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ON THE CRITICAL FUJITA EXPONENT FOR A DEGENERATE PARABOLIC SYSTEM COUPLED VIA NONLINEAR BOUNDARY FLUX

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Abstract In this paper, we deal with the non-negative solutions of a degenerate parabolic system with nonlinear coupled boundary conditions and non-negative non-trivial compactly supported initial data. The critical Fujita exponents are given and the blow-up rates of the non-global solution are obtained.

Keywords: Fujita exponents; degenerate parabolic equations; nonlinear boundary flux; non-global solutions; blow-up rate

2000 *Mathematics subject classification:* Primary 35B33
Secondary 35K65; 35K60

1. Introduction and main results

In this paper, we deal with the following degenerate parabolic system:

$$\left.\begin{aligned} u_t &= (|u_x|^{m-1}u_x)_x, \\ v_t &= (|v_x|^{n-1}v_x)_x, \end{aligned}\right\} \quad x > 0,\ 0 < t < T, \tag{1.1}$$

with nonlinear coupled boundary flux

$$\left.\begin{aligned} -|u_x|^{m-1}u_x(0,t) &= u^{\alpha}(0,t)v^{p}(0,t), \\ -|v_x|^{n-1}v_x(0,t) &= u^{q}(0,t)v^{\beta}(0,t), \end{aligned}\right\} \quad 0 < t < T, \tag{1.2}$$

and initial data

$$\left.\begin{aligned} u(x,0) &= u_0(x), \\ v(x,0) &= v_0(x), \end{aligned}\right\} \quad x > 0, \tag{1.3}$$

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where $m, n > 1$, $p, q > 0$, $\alpha, \beta \geqslant 0$ and $u_0(x)$, $v_0(x)$ are continuous, non-negative and compactly supported in $\mathbb{R}^+$.

Parabolic systems like (1.1) appear in several branches of applied mathematics. They have been used to model, for example, chemical reactions, heat transfer or population dynamics (see [**9**] and the reference therein).

The problems of global existence, blow-up, blow-up rate and blow-up set are considered by many authors (see [**3**, **4**, **7**, **11**]). In particular, critical Fujita exponents are very interesting for various nonlinear parabolic equations of mathematical physics (see [**2**, **10**] and references therein).

The concept of critical Fujita exponents was proposed by Fujita in the 1960s during discussion of the heat conduction equation with a nonlinear source (see [**5**]).

In [**7**], Galaktionov and Levine study the following scalar problem:

$$\left.\begin{aligned} u_t &= (|u_x|^{m-1}u_x)_x, && x > 0,\ 0 < t < T,\\ -|u_x|^{m-1}u_x &= u^p, && x = 0,\ 0 < t < T,\\ u(x,0) &= u_0(x), && x > 0, \end{aligned}\right\} \tag{1.4}$$

where $m > 1$. They show that if $0 < p \leqslant p_0 = 2m/(m+1)$, then for arbitrary initial data the solution is global in time, while for $p > 2m/(m+1)$ there are solutions with finite-time blow-up. Thus, p_0 is the critical global existence exponent. Moreover, they prove that $p_{\mathrm{c}} = 2m$ is a critical exponent of Fujita type. By definition, this means that p_{c} has the following properties:

(i) if $p_0 < p \leqslant p_{\mathrm{c}}$, then non-trivial $u(x,t)$ blows up in a finite time for all non-trivial u_0;

(ii) if $p > p_{\mathrm{c}}$, then $u(x,t)$ is global in time for small and non-trivial u_0.

In [**13**], Rossi considered the following problem:

$$\left.\begin{aligned} &u_t = \Delta u, && v_t = \Delta v, && (x,t) \in B_1(0) \times (0,T),\\ &\frac{\partial u}{\partial n} = u^{p_{11}}v^{p_{12}}, && \frac{\partial v}{\partial n} = u^{p_{21}}v^{p_{22}}, && (x,t) \in \partial B_1(0) \times (0,T),\\ &u(x,0) = u_0(x) > 0, && v(x,0) = v_0(x) > 0, && x \in B_1(0). \end{aligned}\right\} \tag{1.5}$$

