HEREDITARY RADICALS AND DERIVATIONS
OF ALGEBRAS

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Introduction. In the Amitsur-Kurosch theory of radicals in rings (**2**), an important problem is to determine the relationship between the radical of a ring and the radical of each of its ideals. The first result on this problem was by Amitsur who proved that if β is a hereditary radical in the sense that ideals of β-radical rings are β-radical, then for each associative ring R and ideal I of R, $\beta(I) = I \cap \beta(R)$, where $\beta(R)$ denotes the β-radical of R; see (**2**).

Later, Suliński, Divinsky, and the author proved that if β is any radical and R is an associative or alternative ring, then $\beta(I) \subseteq I \cap \beta(R)$ for each ideal I of R; see (**3**). Since every hereditary radical β has the property $\beta(I) \supseteq I \cap \beta(R)$, this result provided another proof of Amitsur's theorem and extended that theorem to alternative rings. Of course, this raises the question of whether Amitsur's theorem is true for Lie or Jordan rings.

To answer this question, it is necessary to study the behaviour of a hereditary radical under derivations, and that is the subject of this paper. However, since the nature of the additive group of a ring plays an important role here, it is expedient to restrict our attention to algebras.

The main result is that if A is an algebra (not necessarily associative or finite-dimensional) over a non-modular field and if A satisfies the descending chain condition (D.C.C.) for ideals, then for each hereditary radical β and derivation D of A, $(\beta(A))D \subseteq \beta(A)$.

In § 3 we give some applications of this theorem to Lie, Jordan, and flexible algebras. The result in the Jordan case may be of particular interest because of Jacobson's recent work (**5**) on Jordan algebras with chain condition.

1. Preliminary definitions. For algebras, the definition of radical property may be formulated as follows.

Let U_F be a class of algebras, not necessarily associative or finite-dimensional over a field F, which is *universal* in the sense that ideals and homomorphic images of algebras in U_F are again in U_F.

Let β be a property that a given algebra A in U_F might have. In the case that A has property β we say that A is a β-algebra. Then β is said to be a *radical* property in U_F if the following conditions are met:

(I) Homomorphic images of β-algebras in U_F are β-algebras;

(II) Each $A \in U_F$ has a maximal β-ideal (i.e., an ideal which is a β-algebra), $\beta(A)$, which contains all the β-ideals of A. The ideal $\beta(A)$ is called the β-radical of A;

(III) For each $A \in U_F$, $\beta(A/\beta(A)) = 0$. That is, $A/\beta(A)$ is β-semi-simple.

A radical β is said to be *hereditary* if ideals of β-algebras are β-algebras. This is equivalent to saying that $\beta(I) \supseteq I \cap \beta(A)$ for each $A \in U_F$ and ideal I of A.

Let $A \in U_F$ and define $A_0 = A$, $A_n = (A_{n-1})^2$ for $n > 0$. The sequence

$$A = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n \supseteq A_{n+1} \supseteq \ldots$$

is called the derived series for A, and A is said to be solvable if $A_n = 0$ for some integer n. Note that in the above sequence, A_k is an ideal of A_{k-1}, $k \geq 1$.

A derivation of an algebra A is a linear transformation D of A with the property that $(xy)D = (xD)y + x(yD)$ for all $x, y \in A$.

2. Main theorem. We require the following lemma, whose proof is obvious.

LEMMA 2.1. *Let I be an ideal of an algebra A and D a derivation of A. Then $I_D = \{t \in I \mid tD \in I\}$ is an ideal of A and $I_D \supseteq I^2$.*

THEOREM 2.2. *Let U_F be a universal class of algebras defined over a field F of characteristic 0, and suppose that $A \in U_F$ has D.C.C. on ideals. Then for each hereditary radical β defined in U_F and derivation D of A, $(\beta(A))D \subseteq \beta(A)$.*

Proof. We first prove the theorem for $\beta(A)$ a solvable ideal of A.

There is nothing to prove if $\beta(A) = 0$; hence, assume that $I = \beta(A)$ is a non-zero solvable ideal of A and let

$$I = I_0 \supset \ldots \supset I_n \supset I_{n+1} = 0$$

be the derived series for I.

