^1_{W'})^{[\otimes m]})$ for any $m > 0$. Moreover, since W is an open subset of X, we obtain $H^0(X, (\Omega^1_X)^{[\otimes m]}) \hookrightarrow H^0(W, (\Omega^1_W)^{[\otimes m]}) \cong H^0(W', (\Omega^1_{W'})^{[\otimes m]}) \cong H^0(X', (\Omega^1_{X'})^{[\otimes m]})$ for all $m > 0$. ∎

The proof of the following lemma is left to the reader.

LEMMA 2.3. *Let $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ be an exact sequence of locally free sheaves on a variety X. Then, for any $m > 0$, we have a filtration of locally free sheaves $\mathscr{B}^{\otimes m} = \mathscr{R}_0 \supseteq \cdots \supseteq \mathscr{R}_{m+1} = 0$ such that $\mathscr{R}_i/\mathscr{R}_{i+1}$ is isomorphic to the direct sum of copies of $\mathscr{A}^{\otimes i} \otimes \mathscr{C}^{\otimes m-i}$ for all $0 \leqslant i \leqslant m$.*

From this lemma, we can deduce the following lemma which is very important in the paper.

LEMMA 2.4. *Let $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ be a complex of coherent sheaves on a normal variety X. Assume that there is an open subset W of X with* $\operatorname{codim} X \backslash W \geqslant 2$ *such that we have an exact sequence of locally free sheaves $0 \to \mathscr{A}|_W \to \mathscr{B}|_W \to \mathscr{C}|_W \to 0$ on W. If $H^0(X, \mathscr{A}^{\otimes r}[\otimes]\mathscr{C}^{\otimes t}) = \{0\}$ for all $t > 0$ and $r \geqslant 0$, then $H^0(X, \mathscr{B}^{[\otimes m]}) \cong H^0(X, \mathscr{A}^{[\otimes m]})$ for all $m > 0$.*

Proof. On W, we have $H^0(W, \mathscr{A}|_W^{\otimes r} \otimes \mathscr{C}|_W^{\otimes t}) = \{0\}$ for all $t > 0$ and $r \geqslant 0$ by Proposition 2.1. If we fix $m > 0$, by Lemma 2.3, we have a filtration $\mathscr{B}|_W^{\otimes m} = \mathscr{R}_0 \supseteq \cdots \supseteq \mathscr{R}_{m+1} = 0$ such that $\mathscr{R}_i/\mathscr{R}_{i+1}$ is isomorphic to the direct sum of copies of $\mathscr{A}|_W^{\otimes i} \otimes \mathscr{C}|_W^{\otimes m-i}$ for all $0 \leqslant i \leqslant m$. Since $H^0(X, \mathscr{A}|_W^{\otimes r} \otimes \mathscr{C}|_W^{\otimes t}) = \{0\}$ for all $t > 0$ and $r \geqslant 0$, we have $H^0(W, \mathscr{R}_i) \cong H^0(W, \mathscr{R}_{i+1})$ for $0 \leqslant i \leqslant m-1$. Thus $H^0(W, \mathscr{B}|_W^{\otimes m}/\mathscr{A}|_W^{\otimes m}) = \{0\}$ and $H^0(W, B|_W^{\otimes m}) \cong H^0(W, A|_W^{\otimes m})$. By Proposition 2.1, we have $H^0(X, \mathscr{B}^{[\otimes m]}) \cong H^0(X, \mathscr{A}^{[\otimes m]})$. □

One of the applications of the lemma above is the following, which gives a relation between pluri-forms and fibrations.

LEMMA 2.5. *Let $p: X \to Z$ be a morphism between normal varieties. Assume that general fibers of p do not carry any non-zero pluri-form. Then $H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(X, ((p^*\Omega_Z^1)^{sat})^{[\otimes m]})$ for $m > 0$, where $(p^*\Omega_Z^1)^{sat}$ is the saturation of the image of $p^*\Omega_Z^1$ in $\Omega_X^{[1]}$.*

Proof. We have an exact sequence of coherent sheaves $0 \to \mathscr{F} \to \Omega_X^{[1]} \to \mathscr{G} \to 0$ on X, where $\mathscr{F} = (p^*\Omega_Z^1)^{sat}$ and $\mathscr{G}$ is a torsion-free sheaf. In particular, $\mathscr{G}$ is locally free in codimension 1. Let V be the smooth locus of Z and let $W = p^{-1}(V)$. If U is the largest open subset of W on which $\mathscr{F}$, Ω_X^1 and $\mathscr{G}$ are locally free, then $\operatorname{codim} X \backslash U \geqslant 2$ and we have an exact sequence $0 \to \mathscr{F}|_U \to \Omega_U^1 \to \mathscr{G}|_U \to 0$ of locally free sheaves on U.

If F is a general fiber of $p|_U$, then $\mathscr{F}|_F$ is the direct sum of $\mathscr{O}_F$ and $\mathscr{G}|_F$ is isomorphic to Ω_F^1. Since general fibers of p do not carry any non-zero pluri-form, neither does F by Proposition 2.1. Hence, we have $H^0(U, \mathscr{F}|_U^{\otimes r} \otimes \mathscr{G}|_U^{\otimes t}) = \{0\}$ for all $t > 0$ and $r \geqslant 0$. By Lemma 2.4, we have an isomorphism from $H^0(U, \mathscr{F}|_U^{\otimes m})$ to $H^0(U, (\Omega^1_U)^{\otimes m})$ for all $m > 0$. By Proposition 2.1, we have $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(X, \mathscr{F}^{[\otimes m]})$ for all $m > 0$. □

If X is a normal variety and E is a Weil divisor on X, then E is a Cartier divisor on the smooth locus U of X and it induces an invertible sheaf $\mathscr{O}_U(E)$ on U. Let $\mathscr{O}_X(E)$ be the push-forward of $\mathscr{O}_U(E)$ on X. Then, by Proposition 2.1, $\mathscr{O}_X(E)$ is a reflexive sheaf on X since X is smooth in codimension 1. Conversely, if $\mathscr{F}$ is a reflexive sheaf of rank 1 on X, it is an invertible sheaf in codimension 1. There is a Weil divisor D on X such that $\mathscr{F} \cong \mathscr{O}_X(D)$. If X is $\mathbb{Q}$-factorial, then for any 1-cycle α on X, we can define the intersection number $\mathscr{F} \cdot \alpha = D \cdot \alpha = (1/k)(kD) \cdot \alpha$, where k is a positive integer such that kD is Cartier. This expression is independent of the choice of D (see [Rei80, Appendix to Section 1] for more details).

2.2 Minimal model program

We recall some basic definitions and properties of the minimal model program (MMP). A pair (X, Δ) consists of a normal quasi-projective variety X and a boundary Δ, that is a $\mathbb{Q}$-Weil divisor $\Delta = \Sigma_{j=1}^k d_j D_j$ on X such that the D_j are pairwise distinct prime divisors and all d_j are contained in $[0, 1]$. Recall the definition of singularities of pairs.

DEFINITION 2.6. [KM98, Definition 2.34] Let (X, Δ) be a pair with $\Delta = \Sigma_{j=1}^k d_j D_j$. Let $r : \widetilde{X} \to X$ be a log resolution of singularities of (X, Δ). Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then we can write

$$K_{\widetilde{X}} + r_*^{-1}\Delta = r^*(K_X + \Delta) + \Sigma a_i E_i,$$

where the E_i are r-exceptional divisors. We call a_i the discrepancy of E_i with respect to (X, Δ). Then (X, Δ) is terminal (resp. canonical, klt) if $a_i > 0$ (resp. $a_i \geqslant 0$, $a_i > -1$ and $d_j < 1$ for all j) for all i.

REMARK 2.7. If X is terminal, then it is smooth in codimension 2. If X is canonical, then K_X is Cartier in codimension 2 (see [KM98, Corollary 5.18]).

For a klt pair (X, Δ) such that X is $\mathbb{Q}$-factorial, we can run a $(K_X + \Delta)$-MMP and obtain a sequence of rational maps $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots$ such that (X_i, Δ_i) is a klt pair and X_i is $\mathbb{Q}$-factorial, where Δ_i is the strict transform of Δ. Every elementary step in an MMP is either a divisorial contraction which is a morphism contracting an irreducible divisor or a flip which is an isomorphism in codimension 1. More generally, if $h : X \to T$ is a morphism, we can run an h-relative MMP such that for all i, there is

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a morphism $h_i : X_i \to T$ with $h_i \circ \varphi_i = h$, where φ_i is the birational map $X \dashrightarrow X_i$. For more details on the MMP, we refer to [KM98, Section 3]. One of the most important problems in the MMP is that if the sequence of rational maps above terminates. If it does, we get the result of this h-relative MMP $X \dashrightarrow X'$. Let Δ' be the direct image of Δ. Then either $K_{X'} + \Delta'$ is relatively nef over T (that is, $(K_{X'} + \Delta') \cdot C \geqslant 0$ for any curve C in X' contracted by $X' \to T$) or we have a $(K_{X'} + \Delta')$-Mori fibration $g : X' \to Z$ over T such that $-(K_{X'} + \Delta')$ is g-ample. Thanks to [Kaw92, Theorem 1], we know that any MMP for a klt pair (X, Δ) such that $\dim X \leqslant 3$ terminates.