Under some assumptions the author proved that there exist positive constants c and C, such that

$$c \leqslant \max_{x \in B_1(0)} u(x,t)(T-t)^{\alpha_1/2} \leqslant C, \quad c \leqslant \max_{x \in B_1(0)} v(x,t)(T-t)^{\alpha_2/2} \leqslant C \quad \text{for } 0 < t < T,$$

where

$$\alpha_1 = \frac{p_{12} - p_{22} + 1}{(p_{11}-1)(p_{22}-1) - p_{12}p_{21}}, \qquad \alpha_2 = \frac{p_{21} - p_{11} + 1}{(p_{11}-1)(p_{22}-1) - p_{12}p_{21}}.$$

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In [**14**], Wang *et al*. considered the following problem:

$$\left.\begin{aligned} u_t &= u_{xx}, & v_t &= v_{xx}, & x>0,\ t>0,\\ -\frac{\partial u}{\partial x}(0,t) &= v^p(0,t), & -\frac{\partial v}{\partial x}(0,t) &= u^q(0,t), & t>0,\\ u(x,0) &= u_0(x), & v(x,0) &= v_0(x), & x>0. \end{aligned}\right\} \tag{1.6}$$

Under some assumptions they established the blow-up estimate near the blow-up time. That is

$$c(T-t)^{-\tau_1} \leqslant u(0,t) \leqslant C(T-t)^{-\tau_1} \quad \text{and} \quad c(T-t)^{-\tau_2} \leqslant v(0,t) \leqslant C(T-t)^{-\tau_2},$$

where

$$\tau_1 = \frac{p+1}{2(pq-1)}, \qquad \tau_2 = \frac{q+1}{2(pq-1)}.$$

In [**15**], Wang *et al*. considered the following problem:

$$\left.\begin{aligned} u_t &= u_{xx}, & v_t &= v_{xx}, & x>0,\ 0<t<T,\\ -u_x(0,t) &= u^{\alpha}(0,t)v^p(0,t), & -v_x(0,t) &= u^q(0,t)v^{\beta}(0,t), & 0<t<T,\\ u(x,0) &= u_0(x), & v(x,0) &= v_0(x), & x>0. \end{aligned}\right\} \tag{1.7}$$

The global existence and blow-up conditions for solutions of (1.7) are $pq \leqslant (1-\alpha)(1-\beta)$ and $pq > (1-\alpha)(1-\beta)$, respectively. The blow-up rate of the solution (u,v) is $(O((T-t)^{-\gamma_1}), O((T-t)^{-\gamma_2}))$ as $t \to T$ with $\alpha < 1$, $\beta < 1$ and $pq > (1-\alpha)(1-\beta)$, where

$$\gamma_1 = \frac{1}{2}\frac{p+1-\beta}{pq-(1-\alpha)(1-\beta)}, \qquad \gamma_2 = \frac{1}{2}\frac{q+1-\alpha}{pq-(1-\alpha)(1-\beta)}.$$

In [**12**], Quirós and Rossi considered the degenerate equation

$$\left.\begin{aligned} u_t &= (u^m)_{xx}, & v_t &= (v^n)_{xx}, & x>0,\ 0<t<T,\\ -(u^m)_x(0,t) &= v^p(0,t), & -(v^n)_x(0,t) &= u^q(0,t), & 0<t<T,\\ u(x,0) &= u_0(x), & v(x,0) &= v_0(x), & x>0, \end{aligned}\right\} \tag{1.8}$$

with notation

$$\alpha_1 = \frac{2p+n+1}{(m+1)(n+1)-4pq}, \qquad \alpha_2 = \frac{2q+m+1}{(m+1)(n+1)-4pq},$$

$$\beta_1 = \frac{p(m-1-2q)+(n+1)m}{(m+1)(n+1)-4pq}, \qquad \beta_2 = \frac{q(n-1-2p)+(m+1)n}{(m+1)(n+1)-4pq}.$$

They proved that the solutions of (1.8) are global if $pq \leqslant \frac{1}{4}(m+1)(n+1)$, and may blow up in finite time if $pq > \frac{1}{4}(m+1)(n+1)$. In the case of $pq > \frac{1}{4}(m+1)(n+1)$, if $\alpha_1+\beta_1 \leqslant 0$, or $\alpha_2+\beta_2 \leqslant 0$, then every non-negative, non-trivial solution of (1.8) blows up in finite time; if $\alpha_1+\beta_1 > 0$ and $\alpha_2+\beta_2 > 0$, then there exist blow-up solutions for large initial data and global solutions for small initial data. The critical Fujita exponents to (1.8) are described by $\alpha_i + \beta_i = 0$, $i = 1, 2$, while the blow-up rate of the positive solution is $O((T-t)^{-\alpha_1})$ for component u and $O((T-t)^{-\alpha_2})$ for v as $t \to T$.