Select $x \neq 0$ in I_n. Since $I_n{}^2 = I_{n+1} = 0$, the one-dimensional subspace, Fx, spanned by x is an ideal of I_n. Then since each of the subalgebras in the derived series is an ideal of its predecessor, the fact that β is hereditary implies that Fx is a β-algebra.

We claim now that if B is any algebra such that $B^2 = 0$, then B is β-radical. To show this, it is sufficient to show that each non-zero homomorphic image B' of B contains a non-zero β-ideal (**4**, p. 4). However, if b' is a non-zero element of B', then Fb' is a non-zero ideal of B' and the mapping $\alpha x \to \alpha b'$, $\alpha \in F$, is an isomorphism from the β-algebra Fx onto Fb'. Hence, Fb' is a non-zero β-ideal of B', as required.

Now we can show that I contains all the solvable ideals of A. Indeed, suppose that W is a solvable ideal of A with derived series

$$W = W_0 \supseteq \ldots \supseteq W_m \supseteq W_{m+1} = 0.$$

Since for each factor W_k/W_{k+1} we have that $(W_k/W_{k+1})^2 = 0$, it follows that W_k/W_{k+1} is β-radical; hence, if W_{k+1} is β-radical, so is W_k a β-radical algebra.

Since W_m is β-radical, we have that W is a β-radical ideal of A; hence $W \subseteq I$.

It is easy to see that $I + (I)D$ is an ideal of A; thus, it follows from the above remark that if $I + (I)D$ is solvable, then $(I)D \subseteq I$.

For each integer of the form $m = 2^k$, where k is a positive integer, we define products $\{a_1, \ldots, a_m\}$ of elements $a_1, \ldots, a_m \in A$ by the rule: $\{a_1, a_2\} = a_1 a_2$, $\{a_1, \ldots, a_{2p}\} = \{a_1, \ldots, a_p\}\{a_{p+1}, \ldots, a_{2p}\}$. Evidently, to prove that $I + (I)D$ is solvable, it is sufficient to prove that there is an integer m such that $\{a_1 D, \ldots, a_m D\} \in I$ for all $a_1, \ldots, a_m \in I$.

We may generalize the Leibniz rule,

$$(xy)D^n = \sum_{r=0}^{n} \binom{n}{r} x D^r y D^{n-r}$$

to obtain the formula

$$(1) \quad \{a_1, \ldots, a_m\} D^m = \sum \binom{m}{\alpha_1}\binom{m - \alpha_1}{\alpha_2} \ldots \binom{m - \alpha_1 - \ldots - \alpha_{m-2}}{\alpha_{m-1}}$$

$$\times \{a_1 D^{\alpha_1}, a_2 D^{\alpha_2}, \ldots, a_m D^{\alpha_m}\},$$

where $\alpha_m = m - \alpha_1 - \ldots - \alpha_{m-1}$ and the summation is taken over all choices of $\alpha_1, \ldots, \alpha_{m-1}$ such that $0 \leqq \alpha_1 \leqq m, 0 \leqq \alpha_i \leqq m - \alpha_1 - \ldots - \alpha_{i-1}$, $1 < i \leqq m - 1$. Of course, xD^0 is defined to be x.

Since I is solvable, there exists an integer m such that $\{a_1, \ldots, a_m\} = 0$ for all $a_1, \ldots, a_m \in I$; hence, $\{a_1, \ldots, a_m\} D^m = 0$. Moreover, for $a_1, \ldots, a_m \in I$, $\{a_1 D^{\alpha_1}, \ldots, a_m D^{\alpha_m}\} \in I$ if some $\alpha_i = 0, 1 \leqq i \leqq m$. On the other hand, if $\alpha_i \geqq 1$ for all $i, 1 \leqq i \leqq m$, then, necessarily, $\alpha_1 = \ldots = \alpha_m = 1$. Therefore, it follows from (1) that

$$\binom{m}{1}\binom{m-1}{1} \ldots \binom{1}{1}\{a_1 D, \ldots, a_m D\} \in I,$$

and since F has characteristic 0, we have that $\{a_1 D, \ldots, a_m D\} \in I$.