LEMMA 2.8. *Let $\varphi : X \dashrightarrow X'$ be an extremal divisorial contraction or a flip. Assume that X has $\mathbb{Q}$-factorial canonical singularities. If Y is an irreducible closed subvariety in X' such that it is the center of a divisor E over X' which has discrepancy 0, then φ^{-1} is a morphism around the generic point of Y.*

Proof. Assume the opposite. Then the discrepancy of E in X is strictly smaller than the one in X' (see [KM98, Lemma 3.38]). This implies that X does not have canonical singularities along the center of E in X, which is a contradiction. □

PROPOSITION 2.9. *Let X be a projective threefold which has canonical singularities. Let X' be the result of an MMP for X and denote the birational map $X \dashrightarrow X'$ by f. Let Γ be the normalization of the graph of f. Then there is a natural isomorphism $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(\Gamma, (\Omega^1_\Gamma)^{[\otimes m]})$ for all $m > 0$.*

Proof. Since there is a natural birational projection $p_1 : \Gamma \to X$, we have an injection from $H^0(\Gamma, (\Omega^1_\Gamma)^{[\otimes m]})$ to $H^0(X, (\Omega^1_X)^{[\otimes m]})$ by Lemma 2.2. Let $p_2 : \Gamma \to X'$ be the natural projection. Let $\sigma_X \in H^0(X, (\Omega^1_X)^{[\otimes m]})$ be a nonzero element. Since X, Γ and X' are birational, σ_X induces a rational section σ_Γ of $(\Omega^1_\Gamma)^{[\otimes m]}$ and an element $\sigma_{X'}$ of $H^0(X', (\Omega^1_{X'})^{[\otimes m]})$ (see Lemma 2.2). In order to prove that σ_Γ is a regular section it is sufficient to prove that σ_Γ does not have a pole along any p_1-exceptional divisor. Let E be an exceptional divisor for p_1. Let $C \subseteq E$ be a curve that is exceptional for p_1, then C is not contracted by p_2 since the graph of f is included in $X \times X'$ and the normalization map is finite. Hence, $Y = p_2(E) \subseteq X'$ is a curve and f^{-1} is not regular around the generic point of Y. By Lemma 2.8, X' has terminal singularities around the generic point of Y. Hence, it is smooth around the generic point of Y since $\operatorname{codim}_{X'} Y = 2$ (see [KM98, Corollary

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5.18]). Moreover, since T, X and X' are birational, the form σ_Γ is just the pullback of $\sigma_{X'}$ by p_2. Hence, σ_Γ does not have a pole along E. □

2.3 Examples

We give some examples of rationally connected varieties which carry non-zero pluri-forms. First, we give an example of such varieties which have terminal singularities. For the construction of this example, we use the method of Theorem 3.2.

EXAMPLE 2.10. Let $C_1 = \{[a:b:c] \in \mathbb{P}^2 \mid a^3 + (a+c)b^2 + c^3 = 0\}$ be a smooth elliptic curve in $\mathbb{P}^2$. Let $X_1 = \{([a:b:c],[x:y:z:t]) \in \mathbb{P}^2 \times \mathbb{P}^3 \mid a^3 + (a+c)b^2 + c^3 = 0, (a^2+2b^2+c^2)x^2 + (a^2+3b^2+c^2)y^2 + (a^2+4b^2+3c^2)z^2 + (a^2+5b^2+6c^2)t^2 = 0\}$. Then, X_1 is a smooth threefold and we have an induced fibration $p_1 : X_1 \to C_1$ such that all fibers of p_1 are smooth quadric surfaces which are Fano surfaces. Moreover, X_1 has Picard number 2 by the Lefschetz theorem (see [Laz04, Example 3.1.23]). Hence, p_1 is a Mori fibration. Moreover, since p_1 is smooth, we have $H^0(X_1, (\Omega^1_{X_1})^{\otimes 2}) \cong H^0(C_1, (\Omega^1_{C_1})^{\otimes 2})$ by Lemma 2.5. Let G be the group $\mathbb{Z}/2\mathbb{Z}$ and let $g \in G$ be the generator. We have an action of G on $\mathbb{P}^2 \times \mathbb{P}^3$ defined by $g \cdot ([a:b:c],[x:y:z:t]) = ([a:-b:c],[-x:-y:z:t])$. This action induces an action of G on X_1 and an action of G on C_1 such that p_1 in G-equivariant. We have $C_1/G = \mathbb{P}^1$. The action of G on X_1 is free in codimension 2 and it has exactly 16 fixed points. Since $G = \mathbb{Z}/2\mathbb{Z}$, the action of G on X_1 satisfies the condition of the first theorem in [Rei87, Section 5]. Thus, $X = X_1/G$ is a threefold which has $\mathbb{Q}$-factorial terminal singularities. We have a Fano fibration $p : X \to \mathbb{P}^1$ induced by p_1. Moreover, the Picard number of X is not larger than the Picard number of X_1. Hence, p is a Mori fibration. Since general fibers of p are smooth quadric surfaces which are rationally connected, X is rationally connected by [GHS03, Theorem 1.1]. In addition, $H^0(X, (\Omega^1_X)^{[\otimes 2]}) \cong H^0(X_1, (\Omega^1_{X_1})^{\otimes 2})^G$ since $X_1 \to X$ is étale in codimension 1. Moreover, $H^0(C_1, (\Omega^1_{C_1})^{\otimes 2})^G \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) \cong \mathbb{C}$ (see [Ou14, Lemma 7.3]). Hence, we have $H^0(X, (\Omega^1_X)^{[\otimes 2]}) \cong \mathbb{C}$.

REMARK 2.11. If X is a normal rationally connected threefold and if X^* is the result of an MMP for X, then $H^0(X, (\Omega^1_X)^{[\otimes m]})$ may be strictly included in $H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]})$ for some $m > 0$. For example, let $X^* \to \mathbb{P}^1$ be the variety described in the example above. Then X^* carries non-zero pluri-forms. Let $X \to X^*$ be a resolution of singularities of X^*. Then X^* is the result of an MMP for X since X^* has terminal singularities. However, since X is rationally connected and smooth, we have $H^0(X,(\Omega_X^1)^{[\otimes m]})=\{0\}$ for any $m>0$ (see [Kol96, Corollary IV.3.8]).

We give two examples of rationally connected threefolds with canonical singularities which carry non-zero pluri-forms. In both cases, there is always a Mori fibration from the threefold X to a surface Z. In Example 2.12, the base Z is a surface with klt singularities such that $K_Z\sim_{\mathbb{Q}}0$. In Example 2.14, the base Z is isomorphic to $\mathbb{P}^2$, which does not carry any non-zero pluri-forms.

EXAMPLE 2.12. Let C be the curve in $\mathbb{P}^2$ defined by $C=\{[x_1:x_2:x_3]\in\mathbb{P}^2 \mid x_1^3+x_2^3+x_3^3=0\}$. There exists an action of group $G=\mathbb{Z}/3\mathbb{Z}$ on C defined by $g\cdot[x_1:x_2:x_3]=[\xi x_1:x_2:x_3]$, where g is a generator of G and ξ is a primitive third root of unity. Let $Z_1=C\times C$. Then there is an induced action of G on Z_1 which acts diagonally. We have that $Z=Z_1/G$ is a klt rationally connected surface such that K_Z is $\mathbb{Q}$-linearly equivalent to the zero divisor (see [Tot12, Example 10]). Let $X_1=\mathbb{P}^1\times Z_1$ and define an action of G on $\mathbb{P}^1$ by $g\cdot[y_1:y_2]\to[\xi y_1:y_2]$, where $[y_1:y_2]$ are homogeneous coordinates of $\mathbb{P}^1$. Then there is an induced action of G on the smooth threefold X_1 which acts diagonally. Since $G=\mathbb{Z}/3\mathbb{Z}$ and this action is free in codimension 2, this action satisfies the condition of [Rei80, Theorem 3.1]. Hence, $X=X_1/G$ has canonical singularities. We also have a Mori fibration $X\to Z$ whose general fibers are isomorphic to $\mathbb{P}^1$. Since Z is rationally connected, so is X by [GHS03, Theorem 1.1]. Moreover, since K_Z is $\mathbb{Q}$-effective, we have $H^0(X,(\Omega_X^1)^{[\otimes m]})\neq\{0\}$ for some $m>0$ (see Section 4.2).

In the example above, the non-zero pluri-forms come from K_Z, the canonical divisor of the base surface. In the following example, $-K_Z$ is ample. However, X still carries non-zero pluri-forms. These forms come from the multiple fibers of the fibration $X\to Z$. Before giving the example, we first introduce a method to construct non-reduced fibers.

Construction 2.13. We want to construct a fibration p from a normal threefold T with canonical singularities to a smooth surface S such that $p^*C=2(p^*C)_{red}$, where C is a smooth curve in S.

Let $T_0=\mathbb{P}^1\times S$, where S is a smooth projective surface. Let p_1, p_2 be the natural projections $T_0\to\mathbb{P}^1$ and $T_0\to S$. Let $C\subseteq S$ be a smooth curve, and let $C_0=\{z\}\times C$ be a section of p_2 over C in T_0, where z is a point of $\mathbb{P}^1$. Let $E_0=p_2^*C$, which is a smooth divisor in T_0. We perform a sequence of birational transformations of T_0.

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First, we blow up C_0, and we obtain a morphism $T_1 \to T_0$ with exceptional divisor E_1. Denote still by E_0 the strict transform of E_0 in T_1. If F is a fiber of $T_1 \to S$ over a point of C, then $F = F_1 \cup F_0$, where $F_i \subseteq E_i$ are rational curves and $K_{T_1} \cdot F_i = -1$, $E_i \cdot F_i = -1$.

Now we blow up C_1, the intersection of E_0 and E_1. We obtain a morphism $T_2 \to T_1$ with exceptional divisor E_2. Denote still by E_0, E_1 the strict transforms of E_0 and E_1 in T_2. If F is any set-theoretic fiber of $T_2 \to S$ over a point of C, then $F = F_0 \cup F_1 \cup F_2$, where $F_i \subseteq E_i$ are rational curves and $K_{T_2} \cdot F_i = 0$, $E_i \cdot F_i = -2$ for $i = 0, 1$, $K_{T_2} \cdot F_2 = -1$, $E_2 \cdot F_2 = -1$. Let q be the fibration $T_2 \to S$.

We blow down E_1 and E_0 in T_2. Let H be an ample $\mathbb{Q}$-divisor such that $(H + E_0) \cdot F_0 = 0$. Since $E_0 \cdot F_1 = 0$, there is a positive rational number b such that $(H + E_0 + bE_1) \cdot F_1 = 0$. Moreover, $(H + E_0 + bE_1) \cdot F_0 = 0$ since $E_1 \cdot F_0 = 0$. Let A be a sufficiently ample divisor on S. Then the $\mathbb{Q}$-divisor $D = q^*A + H + E_0 + bE_1$ is nef and big. Moreover, any curve B which has intersection number 0 with D must be contracted by $T_2 \to S$ since A is sufficiently ample. The curve B is also contained in $E_0 \cup E_1$ since H is ample. Since $K_{T_2} \cdot F_0 = K_{T_2} \cdot F_1 = 0$, there exists a large integer k such that $kD - K_{T_2}$ is nef and big. Then, by the basepoint-free theorem (see [KM98, Theorem 3.3]), there is a large enough integer a such that aD is Cartier and $|aD|$ is basepoint-free. The linear system $|aD|$ induces a contraction which contracts E_0 and E_1.

By contracting E_0 and E_1, we obtain a threefold T. Moreover, there is an induced fibration $p : T \to S$ such that $p^*C = 2(p^*C)_{red}$. Note that $T_2 \to T$ is a resolution of singularities and $K_{T_2} = f^*K_T$. Hence, V has canonical singularities.