In [**1**], Audreu *et al.* consider the behaviour of solutions of the following parabolic problem:

$$\left.\begin{aligned} u_t &= \Delta(|u|^{m-1}u) - \lambda|u|^{p-1}u && \text{in } \Omega \times (0,T), \\ \frac{\partial(|u|^{m-1}u)}{\partial n} &= |u|^{q-1}u && \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) && \text{in } \Omega. \end{aligned}\right\} \tag{1.9}$$

By constructing adequate supersolutions and subsolutions, they obtain the existence of a globally bounded weak solution or blow-up solution that depends on the relationship between the parameters m, p, q and λ. They also prove the results about uniqueness and non-uniqueness in the case of null initial data.

In [**18**], Zheng *et al.* considered the degenerate equations coupled via nonlinear boundary flux:

$$\left.\begin{array}{lll} u_t = (u^m)_{xx}, & v_t = (v^n)_{xx}, & x > 0,\ 0 < t < T, \\ -(u^m)_x(0,t) = u^\alpha(0,t)v^p(0,t), & -(v^n)_x(0,t) = u^q(0,t)v^\beta(0,t), & 0 < t < T, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x > 0, \end{array}\right\} \tag{1.10}$$

with notation

$$r_1 = \frac{2p+n+1-2\beta}{4pq-(m+1-2\alpha)(n+1-2\beta)}, \qquad r_2 = \frac{2q+m+1-2\alpha}{4pq-(m+1-2\alpha)(n+1-2\beta)},$$
$$s_1 = \frac{1-r_1(m-1)}{2}, \qquad s_2 = \frac{1-r_2(n-1)}{2}.$$

They proved that the solutions of (1.10) are global if $\alpha < \frac{1}{2}(m+1)$, $\beta < \frac{1}{2}(n+1)$ and $pq \leqslant (\frac{1}{2}(m+1)-\alpha)(\frac{1}{2}(n+1)-\beta)$ and may blow up in finite time if $\alpha > \frac{1}{2}(m+1)$ or $\beta > \frac{1}{2}(n+1)$. In the case when $\alpha \leqslant \frac{1}{2}(m+1)$, $\beta \leqslant \frac{1}{2}(n+1)$ and $pq > (\frac{1}{2}(m+1)-\alpha)(\frac{1}{2}(n+1)-\beta)$, if $s_1 < r_1$ or $s_2 < r_2$, or $s_1 = r_1$ and $s_2 = r_2$, then every non-negative, non-trivial solution of (1.10) blows up in finite time; if $s_1 > r_1$ and $s_2 > r_2$, then the solution of (1.10) is global for small initial data and blows up in finite time with large initial data. The critical Fujita exponents to (1.10) are described by $r_i = s_i$, $i = 1, 2$, while the blow-up rate of the positive solution is $O((T-t)^{-r_1})$ for component u and $O((T-t)^{-r_2})$ for v as $t \to T$.

The purpose of this paper is to extend the main results of [**7**] into the more general form (1.1)–(1.3). To state our results, we need to introduce parameters k_i, l_i, $i = 1, 2$, satisfying

$$\begin{pmatrix} \alpha - \dfrac{2m}{m+1} & p \\ q & \beta - \dfrac{2n}{n+1} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \dfrac{m}{m+1} \\ \dfrac{n}{n+1} \end{pmatrix}. \tag{1.11}$$

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By (1.11), we have

$$\left.\begin{aligned} k_1 &= \frac{n(m+1)p - m(n\beta+\beta-2n)}{(m+1)(n+1)pq - (m\alpha+\alpha-2m)(n\beta+\beta-2n)}, \\ k_2 &= \frac{m(n+1)q - n(m\alpha+\alpha-2m)}{(m+1)(n+1)pq - (m\alpha+\alpha-2m)(n\beta+\beta-2n)}. \end{aligned}\right\} \tag{1.12}$$

Set

$$l_1 = \frac{1-k_1(m-1)}{m+1} \quad \text{and} \quad l_2 = \frac{1-k_2(n-1)}{n+1}. \tag{1.13}$$

Linear algebraic systems such as (1.11) were also introduced in [**4**, **16**, **17**] for a semilinear parabolic system.