This completes the proof of the theorem when $\beta(A)$ is solvable, and we proceed to consider the general case.

Again, set $I = \beta(A)$. We define a sequence

$$I = I^{(1)} \supseteq \ldots \supseteq I^{(n)} \supseteq I^{(n+1)} \supseteq \ldots$$

by setting $I^{(n+1)} = (I^{(n)})_D$ for $n \geqq 1$. From Lemma 2.1, we know that $I^{(n)}$ is an ideal of A and $I^{(n+1)} \supseteq (I^{(n)})^2$ for all $n \geqq 1$.

From our assumption of D.C.C., it follows that $I^{(m)} = I^{(m+1)} = \ldots$ for some integer m. Then $(I^{(m)})D \subseteq I^{(m)}$, so that D induces a derivation on the algebra $A' = A/I^{(m)}$. Since β is hereditary, $I^{(m)}$ is a β-ideal of A; hence, $\beta(A') = I/I^{(m)}$. From the property $I^{(n+1)} \supseteq (I^{(n)})^2$, it follows that $\beta(A')$ is solvable. Using the result from the first part of the proof, we conclude that $\beta(A')$ is invariant under the derivation of A' which is induced by D. Since $\beta(A') = \beta(A)/I^{(m)}$, this implies that $(\beta(A))D \subseteq \beta(A)$.

Counterexamples. It is well-known that the above theorem is false if F has characteristic not equal to 0. For counterexamples in the case U_F consists of associative or Lie algebras, see (**6**, p. 75). On the other hand, the lone assumption that F be of characteristic 0 is not sufficient to give the above result. As an example, let A be the algebra of formal power series $\sum_{n=0}^{\infty} a_n x^n$ in the variable x with coefficients a_n taken from an arbitrary field F. Then the Jacobson radical, $J(A)$, is the ideal consisting of all $\sum_{n=0}^{\infty} a_n x^n$ with $a_0 = 0$; see (**7**, p. 21). If we take $D = d/dx$, then $x \in J(A)$ but $xD = 1 \notin J(A)$.

3. Applications. *Lie algebras.* Jacobson's argument in (**6**, p. 75) shows that a hereditary radical β has the property $\beta(I) = I \cap \beta(L)$ for each Lie algebra L and ideal I of L if and only if β is invariant under all derivations. Therefore, we have the following theorem as a corollary to Theorem 2.2.

THEOREM 3.1. *If β is a hereditary radical, then for each Lie algebra L over a field of characteristic 0 and ideal I of L satisfying D.C.C. on its ideals,*

$$\beta(I) = I \cap \beta(L).$$

Jordan algebras. Let A be a Jordan algebra over a field F of characteristic 0. The associator, (a, b, c), $a, b, c \in A$, is defined by $(a, b, c) = (ab)c - a(bc)$.

A well-known identity for Jordan algebras is $(w, xy, z) = x(w, y, z) + (w, x, z)y$. Therefore, for fixed a, $b \in A$, the mapping $x \rightarrow (a, x, b)$ is a derivation of A.

Now, suppose that I is an ideal of A and assume that I has D.C.C. on ideals. Let β be a hereditary radical and put $M = \beta(I)$. For each $a, b \in A$, the map $i \rightarrow (a, i, b)$, $i \in I$, defines a derivation of I, hence leaves M invariant because of Theorem 2.2. Thus, we have that

(1) $$(A, M, A) \subseteq M.$$

This implies that

(2) $$M + Mx \text{ is an ideal of } I \text{ for each } x \in A.$$

Indeed, if $m \in M$, $i \in I$, then $i(mx) = -(i, m, x) + (im)x \in M + Mx$ because of (1) and the fact that M is an ideal of I.