Now we construct the example.

EXAMPLE 2.14. Let $C = \{[x : y : z] \in \mathbb{P}^2 \mid x^6 + y^6 + z^6 = 0\}$ be a smooth curve in $\mathbb{P}^2$. Then $2K_{\mathbb{P}^2} + C$ is linearly equivalent to the zero divisor. Let $X_0 = \mathbb{P}^1 \times \mathbb{P}^2$. By the method of Construction 2.13, we can construct a fiber space $p : X \to \mathbb{P}^2$ such that $p^*C = 2(p^*C)_{red}$. Then X has canonical singularities. Since general fibers of p are isomorphic to $\mathbb{P}^1$, X is rationally connected by [GHS03, Theorem 1.1]. Moreover, since $2(K_{\mathbb{P}^2} + \frac{1}{2}D)$ is an effective divisor and p is equidimensional, we have $H^0(X, (\Omega^1_X)^{[\otimes 4]}) \neq \{0\}$ (see Section 4.2).

Note that in the three examples above, the variety X we constructed is a rationally connected variety with canonical singularities. The divisor K_X

is not pseudo-effective and we always have a Mori fibration from X to a variety Z. The non-zero pluri-forms of X come from the base Z. However, in the following example, the variety we construct is a rationally connected threefold with klt singularities whose canonical divisor is ample. Some non-zero pluri-forms come from its canonical divisor.

EXAMPLE 2.15. Let X_1 be the Fermat hypersurface in $\mathbb{P}^4$ defined $x_0^6 + x_1^6 + \cdots + x_4^6 = 0$. Then X_1 is a smooth threefold such that K_{X_1} is ample. Moreover, the Picard number of X_1 is 1 by the Lefschetz theorem (see [Laz04, Example 3.1.23]). Let G be the group $\mathbb{Z}/6\mathbb{Z}$ with generator g. Define an action of G on $\mathbb{P}^4$ by $g \cdot [x_0 : \cdots : x_4] = [\omega x_0 : \omega x_1 : x_2 : x_3 : x_4]$, where ω is a primitive sixth root of unity. This action induces an action of G on X_1 which is free in codimension 1. Denote the quotient X_1/G by X and the natural morphism $X_1 \to X$ by π. Then X has $\mathbb{Q}$-factorial klt singularities but does not have canonical singularities (see [Rei80, Theorem 3.1]). Now we prove that X is rationally connected. First, we prove that X is uniruled. If w is a general point in X, then there is a point $w_1 \in X_1$ such that $\pi(w_1) = w$ and the first two coordinates of w_1 are both non-zero. There are two hyperplanes in $\mathbb{P}^4$ passing through w_1, $H_1 = \{a_2x_2 + a_3x_3 + a_4x_4 = 0\}$ and $H_2 = \{b_2x_2 + b_3x_3 + b_4x_4 = 0\}$, such that the intersection $C_1 = H_1 \cap H_2 \cap X_1$ is a smooth curve of genus 10. Moreover, there are exactly 6 points on C_1, given by $X_1 \cap \{x_2 = x_3 = x_4 = 0\}$, which are fixed under the action of G. Hence, $\pi|_{C_1} : C_1 \to \pi(C_1) = C$ is ramified exactly at those 6 points with degree 6. By the ramification formula, $g(C) = 1 + 2^{-1} \times 6^{-1} \times (2 \times 10 - 2 - 6 \times 5) = 0$, where $g(C)$ is the genus of C. Hence, C is a smooth rational curve. This implies that X is uniruled. Since X has Picard number 1, from the lemma below, we conclude that X is rationally connected.

LEMMA 2.16. *Let X be a normal projective variety which is $\mathbb{Q}$-factorial. Assume that the Picard number of X is 1. Then X is rationally connected if and only if it is uniruled.*

Proof. Assume that X is uniruled and let $\pi : X \dashrightarrow Z$ be the MRC fibration for X (see, for example, [Kol96, Theorem V.5.2]). We argue by contradiction. Assume that $\dim X > \dim Z > 0$. Then there is a Zariski open subset Z_0 of Z such that π is regular and proper over Z_0. Let C be a curve contained in some fiber of $\pi|_{Z_0}$. Let D be a non-zero effective Cartier divisor on Z_0. Let $E_0 = \pi^* D$. Then E_0 is a non-zero effective divisor on $X_0 = \pi^{-1} Z_0$. Let E be the closure of E_0 in X, then E is a non-zero effective $\mathbb{Q}$-Cartier

Weil divisor on X. However, the intersection number of E and C is zero, which contradicts the fact that X has Picard number 1. ∎

§3. Fibrations over curves

3.1 Construction of rationally connected varieties carrying non-zero pluri-forms

In this subsection, we give a method to construct rationally connected varieties which carry non-zero pluri-forms (Theorem 3.2). Together with Theorem 1.4, we see that every rationally connected threefold with $\mathbb{Q}$-factorial terminal singularities which carries non-zero pluri-forms can be constructed by this method.

By Lemma 2.5, if general fibers of an equidimensional fibration $p: X \to Z$ do not carry any non-zero pluri-forms, then for all $m > 0$, we have $H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(X, ((p^*\Omega_Z^1)^{sat})^{[\otimes m]})$. Moreover,

$$H^0(X, ((p^*\Omega_Z^1)^{sat})^{[\otimes m]}) \cong H^0(Z, p_*((p^*\Omega_Z^1)^{sat})^{[\otimes m]}).$$

Hence, we would like to know what $p_*((p^*\Omega_Z^1)^{sat})^{[\otimes m]}$ is. In the case when Z is a smooth curve, this is not difficult to compute. Note that if R is the ramification divisor of p, then $((p^*\Omega_Z^1)^{sat})^{[\otimes m]} \cong (p^*\Omega_Z^1)^{\otimes m} \otimes \mathscr{O}_X(mR)$ for $m > 0$. By the projection formula, we have

$$p_*((p^*\Omega_Z^1)^{sat})^{[\otimes m]} \cong (\Omega_Z^1)^{\otimes m} \otimes p_*\mathscr{O}_X(mR).$$

Hence, it is sufficient to compute $p_*\mathscr{O}_X(mR)$.

LEMMA 3.1. *Let $p: X \to Z$ be a fibration from a normal variety to a smooth curve. Let z be a point in Z. Let $D = p^*z - (p^*z)_{red}$. Then $p_*\mathscr{O}_X(mD) \cong \mathscr{O}_Z([(m(m(p,z)-1))/m(p,z)]z)$ for any $m \geqslant 0$.*

Proof. The problem is local and we may assume that Z is affine. We know that $p_*\mathscr{O}_W(mD)$ is an invertible sheaf. If θ is a section of $p_*\mathscr{O}_W(mD)$, then it is a rational function on Z whose pullback on W is a rational function which can only have a pole along p^*z. Hence, θ can only have a pole at z. Let d be the degree of the pole, then the pullback of θ in V is a section of $\mathscr{O}_W(mD)$ if and only if $m(p,z)d \leqslant m(m(p,z)-1)$, that is, $d \leqslant [(m(m(p,z)-1))/m(p,z)]$. ∎

This relation between ramification divisor and pluri-forms gives us an idea of how to construct rationally connected varieties which carry non-zero pluri-forms. Note that if $p: X \to \mathbb{P}^1$ is a fibration such that general

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fibers of p are rationally connected, then X is rationally connected by [GHS03, Theorem 1.1]. Moreover, if general fibers of p do not carry any non-zero pluri-forms, then from the discussion above, X carries non-zero pluri-forms if and only if $(\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes p_*\mathscr{O}_X(mR)$ has non-zero sections for some $m > 0$, where R is the ramification divisor of p. However, $\Omega^1_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}(-2)$ and $p_*\mathscr{O}_X(mR) \cong \mathscr{O}_{\mathbb{P}^1}(\sum_{z\in\mathbb{P}^1}[((m(p,z)-1)m)/m(p,z)])$, since any two points in $\mathbb{P}^1$ are linearly equivalent. Hence, X carries non-zero pluri-forms if and only if $\sum_{z\in\mathbb{P}^1}((m(p,z)-1)/m(p,z)) \geqslant 2$.

Now we try to construct this kind of varieties by taking quotients. Let T be a normal projective variety, let B be a smooth projective curve, and let G be a finite commutative group. Assume that there are actions of G on T and B. Assume that there is a G-equivariant fibration $q: T \to B$. Let $p_T : T \to T/G$, $p_B : B \to B/G$ be natural projections and let $p: T/G \to B/G$ be the induced fibration. Let S_i, $i = 1, \ldots, r$ be all the G-orbits in B whose cardinality is less than the cardinality of G. Let z_i be the image of S_i under the map $B \to B/G$. Then the z_i are the points in B/G over which $B \to B/G$ is ramified. Let G_i be the stabilizer of a point b_i in S_i. Then G_i acts on the set-theoretic fiber $q^{-1}\{b_i\} = F_i$. If A_i is a component in F_i and the stabilizer of a general point in A_i has cardinality d_i, then p_T is ramified along $G_i \cdot A_i$ of degree d_i, where $G_i \cdot A_i$ is the orbit of A_i under G_i. In this case, $p_T(A_i)$ has coefficient $e_i f_i/d_i$ in p^*z_i, where e_i is the coefficient of A_i in q^*b_i and f_i is the cardinality of G_i. Denote $\min\{e_i f_i/d_i \mid A_i \text{ a component in } F_i\}$ by s_i. Then s_i is equal to $m(p, z_i)$. We have the following theorem.

THEOREM 3.2. *Let T be a projective normal variety, let B be a projective smooth curve with positive genus, and let G be a finite commutative group. Assume that they satisfy the following conditions.*

(1) *There is a fibration q from T to B such that for any general fiber F_q of q, F_q is rationally connected and does not carry any non-zero pluri-forms. Moreover, $m(q, b) = 1$ for all $b \in B$.*
(2) *There exist actions of G on B and on T such that q is G-equivariant.*
(3) *The quotient B/G is isomorphic to $\mathbb{P}^1$.*

Then the quotient $X = T/G$ is a normal projective rationally connected variety and there is a fibration $p: X \to \mathbb{P}^1$. Moreover, if we define s_i, $i = 1, \ldots, r$ as above, then X carries non-zero pluri-forms if and only if

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$\Sigma_{i=1}^{r}((s_i-1)/s_i) \geqslant 2$. *More precisely,*

$$H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0\left(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}\left(-2m + \sum_{i=1}^{r}\left[\frac{(s_i-1)m}{s_i}\right]\right)\right)$$

for $m > 0$.