In this paper, motivated by [**7**, **18**], by seeking a self-similar solution, we obtain our main results as follows.

Theorem 1.1. *Let*

$$\alpha > \frac{2m}{m+1} \quad \text{or} \quad \beta > \frac{2n}{n+1}.$$

Then the solution of (1.1)–(1.3) may blow up in finite time.

Theorem 1.2. *Let*

$$\alpha \leqslant \frac{2m}{m+1}, \quad \beta \leqslant \frac{2n}{n+1} \quad \text{and} \quad pq \leqslant \left(\frac{2m}{m+1} - \alpha\right)\left(\frac{2n}{n+1} - \beta\right).$$

Then every solution of (1.1)–(1.3) exists globally.

Theorem 1.3. *Let*

$$\alpha \leqslant \frac{2m}{m+1}, \quad \beta \leqslant \frac{2n}{n+1} \quad \text{and} \quad pq > \left(\frac{2m}{m+1} - \alpha\right)\left(\frac{2n}{n+1} - \beta\right).$$

(i) *If $l_1 > k_1$ and $l_2 > k_2$, then the solutions of (1.1)–(1.3) are global for small initial data and blow up in finite time with large initial data.*

(ii) *If $l_1 < k_1$ or $l_2 < k_2$ or $l_1 = k_1$ and $l_2 = k_2$, then every non-negative, non-trivial solution of (1.1)–(1.3) blows up in finite time.*

Theorem 1.4. *Assume that $k_1, k_2 > 0$ and that (u, v) is a solution of (1.1)–(1.3) increasing in time ($u_t, v_t \geqslant 0$) which blows up in finite time T. There then exist positive constants c and C such that*

$$c(T-t)^{-k_1} \leqslant \|u(\cdot\,,t)\|_\infty \leqslant C(T-t)^{-k_1},$$
$$c(T-t)^{-k_2} \leqslant \|v(\cdot\,,t)\|_\infty \leqslant C(T-t)^{-k_2}.$$

Remark 1.5. The results of Theorems 1.1–1.4 for problem (1.1)–(1.3) coincide with those for the single equation case (see [**7**, (1.4)]). The critical Fujita exponent of (1.1)–(1.3) obtained in this paper can be described as $l_i = k_i$, $i = 1, 2$: if $l_1 < k_1$ or $l_2 < k_2$, or $l_1 = k_1$ and $l_2 = k_2$, every non-negative, non-trivial solution of (1.1)–(1.3) is non-global, while if $l_1 > k_1$ and $l_2 > k_2$, there are both non-trivial global and non-global solutions.

Remark 1.6. The word 'large' in Theorem 1.3 means that at least one of the altitudes and the supports of the initial data is sufficiently large; see the proof of Corollary 3.2, below. As will be shown in the proof of Lemma 3.3, the word 'small' here requires that both the altitudes and the supports of the initial data are sufficiently small.

Remark 1.7. The classification for the parameters m, n, p, q, α and β in Theorems 1.1–1.3 is complete. In fact, the coupled condition $p, q > 0$ together with the assumption that

$$pq \leqslant \left(\frac{2m}{m+1} - \alpha\right)\left(\frac{2n}{n+1} - \beta\right)$$

in Theorem 1.2 rules out the possibility that $\alpha = 2m/(m+1)$ or $\beta = 2n/(n+1)$.

Remark 1.8. The assumption that $k_1, k_2 > 0$ in Theorem 1.4, together with (1.12), implies that either

(i) $n(m+1)p - m(n\beta + \beta - 2n) > 0$, $m(n+1)q - n(m\alpha + \alpha - 2m) > 0$ or

(ii) $n(m+1)p - m(n\beta + \beta - 2n) < 0$, $m(n+1)q - n(m\alpha + \alpha - 2m) < 0$.