Next we prove that

(3) $$(Mx \cdot Mx)I \subseteq M \quad \text{for all } x \in A.$$

To do this, we recall that in any algebra we have the identity

$$0 = (xm, y, z) - (x, my, z) + (x, m, yz) - x(m, y, z) - (x, m, y)z.$$

In particular, if $m \in M$, $x \in A$, $y, z \in I$, then $(x, my, z) \in (A, M, A) \subseteq M$, $(x, m, yz) \in (A, M, A) \subseteq M$, and $(x, m, y)z \in (A, M, A)I \subseteq MI \subseteq M$; hence, $(xm, y, z) - x(m, y, z) \in M$. Using the fact that $x \rightarrow (m, x, z)$ is a

derivation, we can write $x(m, y, z) = (m, xy, z) - (m, x, z)y \equiv -(m, x, z)y \pmod{M}$ since $(m, xy, z) \in (M, AI, I) \subseteq (M, I, I) \subseteq M$. Altogether, we have that $(xm, y, z) + (m, x, z)y \equiv 0 \pmod{M}$. In this relation, put $z = mx$. Then $(xm, y, mx) = 0$ since A is commutative; hence, $(m, x, mx)y \equiv 0 \pmod{M}$. However, $(m, x, mx)y = (mx \cdot mx)y - (m(x \cdot mx))y$, and $(m(x \cdot mx))y \in M$. Therefore, $(mx \cdot mx)y \in M$ for all $m \in M$, $x \in A$, $y \in I$. If we now replace m by $m + m'$, where $m' \in M$, then we obtain $2(mx \cdot m'x)y \in M$; hence, $(mx \cdot m'x)y \in M$. Since the elements of $(Mx \cdot Mx)I$ are sums of terms of the type $(mx \cdot m'x)y$, $m, m' \in M$, $y \in I$, it follows that $(Mx \cdot Mx)I \subseteq M$.

Finally, let us note that $M^2A \subseteq M$. Indeed, using (1) we obtain $M^2A = (MM)A \subseteq (M, M, A) + M(MA) \subseteq M$.

The next lemma is an easy consequence of (2) and (3).

LEMMA 3.2. *Let A be a non-modular Jordan algebra and β a hereditary radical. Assume the ideal I has D.C.C. on its ideals and put $M = \beta(I)$. Then for each $x \in A$, $M + Mx$ and $M + Mx \cdot Mx$ are ideals of I and $(Mx \cdot Mx)(Mx \cdot Mx) \subseteq M$. Moreover, $M^2A \subseteq M$.*

THEOREM 3.3. *Let A be a non-modular Jordan algebra and β a hereditary radical. Then, if I is an ideal of A and has D.C.C. on its ideals,*

$$\beta(I) = I \cap \beta(A).$$

Proof. It is sufficient to prove that $\beta(I)$ is an ideal of A. However, having Lemma 3.2, this may be done simply by repeating the argument for alternative rings in (**3**).

We remark that if every algebra B such that $B^2 = 0$ is β-radical, then it is not necessary to refer to (**3**) to complete the proof. Indeed, for each $x \in A$, $M + Mx \cdot Mx/M$ is an ideal in the β-semi-simple algebra I/M. According to Lemma 3.2, $(M + Mx \cdot Mx/M)^2 = 0$; hence, $Mx \cdot Mx \subseteq M$. Then, applying the same argument to $M + Mx/M$, we find that $Mx \subseteq M$. That is, M is an ideal of A.

Flexible algebras. A flexible algebra A is one in which $(xy)z + (zy)x = x(yz) + z(yx)$ for all $x, y, z \in A$.

In determining the structure of A, it is useful to study the commutative algebra A^+ which is obtained from A by letting A^+ be the vector space of A in which a new product $x \circ y$ is defined in terms of the product xy of A by the equation $x \circ y = xy + yx$; see, for example (**1**; **8**; or **9**).

Let $(x, y) = xy - yx$. Then it is easy to see that the mapping $x \to (x, z)$ is a derivation of A^+ for each $z \in A$. Therefore, we have the following result by application of Theorem 2.2.

THEOREM 3.4. *Let A be a flexible algebra over a field of characteristic 0. Then for every hereditary radical β defined in the class of commutative algebras, $\beta(A^+)$ is an ideal of A if A^+ satisfies D.C.C. on ideals.*

We remark here that for power-associative algebras, i.e., algebras such that each element generates an associative subalgebra, the nil radical η certainly is hereditary. Therefore, we have the result that if A is a flexible, power-associative algebra over a non-modular field which is η-semi-simple, and if A^+ has D.C.C. on ideals, then so is A^+ an η-semi-simple algebra.

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