Proof. The fibration $p: X \to \mathbb{P}^1$ is induced by q. Since general fibers of q are rationally connected, general fibers of p are also rationally connected. Hence, X is rationally connected by [GHS03, Theorem 1.1]. If F_p is a general fiber of p, then F_p does not carry any non-zero pluri-forms either. Hence, $H^0(X, (\Omega_X^1)^{[\otimes m]})$ is isomorphic to $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m) \otimes p_*\mathscr{O}_X(mR))$ for $m > 0$, where R is the ramification divisor of p. However, since $m(q, b) = 1$ for all $b \in B$, we have $\sum_{i=1}^{r}[((s_i-1)m)/s_i] = \sum_{z\in\mathbb{P}^1}[((m(p, z)-1)m)/m(p, z)]$ by the definition of s_i. Finally, we have $p_*\mathscr{O}_X(mR) \cong \mathscr{O}_{\mathbb{P}^1}(\sum_{i=1}^{r}[((s_i-1)m)/s_i])$ and $H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + \sum_{i=1}^{r}[((s_i-1)m)/s_i]))$ by Lemma 3.1. □

REMARK 3.3. Conversely, let $p: X \to \mathbb{P}^1$ be a fibration such that general fibers of p are rationally connected and do not carry non-zero pluri-forms. If X carries non-zero pluri-forms, then X can be constructed by the method described above. In fact, by the discussion above, we have $\sum_{z\in\mathbb{P}^1}((m(p, z)-1)/m(p, z)) \geqslant 2$. In particular, there are at least three points in $\mathbb{P}^1$ such that the multiplicity of p is larger than 1 over these points. Let $z_1, \ldots, z_r$ be all the points in $\mathbb{P}^1$ such that $m(p, z_i) > 1$ for all i. Since $r \geqslant 3$, there is a smooth curve B and a Galois cover $p_B: B \to \mathbb{P}^1$ with Galois group G such that p_B is ramified exactly over the z_i and the degree of ramification is m_i at each point over z_i (see [KO82, Lemma 6.1]). Let T be the normalization of the fiber product $X \times_{\mathbb{P}^1} B$. Then we obtain a natural fibration $q: T \to B$ such that $m(q, b) = 1$ for all $b \in B$. Moreover, G acts naturally on T and $T/G \cong X$.

3.2 The case of threefolds with canonical singularities

If $p: X \to Z$ is a Mori fibration from a threefold which has $\mathbb{Q}$-factorial canonical singularities to a smooth curve, then general fibers of p are Fano surfaces which have canonical singularities. These surfaces do not carry any non-zero pluri-forms by the following theorem.

THEOREM 3.4. *If S is a Fano surface which has canonical singularities, then* $H^0(S, (\Omega_S^1)^{[\otimes m]}) = \{0\}$ *for all* $m > 0$.

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Proof. Assume the opposite. If S' is the result of an MMP for S, then S' is a Mori fiber space. By [Ou14, Theorem 3.1], S' has Picard number 2 and we have a Mori fibration $S' \to \mathbb{P}^1$. Let p_S be the composition of $S \to S' \to \mathbb{P}^1$. By [Ou14, Theorem 1.5], there is a smooth curve B of positive genus and a finite morphism $B \to \mathbb{P}^1$ such that the natural morphism $S_B \to S$ is étale in codimension 1, where S_B is the normalization of $S \times_{\mathbb{P}^1} B$. Hence, S_B is a Fano surface. Moreover, it has canonical singularities (see [KM98, Proposition 5.20]). Therefore, it is rationally connected by [HM07, Corollaries 1.3 and 1.5]. Hence, B is also rationally connected, which is a contradiction since its genus is positive. □

Thanks to this theorem, we obtain the following result.

PROPOSITION 3.5. *Let X be a rationally connected threefold which has $\mathbb{Q}$-factorial canonical singularities. Assume that X carries non-zero pluri-forms. Let X^* be the result of an MMP for X and assume that there is a Mori fibration $p : X^* \to \mathbb{P}^1$. Let Y be the normalization of the graph of $X \dashrightarrow X^*$. Then Y can be constructed by the method of Theorem 3.2.*

Proof. By Theorem 3.4 and Lemma 2.5, we know that every element in the space $H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]})$ comes from the base $\mathbb{P}^1$, and so does every element in $H^0(X, (\Omega^1_X)^{[\otimes m]})$ by Lemma 2.2. We have a rational map $X \dashrightarrow \mathbb{P}^1$ induced by p. By Proposition 2.9, we know that $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(Y, (\Omega^1_Y)^{[\otimes m]})$. Moreover, we have a natural fibration $g : Y \to \mathbb{P}^1$ which is the composition of $Y \to X^* \to \mathbb{P}^1$. If F_g is a general fiber of g, then there is a birational morphism $F_g \to S$, where S is a general fiber of $p : X^* \to \mathbb{P}^1$. Since S does not carry any non-zero pluri-forms by Theorem 3.4, neither does F_g by Lemma 2.2. From Remark 3.3, we know that Y can be constructed by the method of Theorem 3.2. □

§4. Mori fibrations and non-zero pluri-forms

In this section, we study relations between Mori fibrations and non-zero pluri-forms. We consider Mori fibrations from a normal threefold to a normal surface. First, we recall the definition and some properties of slopes. Let X be a normal projective $\mathbb{Q}$-factorial variety. Let α be a class of 1-cycles in X. For a coherent sheaf $\mathscr{F}$ of positive rank, we define the slope $\mu_\alpha(\mathscr{F})$ of $\mathscr{F}$ with respect to α by

$$\mu_\alpha(\mathscr{F}) := \frac{\det(\mathscr{F}) \cdot \alpha}{\operatorname{rank}(\mathscr{F})},$$

where $\det(\mathscr{F})$ is the reflexive hull of $\bigwedge^{rank\,\mathscr{F}}\mathscr{F}$. A class of movable curves is a class of 1-cycles that has non-negative intersection number with any pseudo-effective divisor. If α is a class of movable curves, then $\mu_\alpha^{max}(\mathscr{F}) = \sup\{\mu_\alpha(\mathscr{G}) \mid \mathscr{G} \subseteq \mathscr{F}$ a coherent subsheaf of positive rank$\}$ is well defined. For any coherent sheaf $\mathscr{F}$, there is a saturated coherent subsheaf $\mathscr{G} \subseteq \mathscr{F}$ such that $\mu_\alpha^{max}(\mathscr{F}) = \mu_\alpha(\mathscr{G})$. If $\mathscr{E}$ and $\mathscr{F}$ are two coherent sheaves of positive rank, then $\mu_\alpha(\mathscr{F} \otimes \mathscr{E}) = \mu_\alpha(\mathscr{F}) + \mu_\alpha(\mathscr{E})$. For more detail, see [GKP14, Appendix A]

4.1 General properties

Consider a Mori fibration $p : X \to Z$ from a normal rationally connected threefold to a normal variety Z of positive dimension. Assume that X has $\mathbb{Q}$-factorial klt singularities. Let $D_1, \ldots, D_k$ be all prime Weil divisors in Z such that $m_i = m(p, D_i) > 1$. Let $\Delta = \sum_{i=1}^{k} ((m_i - 1)/m_i)D_i$. Then $\det((p^*\Omega_Z^1)^{sat}) \cong \mathscr{O}_X(p^*(K_Z + \Delta))$ (see Remark 4.5). Assume that X carries non-zero pluri-forms and general fibers of p do not carry any non-zero pluri-forms. If $\dim Z = 1$, then $K_Z + \Delta$ is an effective $\mathbb{Q}$-divisor of degree $-2 + \sum_{i=1}^{k} ((m_i - 1)/m_i)$ on $\mathbb{P}^1$ (see Section 3). The aim of this subsection is to prove something analogous in the case when $\dim Z = 2$. We prove that if $K_Z + \Delta$ is not pseudo-effective, then there will be a fibration from Z to $\mathbb{P}^1$. In this case, we have an induced fibration $X \to \mathbb{P}^1$ and we are in the same situation as in Section 3 (see Lemma 4.10). In order to do this, we run an MMP for the pair (Z, Δ). To this end, we prove the following proposition which implies that the pair (Z, Δ) is klt.

PROPOSITION 4.1. *Let $p : X \to Z$ be a Mori fibration from a $\mathbb{Q}$-factorial klt quasi-projective variety X to a normal variety Z. Let $D_1, \ldots, D_k$ be pairwise distinct prime Weil divisors in Z such that $p^*D_i = m_i(p^*D_i)_{red}$ with $m_i \geqslant 2$. Then the pair $(Z, \sum_{i=1}^{k} ((m_i - 1)/m_i)D_i)$ is klt.*

Proof. Let $D = D_1 + \cdots + D_k$. By [KMM87, Lemma 5-1-5], Z is $\mathbb{Q}$-factorial. Note that the problem is local in Z, and we may assume that Z is affine.

We construct by induction a finite morphism $c_k : Z_k \to Z$ which is ramified over D such that $c_k^*D_i = m_i(c_k^*D_i)_{red}$ for all i. Let k_1 be the smallest positive integer such that k_1D_1 is a Cartier divisor. By taking the k_1th root of the function defining the Cartier divisor k_1D_1, we can construct a finite morphism $c_{1,1} : Z_{1,1} \to Z$ which is étale in codimension 1. Moreover, $c_{1,1}^*D_1$ is a reduced Cartier divisor (see [Mor88, Proposition-Definition 1.11]). By

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taking the m_1th root of the function defining the Cartier divisor $c_{1,1}^*D_1$, we can find a finite morphism $c_{1,2}: Z_1 \to Z_{1,1}$ which is ramified exactly over D_1 with ramification degree m_1 (see [Laz04, Section 4.1B]). Let $c_1 = c_{1,2} \circ c_{1,1}: Z_1 \to Z$. Then $c_1^*D_1 = m_1(c_1^*D_1)_{red}$. Assume that we have constructed a finite morphism $c_j: Z_j \to Z$ such that $c_j^*D_i = m_i(c_j^*D_i)_{red}$ for any $1 \leqslant i \leqslant j$ and $c_j^*D_i$ is reduced for any $i > j$, where $1 \leqslant j < k$. Let $E = c_j^*D_{j+1}$. Then E is a reduced divisor. By the same method as above, we can construct a finite morphism $d: Z_{j+1} \to Z_j$ which is ramified exactly over E with ramification degree m_{j+1}. Let $c_{j+1}: Z_{j+1} \to Z$ be the composition $c_j \circ d$. Then $c_{j+1}^*D_i = m_i(c_{j+1}^*D_i)_{red}$ for any $1 \leqslant i \leqslant j+1$ and $c_{j+1}^*D_i$ is reduced for any $i > j+1$. By induction, we can construct the finite morphism c_k.