For (i), the assumption $k_1, k_2 > 0$ requires $(m+1)(n+1)pq - (m\alpha + \alpha - 2m)(n\beta + \beta - 2n) > 0$ if $\alpha \leqslant 2m/(m+1)$, $\beta \leqslant 2n/(n+1)$; the assumption $k_1, k_2 > 0$ is automatically satisfied if at least one of

$$\frac{2n}{n+1} < \beta < \frac{n(m+1)p + 2mn}{m(n+1)} \quad \text{and} \quad \frac{2m}{m+1} < \alpha < \frac{m(n+1)q + 2mn}{n(m+1)}$$

holds.

Case (ii) implies that $\alpha > 2m/(m+1)$, $\beta > 2n/(n+1)$. We clearly have $(m+1)(n+1)pq - (m\alpha + \alpha - 2m)(n\beta + \beta - 2n) < 0$.

By using Theorems 1.1 and 1.3, we know that both (i) and (ii) for $k_1, k_2 > 0$ do indeed correspond to the finite-time blow-up situation of the solution.

This paper is organized as follows. In the next section we study the conditions of blow-up and global existence (Theorems 1.1 and 1.2). In § 3 we obtain the critical Fujita exponents (Theorem 1.3). Section 4 is devoted to computation of the blow-up rate in the case of solutions which are monotonic in time (Theorem 1.4).

2. Blow-up and global existence

Definition 2.1. The pair $(\underline{u}, \underline{v})$ is a subsolution of (1.1), (1.2) if it satisfies

$$\left.\begin{aligned} \underline{u}_t \leqslant (|\underline{u}_x|^{m-1}\underline{u}_x)_x, \quad \underline{v}_t \leqslant (|\underline{v}_x|^{n-1}\underline{v}_x)_x, \quad x > 0,\ 0 < t < T, \\ -|\underline{u}_x|^{m-1}\underline{u}_x(0,t) \leqslant \underline{u}^{\alpha}(0,t)\underline{v}^{p}(0,t), \\ -|\underline{v}_x|^{n-1}\underline{v}_x(0,t) \leqslant \underline{u}^{q}(0,t)\underline{v}^{\beta}(0,t), \quad 0 < t < T. \end{aligned}\right\} \tag{2.1}$$

Definition 2.2. We call $(\bar{u}, \bar{v})$ a supersolution of (1.1), (1.2) if it satisfies (2.1) with the opposite inequalities.

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Lemma 2.3. *Let (u_0, v_0) be smooth and satisfy the compatibility condition at the boundary and $(|u_0'|^{m-1}u_0')' \geqslant 0$, $(|v_0'|^{n-1}v_0')' \geqslant 0$. Then the solution of (1.1)–(1.3) increase in time, i.e. $u_t \geqslant 0$, $v_t \geqslant 0$.*

Proof. Set $Z = u_t$, $W = v_t$. We can show that (Z, W) is a solution of

$$\begin{aligned}
Z_t &= m(|u_x|^{m-1}Z_x)_x, \\
W_t &= n(|v_x|^{n-1}W_x)_x, \\
-m|u_x|^{m-1}Z_x(0,t) &= \alpha u^{\alpha-1}(0,t)v^p(0,t)Z(0,t) + pv^{p-1}(0,t)u^\alpha(0,t)W(0,t), \\
-n|v_x|^{n-1}W_x(0,t) &= qu^{q-1}(0,t)v^\beta(0,t)Z(0,t) + \beta u^q(0,t)v^{\beta-1}(0,t)W(0,t),
\end{aligned}$$

with $Z(x,0) \geqslant 0$, $W(x,0) \geqslant 0$.

To end the proof we apply the maximum principle. Due to the degeneration of the equations, this cannot be done directly. By a similar regularization procedure to that used in [**7**] we can prove it easily, so we shall omit it. The proof of Lemma 2.3 is complete. □

Proof of Theorem 1.1. Without loss of generality, we assume that $\alpha > 2m/(m+1)$. We know from Lemma 2.3 that $u_t \geqslant 0$, $v_t \geqslant 0$. Thus, $u^\alpha(0,t)v^p(0,t) \geqslant u^\alpha(0,t)v_0^p(0)$. Consider the single equation problem:

$$\left.\begin{aligned}
W_t &= (|W_x|^{m-1}W_x)_x, && x > 0,\ 0 < t < T, \\
-|W_x|^{m-1}W_x(0,t) &= W^\alpha(0,t)v_0^p(0), && 0 < t < T, \\
W(x,0) &= u_0(x), && x > 0.
\end{aligned}\right\} \tag{2.2}$$