We have $K_{Z_k} = c_k^*(K_Z + \sum_{i=1}^k ((m_i - 1)/m_i)D_i)$. Let X_k be the normalization of $X \times_Z Z_k$. Then the natural projection $c_X: X_k \to X$ is étale in codimension 1 and $K_{X_k} = c_X^*K_X$. Hence, X_k is klt by [KM98, Proposition 5.20]. Moreover, $X_k \to Z_k$ is a Fano fibration since $X \to Z$ is a Mori fibration. Hence, there is a $\mathbb{Q}$-divisor Δ_k such that the pair (Z_k, Δ_k) is klt by [Fuj99, Corollary 4.7]. Since K_{Z_k} is $\mathbb{Q}$-Cartier, Z_k is klt (see [KM98, Corollary 2.35]). Therefore, the pair $(Z, \sum_{i=1}^k ((m_i - 1)/m_i)D_i)$ is also klt by [KM98, Proposition 5.20]. □

In the remainder of this subsection, our aim is to prove the following theorem.

THEOREM 4.2. *Let $p: X \to Z$ be a Mori fibration from a projective normal threefold to a normal projective surface. Assume that X has $\mathbb{Q}$-factorial klt singularities. Let $D_1, \ldots, D_k$ be all prime Weil divisors in Z such that $m_i = m(p, D_i) > 1$. Let $\Delta = \sum_{i=1}^k ((m_i - 1)/m_i)D_i$. If X carries non-zero pluri-forms and $K_Z + \Delta$ is not pseudo-effective, then the result Z' of any MMP for the pair (Z, Δ) has Picard number 2.*

First, we would like to illustrate the idea of the proof in a simple case. Let $f: (Z, \Delta) \to (Z', \Delta')$ be the result of an MMP for the pair (Z, Δ). Assume in a first stage that $Z = Z'$. We argue by contradiction. Assume that Z' has Picard number 1. Since general fibers of p are isomorphic to $\mathbb{P}^1$, they do not carry any non-zero pluri-forms. By Lemma 2.5, the non-zero pluri-forms of X come from $(p^*\Omega_Z^1)^{sat}$. Since $K_Z + \Delta$ is not pseudo-effective, we can prove that there is a rank-1 coherent subsheaf $\mathscr{H}$ of $(p^*\Omega_Z^1)^{sat}$ such that $\mathscr{H}^{[\otimes l]}$ is an invertible sheaf which has non-zero global sections for some positive integer l. Next, we can prove that $\mathscr{H}^{[\otimes l]}$ is isomorphic to $\mathscr{O}_X$ under the assumption that Z has Picard number 1 (by using Lemma 4.8). Let $W \to X$ be the normalization of the cyclic cover with respect to the isomorphism $\mathscr{H}^{[\otimes l]} \cong \mathscr{O}_X$ (see [KM98, Definition 2.52]). Let $W \to V \to Z$ be the Stein factorization, then V will be a klt Fano variety (by Lemma 4.7). Moreover, $H^0(V, \Omega_V^{[1]}) \neq \{0\}$ (we use Lemma 4.9). This contradicts [GKKP11, Theorem 5.1].

However, in general, Z' is different from Z and the fibration $X \to Z'$ is not equidimensional. The idea of the proof is the same as above but the details are more complicated. We work over an open subset Z_0 of Z such that $f|_{Z_0}$ is an isomorphism and $\operatorname{codim} Z' \backslash f(Z_0) \geqslant 2$. For the complete proof of the theorem, we need several lemmas.

LEMMA 4.3. *Let $p : X \to Z$ be an equidimensional morphism between smooth varieties. Let n be the dimension of X, and let d be the dimension of Z. Let D be a prime divisor in Z, and let E be a prime divisor in X such that E is a component of p^*D. Assume that the coefficient of E in p^*D is k. Then for any general point $x \in E$, there is an open neighborhood $U \subseteq X$ of x such that $(p^*\Omega_Z^d)^{sat}|_U \cong \mathscr{O}_X(p^*K_Z + (k-1)E)|_U$, where $(p^*\Omega_Z^d)^{sat}$ is the saturation of $p^*\Omega_Z^d$ in Ω_X^d.*

Proof. We may assume that E is smooth around x and D is smooth around $p(x)$. There exist local coordinates $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_d)$ of X and Z around x and $p(x)$ such that E is defined by $\{a_1 = 0\}$, D is defined by $\{b_1 = 0\}$ and p is given by $(a_1, a_2, \ldots, a_n) \mapsto (a_1^k, a_2, \ldots, a_d)$. With these coordinates, the natural morphism $p^*\Omega_Z^1 \to \Omega_X^1$ is given by

$$(\mathrm{d}b_1, \mathrm{d}b_2, \ldots, \mathrm{d}b_d) \mapsto (ka_1^{k-1}\mathrm{d}a_1, \mathrm{d}a_2, \ldots, \mathrm{d}a_d)$$

and the image of the natural morphism $p^*\Omega_Z^d \to \Omega_X^d$ is generated by $p^*(\mathrm{d}b_1 \wedge \mathrm{d}b_2 \wedge \cdots \wedge \mathrm{d}b_d) = (ka_1^{k-1}\mathrm{d}a_1) \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_d$. Hence, $(p^*\Omega_Z^d)^{sat}$ is generated by $\mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_d$. Since $\{a_1 = 0\}$ defines the divisor E, we have $(p^*\Omega_Z^d)^{sat}|_U \cong \mathscr{O}_X(p^*K_Z + (k-1)E)|_U$ for some open neighborhood U of x. ∎

LEMMA 4.4. *Let $p : X \to Z$ be an equidimensional morphism between normal varieties. Assume that Z is $\mathbb{Q}$-factorial. Let $D_1, \ldots, D_r$ be all the prime divisors in Z such that $m(p, D_i)$ is larger than 1. Write $\det((p^*(\Omega_Z^1))^{sat}) \cong \mathscr{O}_X(M)$, where M is a divisor on X. Then $M - (p^*(K_Z + \sum_{i=1}^r ((m(p, D_i) - 1)/(m(p, D_i)))D_i))$ is $\mathbb{Q}$-effective.*

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Proof. By Proposition 2.1, we only have to prove the assertion on an open subset of X whose complement is of codimension at least 2. Hence, we may assume that both X and Z are smooth and that $\sum_{i=1}^{r} D_i$ is smooth. In this case, by Lemma 4.3, the divisor M is linearly equivalent to $p^*K_Z + \sum_{i=1}^{r}\sum_{j=1}^{s_i}(n_{i,j}-1)E_{i,j}$, where the $E_{i,1},\ldots,E_{i,s_i}$ are the components of p^*D_i and $n_{i,j}$ is the coefficient of $E_{i,j}$ in p^*D_i. Since $m(p,D_i)$ is the smallest integer among $n_{i,1},\ldots,n_{i,s_i}$, we have $n_{i,j}-1 \geqslant ((m(p,D_i)-1)/m(p,D_i))\cdot n_{i,j}$. Thus,

$$\sum_{i=1}^{r}\sum_{j=1}^{s_i}(n_{i,j}-1)E_{i,j} \geqslant \sum_{i=1}^{r}\frac{m(p,D_i)-1}{m(p,D_i)}p^*D_i. \qquad \square$$

REMARK 4.5. With the notation above, if in addition p is a Mori fibration, then p^*D_i is irreducible for all i since p has relative Picard number 1. In this case we obtain that

$$\det\left((p^*(\Omega_Z^1))^{sat}\right) \cong \mathscr{O}_X\left(p^*\left(K_Z + \sum_{i=1}^{r}\frac{m(p,D_i)-1}{m(p,D_i)}D_i\right)\right).$$

LEMMA 4.6. *Let $p: X \to Z$ be an equidimensional fibration between normal varieties. Let $(p^*\Omega_Z^1)^{sat}$ be the saturation of the image of $p^*\Omega_Z^1$ in $\Omega_X^{[1]}$. Then $p_*(((p^*\Omega_Z^1)^{sat})^{[\wedge r]}) \cong \Omega_Z^{[r]}$ for $r>0$.*

Proof. Let D be the sum of the divisors in Z over which p does not have reduced fibers. Applying Proposition 2.1 several times, we can suppose without loss of generality that Z is smooth and D is a simple normal crossing divisor. Let $\mathscr{M} = p_*(((p^*\Omega_Z^1)^{sat})^{[\wedge r]})$.

First, we prove that there is a natural injection from Ω_Z^r to $\mathscr{M}$. We have an injection from $p^*\Omega_Z^1$ to $(p^*\Omega_Z^1)^{sat}$. Hence, the natural morphism from $(p^*\Omega_Z^1)^{\wedge r}$ to $((p^*\Omega_Z^1)^{sat})^{[\wedge r]}$ is generically injective. Since $(p^*\Omega_Z^1)^{\wedge r}$ is without torsion, the natural morphism from $(p^*\Omega_Z^1)^{\wedge r}$ to $((p^*\Omega_Z^1)^{sat})^{[\wedge r]}$ is injective. Since $p(X)=Z$, we have an injection from $p_*((p^*\Omega_Z^1)^{\wedge r})$ to $\mathscr{M}$. By the projection formula, this implies that Ω_Z^r is a subsheaf of $\mathscr{M}$.

Now we prove that $\Omega_Z^r \cong \mathscr{M}$. Let $W = p^{-1}(Z\backslash D)$. Then the morphism $p|_W : W \to Z\backslash D$ is smooth in codimension 1. Thus,

$$((p^*\Omega_Z^1)^{sat})^{[\wedge r]}|_W \cong p^*\Omega_Z^r|_W$$

(see the proof of Lemma 4.3). Then we obtain $\mathscr{M}|_{Z\backslash D} \cong \Omega_{Z\backslash D}^r$ by the projection formula. Let U be any open set in Z, and let β be any element of $\mathscr{M}(U)$, that is, a section of $\mathscr{M}|_U$. Then β is a rational section of $\Omega^r_Z|_U$ which can only have a pole along D since $\mathscr{M}|_{Z\backslash D} \cong \Omega^r_{Z\backslash D}$. However, by the definition of $\mathscr{M}$, β induces a regular section of Ω^r_X on the smooth locus of $p^{-1}(U)$. This implies that β does not have pole along D. Thus, $\mathscr{M}(U) = \Omega^r_Z(U)$. Hence, $\Omega^r_Z \cong \mathscr{M}$. □

LEMMA 4.7. *Let $p: X \to Z$ be an equidimensional fibration from a normal variety X to a smooth variety Z. Let $D_1, \ldots, D_r$ be all the prime divisors in Z such that the multiplicity $m(p, D_i)$ is larger than 1. If $c_X: X_1 \to X$ is a finite morphism which is étale in codimension 1 and $X_1 \xrightarrow{p_1} Z_1 \xrightarrow{c_Z} Z$ is the Stein factorization, then $K_{Z_1} \leqslant c_Z^*(K_Z + \sum_{i=1}^r ((m(p, D_i) - 1)/m(p, D_i))D_i)$. That is, there is an effective $\mathbb{Q}$-divisor Δ_1 on Z_1 such that*

$$K_{Z_1} + \Delta_1 = c_Z^*\Big(K_Z + \sum_{i=1}^r \frac{m(p, D_i) - 1}{m(p, D_i)} D_i\Big).$$

Proof. If D is a prime divisor in Z, then $p_1^* c_Z^* D = c_X^* p^* D$. Let E be any irreducible component of $c_Z^* D$. Since c_X is étale in codimension 1, by comparing the coefficients of E in $p_1^* c_Z^* D = c_X^* p^* D$, the coefficient m_E of E in $c_Z^* D$ is not larger than $m(p, D)$. This implies that $(m_E - 1) \leqslant m_E \cdot ((m(p, D) - 1)/m(p, D))$. Now, from Lemma 4.3, we obtain $K_{Z_1} = c_Z^* K_Z + \Sigma_{j=1}^s (m_j - 1)E_j$, where the E_j are all prime divisors along which c_Z^* is ramified, and m_j is the degree of ramification of c_Z along E_j. By the discussion above, we have $(m_j - 1) \leqslant m_j \cdot ((m(p, p(E_j)) - 1)/m(p, p(E_j)))$. Hence, $K_{Z_1} \leqslant c_Z^*(K_Z + \sum_{i=1}^r ((m(p, D_i) - 1)/m(p, D_i))D_i)$. □

LEMMA 4.8. *Let $p: X \to Z$ be a Mori fibration from a normal threefold X to a smooth surface Z. Let $D_1, \ldots, D_r$ be all the prime divisors in Z such that the multiplicity $m_i = m(p, D_i)$ is larger than 1. Assume that there is a projective $\mathbb{Q}$-factorial variety V such that we have an open embedding $j: Z \to V$ with* $\operatorname{codim} V\backslash Z \geqslant 2$. *Assume further that the pair $(V, \sum_{i=1}^r ((m_i - 1)/m_i)\bar{D}_i)$ is klt, where $\bar{D}_i$ is the closure of $\bar{D}_i$ in V. If $\mathscr{F}$ is a rank-1 reflexive subsheaf of $\Omega_X^{[1]}$ such that $\mathscr{F}^{[\otimes m_0]} \cong \mathscr{O}_X(p^* D_0)$ for some positive integer m_0 and some divisor D_0 on Z, then $\bar{D}_0$, the closure of D_0 in V, is not ample.*

Proof. Assume the opposite. First, we assume further that $m_0 = 1$ and $\mathscr{O}_Z(D_0)$ has non-zero global sections. Let $E = p^* D_0$, which is effective. Then $\mathscr{F} = \mathscr{O}_X(E)$. Since general fibers of p are isomorphic to $\mathbb{P}^1$, which does not

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carry any non-zero pluri-forms, by Lemma 2.5, the injection $\mathscr{O}_X(E) \hookrightarrow \Omega_X^{[1]}$ factorizes through $(p^*\Omega_Z)^{sat}$. Hence, by the projection formula, we have an injection from $\mathscr{O}_Z(D_0)$ to $p_*((p^*\Omega_Z^1)^{sat})$. However, by Lemma 4.6, we have $p_*((p^*\Omega_Z^1)^{sat}) = \Omega_Z^1$. By Proposition 2.1, we get an injection from $\mathscr{O}_V(\bar{D}_0)$ to $\Omega_V^{[1]}$. Hence, by the Bogomolov–Sommese theorem (see [Gra15, Corollary 1.3]), the Kodaira dimension of $\mathscr{O}_V(\bar{D}_0)$ is not larger than 1. Hence, $\bar{D}_0$ is not ample. We obtain a contradiction.

Now we treat the general case. We show that we can reduce to the previous case. By replacing m_0 with a large multiple, we may assume that $\bar{D}_0$ is very ample. We may also assume that both $\bar{D}_0$ and p^*D_0 are prime and that the pair $(V, \sum_{i=0}^r ((m_i - 1)/m_i)\bar{D}_i)$ is klt. Let $c_X : X_1 \to X$ be the normalization of the ramified cyclic cover with respect to $\mathscr{F}$, m_0 and p^*D_0 (see [KM98, Definition 2.52]). Then c_X is ramified over p^*D_0 with degree m and $(c_X^*\mathscr{F})^{**} \cong \mathscr{O}_{X_1}(E)$, where $E = (c_X^*p^*D_0)_{red}$. Moreover, over the smooth locus of X, there is an injection of sheaves from $c_X^*\Omega_X^1$ to $\Omega_{X_1}^1$. Hence, by Proposition 2.1, we have an injection $\mathscr{O}_{X_1}(E) \hookrightarrow \Omega_{X_1}^{[1]}$.

Let $X_1 \xrightarrow{p_1} Z_1 \xrightarrow{c_Z} Z$ be the Stein factorization. Let $c_V : V_1 \to V$ be the normalization of V in the function field of X_1. Then we obtain an open embedding $j_1 : Z_1 \to V_1$ such that $\operatorname{codim} V_1 \backslash Z_1 \geqslant 2$. If F_p is a general fiber of p and if F_{p_1} is a general fiber of p_1 which is mapped to F_p, then $F_{p_1} \to F_p$ is étale. Since p is a Mori fibration, F_p is a smooth rational curve which is simply connected. Hence, $F_{p_1} \to F_p$ is an isomorphism and $F_{p_1} \cong \mathbb{P}^1$. Hence, c_Z is of degree m_0. Let $H = (c_Z^*D_0)_{red}$. Since $c_Z \circ p_1 : X_1 \to Z$ has connected fibers over D_0, we have $c_Z^*D_0 = m_0H$. Hence, $E_1 = p_1^*H$. Let $\bar{H}$ be the closure of H in V_1. Then it is ample since c_V is finite.

Note that $c_X|_{X_1\backslash E}$ is étale in codimension 1 and $c_Z^*D_0 = m_0H$. By Lemma 4.7, we conclude that there is an effective $\mathbb{Q}$-divisor Δ_1' in Z_1 such that $K_{Z_1} + \Delta_1' = p^*(K_Z + \sum_{i=0}^r ((m_i - 1)/m_i)D_i)$. Hence, if Δ_1 is the closure of Δ_1' in V_1, then $K_{V_1} + \Delta_1 = p^*(K_V + \sum_{i=0}^r ((m_i - 1)/m_i)\bar{D}_i)$. This implies that (V_1, Δ_1) is klt by [KM98, Proposition 5.20 and Corollary 2.35].

Hence, $p_1 : X_1 \to Z_1$ satisfies the conditions in the lemma. There is an injection from $\mathscr{O}_{X_1}(E)$ to $\Omega_{X_1}^{[1]}$, and $\mathscr{O}_{X_1}(E) \cong \mathscr{O}_{X_1}(p_1^*H)$, such that $\bar{H}$ is ample in V_1. We are in the same situation as in the first case. This leads to a contradiction. ∎

LEMMA 4.9. *Let $p : X \to Z$ be an equidimensional fibration such that general fibers of p do not carry any non-zero pluri-forms. Assume that Z is smooth and that there is an open embedding $j : Z \to V$ such* *that* $\operatorname{codim} V\backslash Z \geqslant 2$. *Assume that* $H^0(V, \Omega_V^{[r]}) = \{0\}$ *for all* $r > 0$. *Then* $H^0(X, \Omega_X^{[r]}) = \{0\}$ *for all* $r > 0$.

Proof. Assume the opposite. Let $\mathscr{F} = (p^*\Omega_Z^1)^{sat}$ be the saturation of the image of $p^*\Omega_Z^1$ in $\Omega_X^{[1]}$. Since general fibers of p do not carry any non-zero pluri-forms, we have $H^0(X, \Omega_X^{[r]}) \cong H^0(X, \mathscr{F}^{[\wedge r]})$ by Lemma 2.5. Hence, there is an injection from $\mathscr{O}_X$ to $\mathscr{F}^{[\wedge r]}$. By taking the direct image, we have an injection from $\mathscr{O}_Z$ to $p_*(\mathscr{F}^{[\wedge r]})$. By Lemma 4.6, $p_*(\mathscr{F}^{[\wedge r]}) \cong \Omega_Z^r$. This implies that $H^0(V, \Omega_V^{[r]}) \neq \{0\}$ by Proposition 2.1, which is a contradiction. ∎

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. We argue by contradiction. Let $f: Z \to Z'$ be the result of a $(K_Z + \Delta)$-MMP. Assume that Z' has Picard number 1. Set $\Delta' = f_*\Delta$. Then $K_{Z'} + \Delta'$ is not pseudo-effective either. Thus, $-(K_{Z'} + \Delta')$ is ample. We know that there is a smooth open subset $Z'_0 \subseteq Z'$ with $\operatorname{codim} Z'\backslash Z'_0 \geqslant 2$ such that f^{-1} is an isomorphism from Z'_0 onto its image. Let Z_0 be $f^{-1}(Z'_0)$, which is an open subset in Z. Let $X_0 = p^{-1}(Z_0)$.

Since $\operatorname{codim} Z'\backslash Z'_0 = 2$, there is a projective curve C'_0 in Z'_0 such that it is an ample divisor in Z'. Let C_0 be the strict transform of C'_0 in Z_0. Let α be the class of the curve $p^*C_0 \cap H$, where H is a very ample divisor in X. Then the class α is movable and $p_*\alpha$ is proportional to the class of C_0.