Clearly, (W, v_0) is a subsolution of (1.1)–(1.3). By the result of [**7**] we know that the solution of (2.2) may blow up in finite time and so may the solution of (1.1)–(1.3). The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. It is sufficient to construct global supersolutions with initial data as large as needed. We achieve this with the aid of the self-similar solutions of exponential form. Let

$$\begin{aligned}
\bar{u}(x,t) &= \mathrm{e}^{Kt}\left(M + \exp\left(-L_1 x \exp\left(\frac{K(1-m)t}{1+m}\right)\right)\right), \\
\bar{v}(x,t) &= \exp\left(\frac{K(2m - m\alpha - \alpha)t}{(m+1)p}\right) \\
&\quad \times \left(M + \exp\left(-L_2 x \exp\left(\frac{K(2m - m\alpha - \alpha)(1-n)t}{(m+1)(n+1)p}\right)\right)\right),
\end{aligned}$$

with

$$M = \max(\|u_0\|_\infty, \|v_0\|_\infty, 1), \qquad L_1 = (M+1)^{(\alpha/m)+(p/m)},$$
$$L_2 = (M+1)^{(\beta/n)+(q/n)}, \qquad K = \max\left(\frac{mL_1^{m+1}}{M}, \frac{n(m+1)pL_2^{n+1}}{M(2m - m\alpha - \alpha)}\right).$$

So $\bar u(x,0)\geqslant u_0(x)$, $\bar v(x,0)\geqslant v_0(x)$ for $x\in\mathbb{R}^+$. After a computation we have

$$\begin{aligned}
\bar u_t &= K\mathrm{e}^{Kt}\biggl(M+\exp\biggl(-L_1x\exp\biggl(\frac{K(1-m)t}{1+m}\biggr)\biggr)\biggr)\\
&\quad+\mathrm{e}^{Kt}\biggl(\frac{KL_1(m-1)x}{m+1}\exp\biggl(\frac{K(1-m)}{1+m}t-L_1x\exp\biggl(\frac{K(1-m)t}{1+m}\biggr)\biggr)\biggr)\\
&\geqslant K\mathrm{e}^{Kt}\biggl(M+\exp\biggl(-L_1x\exp\biggl(\frac{K(1-m)t}{1+m}\biggr)\biggr)\biggr)\\
&\geqslant KM\mathrm{e}^{Kt},\\
\bar u_x &= -L_1\exp\biggl(\frac{2kt}{1+m}\biggr)\exp\biggl(-L_1x\exp\biggl(\frac{K(1-m)t}{1+m}\biggr)\biggr),\\
|\bar u_x|^{m-1}\bar u_x &= -L_1^m\exp\biggl(\frac{2Kmt}{1+m}\biggr)\exp\biggl(-L_1mx\exp\biggl(\frac{K(1-m)t}{1+m}\biggr)\biggr),\\
(|\bar u_x|^{m-1}\bar u_x)_x &= mL_1^{m+1}\mathrm{e}^{Kt}\exp\biggl(-L_1mx\exp\biggl(\frac{K(1-m)t}{1+m}\biggr)\biggr)\\
&\leqslant mL_1^{m+1}\mathrm{e}^{Kt},
\end{aligned}$$

and

$$\begin{aligned}
\bar v_t &\geqslant \frac{K(2m-m\alpha-\alpha)}{(m+1)p}\exp\biggl(\frac{K(2m-m\alpha-\alpha)t}{(m+1)p}\biggr)\\
&\quad\times\biggl(M+\exp\biggl(-L_2x\exp\biggl(\frac{K(2m-m\alpha-\alpha)(1-n)t}{(m+1)(n+1)p}\biggr)\biggr)\biggr)\\
&\geqslant M\frac{K(2m-m\alpha-\alpha)}{(m+1)p}\exp\biggl(\frac{K(2m-m\alpha-\alpha)t}{(m+1)p}\biggr),\\
\bar v_x &= -L_2\exp\biggl(\frac{2K(2m-m\alpha-\alpha)t}{(m+1)(n+1)p}\biggr)\\
&\quad\times\exp\biggl(-L_2x\exp\biggl(\frac{K(2m-m\alpha-\alpha)(1-n)t}{(m+1)(n+1)p}\biggr)\biggr),\\
|\bar v_x|^{n-1}\bar v_x &= -L_2^n\exp\biggl(\frac{2Kn(2m-m\alpha-\alpha)t}{(m+1)(n+1)p}\biggr)\\
&\quad\times\exp\biggl(-L_2nx\exp\biggl(\frac{K(2m-m\alpha-\alpha)(1-n)t}{(m+1)(n+1)p}\biggr)\biggr),\\
(|\bar v_x|^{n-1}\bar v_x)_x &= nL_2^{n+1}\exp\biggl(\frac{K(2m-m\alpha-\alpha)t}{(m+1)p}\biggr)\\
&\quad\times\exp\biggl(-L_2nx\exp\biggl(\frac{K(2m-m\alpha-\alpha)(1-n)t}{(m+1)(n+1)p}\biggr)\biggr)\\
&\leqslant nL_2^{n+1}\exp\biggl(\frac{K(2m-m\alpha-\alpha)t}{(m+1)p}\biggr)
\end{aligned}$$