Since general fibers of p do not carry any non-zero pluri-forms, by Lemma 2.5, we obtain that

$$H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(X, ((p^*\Omega_Z^1)^{sat})^{[\otimes m]})$$

for any $m > 0$, where $(p^*\Omega_Z^1)^{sat}$ is the saturation of the image of $(p^*\Omega_Z^1)$ in $\Omega_X^{[1]}$. Hence, $\mu_\alpha^{max}((p^*\Omega_Z^1)^{sat}) \geqslant 0$ and there is a coherent sheaf $\mathscr{H}$ saturated in $(p^*\Omega_Z^1)^{sat}$ such that $\mu_\alpha(\mathscr{H}) \geqslant 0$.

However, since $p|_{X_0}$ is a Mori fibration and Z_0 is smooth, the $\mathbb{Q}$-Cartier divisor which associates to the determinant of $(p^*\Omega_Z^1)^{sat}|_{X_0}$ is equal to $p^*(K_Z + \Delta)|_{X_0}$ by Remark 4.5. Hence,

$$\det((p^*\Omega_Z^1)^{sat}) \cdot \alpha = p^*(K_Z + \Delta) \cdot \alpha = (p \circ f)^*(K_{Z'} + \Delta') \cdot \alpha < 0.$$

This implies that $\mathscr{H} \neq (p^*\Omega_Z^1)^{sat}$. If $\mathscr{J}$ is the quotient $(p^*\Omega_Z^1)^{sat}/\mathscr{H}$, then rank $\mathscr{J}$ = rank $\mathscr{H} = 1$ and $\mathscr{H}[\otimes]\mathscr{J} = \det((p^*\Omega_Z^1)^{sat})$. Thus, $\mathscr{J} \cdot \alpha < 0$.

Since $H^0(X, ((p^*\Omega^1_Z)^{sat})^{[\otimes m]}) \neq \{0\}$ for some $m > 0$, we have

$$H^0(X, \mathscr{H}^{\otimes s}[\otimes]\mathscr{J}^{\otimes t}) \neq \{0\}$$

for some $s, t \geqslant 0$ by Proposition 2.1 and Lemma 2.3. In this case, we have $s > t$ for $(\mathscr{H}[\otimes]\mathscr{J}) \cdot \alpha < 0$ and $\mathscr{J} \cdot \alpha < 0$.

Let F_p be a general fiber of p. Then the class of F_p is movable. Hence, $(\mathscr{H}^{[\otimes s]}[\otimes]\mathscr{J}^{[\otimes t]}) \cdot F_p \geqslant 0$. Moreover, we have

$$(\mathscr{H}[\otimes]\mathscr{J}) \cdot F_p = \det((p^*\Omega^1_Z)^{sat}) \cdot F_p = 0.$$

This implies that $\mathscr{H} \cdot F_p \geqslant 0$ since $s > t$. However, since the restriction of $(p^*\Omega^1_Z)^{sat}$ on F_p is isomorphic to $\mathscr{O}_{F_p} \oplus \mathscr{O}_{F_p}$, we have $\mathscr{H} \cdot F_p = 0$.

Let k be the smallest positive integer such that $\mathscr{H}^{[\otimes k]}$ is invertible. Then there is a Cartier divisor L in Z such that $\mathscr{H}^{[\otimes k]} \cong \mathscr{O}_X(p^*L)$. Let $L' = f_*L$. Then $L' \cdot C'_0 \geqslant 0$ since $\mu_\alpha(\mathscr{H}) \geqslant 0$. Note that if $L' \cdot C'_0 > 0$, then L' is ample on Z' since Z' has Picard number 1. Hence, by Lemma 4.8, we can only have $L' \cdot C'_0 = 0$, and L' is numerically equal to the zero divisor since Z' has Picard number 1. By [AD14, Lemma 2.6], there is a positive integer k' such that $k'L'$ is linearly equivalent to the zero divisor. Hence, $\mathscr{O}_Z(k'L)|_{Z_0} \cong \mathscr{O}_{Z_0}$ and $\mathscr{H}^{[\otimes kk']}|_{X_0} \cong \mathscr{O}_{X_0}$. Let l be the smallest positive integer such that $\mathscr{H}^{[\otimes l]}|_{X_0} \cong \mathscr{O}_{X_0}$. Let $c : W_0 \to X_0$ be the normalization of the cyclic cover with respect to the isomorphism $\mathscr{H}^{[\otimes l]}|_{X_0} \cong \mathscr{O}_{X_0}$ (see [KM98, Definition 2.52]). Then c is étale in codimension 1. We have $\mathscr{O}_{W_0} \cong (c^*\mathscr{H})^{**}$ and there is an injection $(c^*\mathscr{H})^{**} \hookrightarrow (c^*\Omega^1_{X_0})^{**} \cong \Omega^{[1]}_{W_0}$. Hence, $H^0(W_0, \Omega^{[1]}_{W_0}) \neq \{0\}$.

Let $W_0 \to V_0 \to Z'_0$ be the Stein factorization. Let $h : V \to Z'$ be the normalization of Z' in the function field of W_0. Then there is an open embedding from V_0 to V such that $\operatorname{codim} V \backslash V_0 \geqslant 2$. By Lemma 4.7, we have $K_{V_0} \leqslant (h^*(K_{Z'} + \Delta'))|_{V_0}$. Since $\operatorname{codim} V \backslash V_0 \geqslant 2$, there is an effective $\mathbb{Q}$-divisor Δ_V in V such that $K_V + \Delta_V = h^*(K_{Z'} + \Delta')$. Thus the pair (V, Δ_V) is klt by [KM98, Proposition 5.20]. Moreover, $-(K_V + \Delta_V)$ is ample since $-(K_{Z'} + \Delta')$ is ample, hence V is rationally connected by [HM07, Corollary 1.13]. By [GKKP11, Theorem 5.1], we have $H^0(V, \Omega^{[1]}_V) = 0$. However, since general fibers of $W_0 \to V_0$ are isomorphic to $\mathbb{P}^1$, which does not carry any non-zero pluri-forms, we conclude that $H^0(W_0, \Omega^{[1]}_{W_0}) = \{0\}$ from Lemma 4.9. This is a contradiction. ∎

4.2 Proof of Theorem 1.6

The object of this subsection is to prove Theorem 1.6. We study Mori fibrations $X \to Z$ such that X is a projective threefold with $\mathbb{Q}$-factorial

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canonical singularities and Z is a projective normal surface. Assume that X carries non-zero pluri-forms. Then by Theorem 4.2, either the result Z' of any $(K_Z+\Delta)$-MMP has Picard number 2 or $(K_Z+\Delta)$ is pseudo-effective, where Δ is the $\mathbb{Q}$-divisor defined in Theorem 4.2. We study the first case in Proposition 4.10 and the second case in Proposition 4.12.

PROPOSITION 4.10. *Let $p: X\to Z$ be a Mori fibration from a projective threefold to a projective surface such that X has $\mathbb{Q}$-factorial canonical singularities. Let Δ be the divisor in Z defined in Theorem 4.2. Assume that $K_Z+\Delta$ is not pseudo-effective. Let $f: Z\to Z'$ be the result of a $(K_Z+\Delta)$-MMP, and let Δ' be the strict transform of Δ in Z'. Then there is a $(K_{Z'}+\Delta')$-Mori fibration $\pi': Z'\to\mathbb{P}^1$. Let $\pi=\pi'\circ f$, and let $q=\pi\circ p: X\to\mathbb{P}^1$. Then we have $H^0(X,(\Omega_X^1)^{[\otimes m]})\cong H^0(X,((q^*\Omega^1_{\mathbb{P}^1})^{sat})^{[\otimes m]})$ for any $m>0$, where $(q^*\Omega^1_{\mathbb{P}^1})^{sat}$ is the saturation of the image of $q^*\Omega^1_{\mathbb{P}^1}$ in $\Omega_X^{[1]}$.*

Proof. Let $\mathscr{H}=(q^*\Omega^1_{\mathbb{P}^1})^{sat}$, and let $\mathscr{F}=(p^*\Omega_Z^{[1]})^{sat}$. Then we have an exact sequence of coherent sheaves $0\to\mathscr{H}\to\mathscr{F}\to\mathscr{J}\to 0$, where $\mathscr{J}$ is a torsion-free sheaf such that $\det\mathscr{F}=\mathscr{H}[\otimes]\mathscr{J}$. Let α be a class of movable curves in X whose image in Z is not zero and is proportional to the class of general fibers of $\pi: Z\to\mathbb{P}^1$. Then $\mathscr{H}\cdot\alpha=0$. Moreover, we have

$$(\mathscr{H}[\otimes]\mathscr{J})\cdot\alpha=\det\mathscr{F}\cdot\alpha=p^*(K_Z+\Delta)\cdot\alpha$$

by Remark 4.5. This intersection number is negative since for a general fiber F_π of $\pi: \mathrm{Z}\to\mathbb{P}^1$, we have $(K_Z+\Delta)\cdot F_\pi=(K_{Z'}+\Delta')\cdot(f_*F_\pi)<0$. Hence, $\mathscr{J}\cdot\alpha<0$.

There is an open subset U of X with codim $X\backslash U\geqslant 2$ such that we have an exact sequence of locally free sheaves over U, $0\to\mathscr{H}|_U\to\mathscr{F}|_U\to\mathscr{J}|_U\to 0$. Since $\mu_\alpha(\mathscr{H}^{\otimes s}\otimes\mathscr{J}^{\otimes t})<0$ if $t>0$, we have $H^0(U,\mathscr{H}|_U^{\otimes s}\otimes\mathscr{J}|_U^{\otimes t})=\{0\}$ if $t>0$ by Proposition 2.1. Hence, by Lemmas 2.5 and 2.4, we have $H^0(X,(\Omega_X^1)^{[\otimes m]})\cong H^0(X,(\mathscr{F})^{[\otimes m]})\cong H^0(X,(\mathscr{H})^{[\otimes m]})$ for any $m>0$. ∎

EXAMPLE 4.11. We give an example of this kind of threefold. Let $Z=\mathbb{P}^1\times\mathbb{P}^1$. Denote by p_1, p_2 the two natural projections from Z to $\mathbb{P}^1$. Let $z_1,\ldots,z_r$ be $r\geqslant 4$ different points in $\mathbb{P}^1$, and let $C_i=p_1^*z_i$ for $i=1,\ldots,r$. Let $X_0=\mathbb{P}^1\times Z$. By the method of Construction 2.13, we can construct a Mori fibration $\pi: X\to Z$ such that $m(\pi,C_i)=2$ for $i=1,\ldots,r$. Note that $K_Z+\frac{1}{2}(C_1+\cdots+C_r)$ is not pseudo-effective since it has negative intersection number with general fibers of p_1. Moreover, we have

$$\begin{aligned} H^0(X, (\Omega^1_X)^{[\otimes 2]}) &\cong H^0(\mathbb{P}^1, (\Omega^1_{\mathbb{P}^1})^{\otimes 2} \otimes \mathscr{O}_{\mathbb{P}^1}(C_1 + \cdots + C_r)) \\ &\cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-4+r)) \neq \{0\}. \end{aligned}$$

Now we treat the second case. Note that this is the case for Examples 2.12 and 2.14.

PROPOSITION 4.12. *Let $p: X \to Z$ be a Mori fibration from a projective threefold to a projective surface. Assume that X has $\mathbb{Q}$-factorial klt singularities. Assume that $(K_Z + \Delta)$ is pseudo-effective, where Δ is the $\mathbb{Q}$-divisor defined in Theorem 4.2. Then X carries non-zero pluri-forms.*

Proof. By the abundance theorem for log surfaces (see [AFKM92, Theorem 11.1.3]), $(K_Z + \Delta)$ is $\mathbb{Q}$-effective. Hence, there is a positive integer l such that $l(K_Z + \Delta)$ is an effective Cartier divisor. This implies that $h^0(X, (\Omega^1_X)^{[\otimes 2l]})$ is non-zero by Lemma 4.4. ∎

We can now conclude Theorem 1.6.

Proof of Theorem 1.6. Let $X^* \to Z$ be a Mori fibration. If Z is a curve, then we are in the second case of the theorem. Assume that $\dim Z = 2$. If $K_Z + \Delta$ is $\mathbb{Q}$-effective, then we are in the situation of Proposition 4.12. If $K_Z + \Delta$ is not $\mathbb{Q}$-effective, then by Proposition 4.10, there is a fibration $p: X^* \to \mathbb{P}^1$ such that $H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) \cong H^0(X, ((p^*\Omega^1_{\mathbb{P}^1})^{sat})^{[\otimes m]})$ for any $m > 0$. By Lemma 3.1, we have $H^0(X^*, (\Omega^1_{X^*})^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + \sum_{z \in \mathbb{P}^1} [((m(p, z) - 1)m)/m(p, z)]))$ for any $m > 0$. ∎

§5. Proof of Theorem 1.4

In this section, we complete the proof of Theorem 1.4. First, we show that if X is a rationally connected projective threefold with $\mathbb{Q}$-factorial *terminal* singularities, which carries non-zero pluri-forms, then there is a dominant rational map from X to $\mathbb{P}^1$ (Lemma 5.2). To this end, we need the following lemma.

LEMMA 5.1. *Let $p: X \to Z$ be a Mori fibration such that X has $\mathbb{Q}$-factorial terminal singularities and* $\dim Z = \dim X - 1$. *Then there exists an open subset $Z_0 \subseteq Z$ with* codim $Z \backslash Z_0 \geqslant 2$ *such that every scheme-theoretic fiber over Z_0 is reduced.*

Proof. By taking general hyperplane sections on Z, we reduce to the case where Z is a smooth curve and p is a Fano fibration. In this case, X is a surface with terminal singularities. Hence, X is smooth and the fibers of p are reduced. ∎

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LEMMA 5.2. *Let X be a rationally connected projective threefold with $\mathbb{Q}$-factorial terminal singularities such that $H^0(X,(\Omega_X^1)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$. Let $f : X \dashrightarrow X^*$ be the result of an MMP for X. Then there is a fibration $p : X^* \to \mathbb{P}^1$.*

Proof. Note that X^* is a Mori fiber space. Then we have a Mori fibration $q : X^* \to Z$, where Z is a normal rationally connected variety. If $\dim Z = 1$, then we are done.

By Lemma 2.2, we know that X^* carries non-zero pluri-forms. Hence, $\dim Z > 0$ by [Ou14, Theorem 3.1]. Assume that $\dim Z = 2$. Then Z has canonical singularities by [MP08, Corollary 1.2.8]. Hence, K_Z is not pseudo-effective by [Kol96, Corollary 1.11]. Moreover, by Lemma 5.1, $m(q, D) = 1$ for any effective divisor D on Z. Hence, by Theorem 4.2, if $Z \to Z'$ is the result of an MMP for Z, then Z' has Picard number 2. Hence, we have a Mori fibration $Z' \to \mathbb{P}^1$. Let p be the composition of $X^* \to Z \to Z' \to \mathbb{P}^1$. Then p is a fibration from X^* to $\mathbb{P}^1$ □

With the notation as above, note that a general fiber F' of $p : X^* \to \mathbb{P}^1$ is a smooth rationally connected surface. Hence, F' does not carry any non-zero pluri-forms. Let U be the largest open subset in X over which $f : X \dashrightarrow X^*$ is regular. Then $\operatorname{codim} X \backslash U \geqslant 2$, $\operatorname{codim} X^* \backslash f(U) \geqslant 2$, and the rational map $p : X \dashrightarrow \mathbb{P}^1$ is regular over U. If F is a general fiber of $U \to \mathbb{P}^1$, then $f(F) \subseteq F'$, where F' is a general fiber of p. Moreover, $\operatorname{codim} F' \backslash f(F) \geqslant 2$. Hence, $f(F)$ does not carry any non-zero pluri-forms and neither does F by Lemma 2.2. The following lemma shows that the rational map $X \dashrightarrow \mathbb{P}^1$ is regular. Moreover, general fibers of $X \to \mathbb{P}^1$ are birational to the ones of $X^* \to \mathbb{P}^1$ which are rationally connected. Hence, general fibers of $X \to \mathbb{P}^1$ are rationally connected.

LEMMA 5.3. *Let X be a projective threefold with $\mathbb{Q}$-factorial terminal singularities. Assume that there is a non-constant rational map $p : X \dashrightarrow \mathbb{P}^1$ which is regular over U such that $\operatorname{codim} X \backslash U \geqslant 2$. Assume that general fibers of $U \to \mathbb{P}^1$ do not carry any pluri-forms. If $H^0(X,(\Omega_X^1)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$, then p is regular.*

Proof. Let Γ be the normalization of the graph of p. Let $p_1 : \Gamma \to X$, $p_2 : \Gamma \to \mathbb{P}^1$ be the natural projections. Then there is a natural injection from $H^0(\Gamma,(\Omega_\Gamma^1)^{[\otimes m]})$ to $H^0(X,(\Omega_X^1)^{[\otimes m]})$ by Lemma 2.2. Let σ be a non-zero element in $H^0(X,(\Omega_X^1)^{[\otimes m]})$. Then σ induces a rational section σ_Γ of $(\Omega_\Gamma^1)^{[\otimes m]}$ on Γ. Let E be a p_1-exceptional divisor. Then there is a curve

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in E which is contracted by p_2 since $\dim E = 2 > 1$. Hence, this curve is not contracted by p_1 since the graph of p is included in $X \times \mathbb{P}^1$ and the normalization map is finite. Thus, $p_1(E)$ is a curve in X, and X is smooth around the generic point of $p_1(E)$ since X is smooth in codimension 2 (see [KM98, Corollary 5.18]). Hence, σ_Γ does not have a pole along E. This implies that we have an isomorphism from $H^0(\Gamma, (\Omega^1_\Gamma)^{[\otimes m]})$ to $H^0(X, (\Omega^1_X)^{[\otimes m]})$ induced by p_1.

Note that $p_1^{-1}|_U$ induces an isomorphism from U onto its image. If F_U is the fiber of $p|_U$ over a general point z, then $p_1^{-1}(F_U)$ is an open subset of F_Γ, where F_Γ is the fiber of $p_2 : \Gamma \to \mathbb{P}^1$ over z. Since F_U does not carry any non-zero pluri-forms, neither does F_Γ. By Lemma 2.5, this implies that $H^0(\Gamma, (\Omega^1_\Gamma)^{[\otimes m]}) \cong H^0(\Gamma, ((p_2^*\Omega^1_{\mathbb{P}^1})^{sat})^{[\otimes m]})$.

We first prove that p is regular in codimension 2. Assume the opposite. Then there is a divisor D in Γ which is exceptional for $p_1 : \Gamma \to X$, and the codimension of $p_1(D)$ in X is 2. Since X is smooth in codimension 2 (see [KM98, Corollary 5.18]), X is smooth around the generic point of $p_1(D)$. Thus, there is a smooth quasi-projective curve C in D such that Γ is smooth along C and C is contracted to a smooth point of X by p_1. Note that C is horizontal over $\mathbb{P}^1$ under the projection $p_2 : \Gamma \to \mathbb{P}^1$ for the same reason as before. Let σ be a non-zero element in $H^0(\Gamma, (\Omega^1_\Gamma)^{[\otimes m]})$. By the exact sequence of locally free sheaves $\Omega^1_\Gamma|_C \to \Omega^1_C \to 0$, σ induce an element σ_C in $H^0(C, (\Omega^1_C)^{\otimes m})$. On the one hand, C is horizontal over $\mathbb{P}^1$ and σ is non-zero in $H^0(\Gamma, ((p_2^*\Omega^1_{\mathbb{P}^1})^{sat})^{[\otimes m]})$, we have $\sigma_C \neq 0$. On the other hand, since C is contracted to a smooth point in X, we obtain $\sigma_C = 0$ for σ is the pullback of certain element in $H^0(X, (\Omega^1_X)^{[\otimes m]})$. This is a contradiction.

Now we prove that p is regular. Let F_1 and F_2 be two different fibers of $U \to \mathbb{P}^1$. Then their closures in X are two Weil divisors and their intersection is included in a closed subset of codimension at most 3. Hence, their intersection is empty since X is $\mathbb{Q}$-factorial. This implies that p is regular. ∎

Together with Lemmas 2.5 and 3.1, we can conclude Theorem 1.4.

Proof of Theorem 1.4. By Lemmas 5.2 and 5.3, there is a fibration $p : X \to \mathbb{P}^1$ such that general fibers of p do not carry any non-zero pluri-forms. Lemma 2.5 shows that all pluri-forms on X come from the base $\mathbb{P}^1$. Finally, we obtain the formula in the theorem from Lemma 3.1. ∎

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Wenhao Ou
Mathematical Institute of the University of Bonn
Endenicher Allee 60
D-53115 Bonn
Germany
wenhaoou@math.uni-bonn.de

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