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in $\mathbb{R}^+ \times \mathbb{R}^+$. On the other hand, We have on the boundary that

$$
\begin{gathered}
-|\bar{u}_x|^{m-1}\bar{u}_x(0,t) = L_1^m \exp\left(\frac{2Kmt}{1+m}\right), \\
-|\bar{v}_x|^{n-1}\bar{v}_x(0,t) = L_2^n \exp\left(\frac{2Kn(2m-m\alpha-\alpha)t}{(m+1)(n+1)p}\right), \\
\bar{u}^\alpha(0,t) = \mathrm{e}^{K\alpha t}(M+1)^\alpha, \quad \bar{v}^p(0,t) = \exp\left(\frac{K(2m-m\alpha-\alpha)t}{(m+1)}\right)(M+1)^p, \\
\bar{u}^q(0,t) = \mathrm{e}^{Kqt}(M+1)^q, \quad \bar{v}^\beta(0,t) = \exp\left(\frac{K\beta(2m-m\alpha-\alpha)t}{(m+1)p}\right)(M+1)^\beta.
\end{gathered}
$$

By the definitions of K, M, L_1, L_2 and the assumption that

$$pq \leqslant \left(\frac{2m}{m+1} - \alpha\right)\left(\frac{2n}{n+1} - \beta\right),$$

we know that $\bar{u}_t \geqslant (|\bar{u}_x|^{m-1}\bar{u}_x)_x$, $\bar{v}_t \geqslant (|\bar{v}_x|^{n-1}\bar{v}_x)_x$ in $\mathbb{R}^+ \times \mathbb{R}^+$ and $-|\bar{u}_x|^{m-1}\bar{u}_x(0,t) \geqslant \bar{u}^\alpha(0,t)\bar{v}^p(0,t)$, $-|\bar{v}_x|^{n-1}\bar{v}_x(0,t) \geqslant \bar{u}^q(0,t)\bar{v}^\beta(0,t)$ for $t > 0$.

Therefore, $(\bar{u}, \bar{v})$ is a supersolution of (1.1)–(1.3), which implies that every solution of (1.1)–(1.3) is global provided that

$$\alpha \leqslant \frac{2m}{m+1}, \quad \beta \leqslant \frac{2n}{n+1} \quad \text{and} \quad pq \leqslant \left(\frac{2m}{m+1} - \alpha\right)\left(\frac{2n}{n+1} - \beta\right).$$

The proof of Theorem 1.2 is complete. □

3. Critical Fujita exponents

Using some ideas in [**7**], in this section, we will prove Theorem 1.3. However, the fact that we are dealing with a system instead of a single equation forces us to develop a significantly different proof. We will organize the proof in several lemmas.

Lemma 3.1. *Let*

$$\alpha \leqslant \frac{2m}{m+1}, \quad \beta \leqslant \frac{2n}{n+1} \quad \textit{and} \quad pq > \left(\frac{2m}{m+1} - \alpha\right)\left(\frac{2n}{n+1} - \beta\right).$$

There then exists a pair of compactly supported functions f_1, f_2, such that

$$
\begin{aligned}
\underline{u}(x,t) &= (T-t)^{-k_1} f_1(\xi), \quad \xi = x(T-t)^{-l_1}, \\
\underline{v}(x,t) &= (T-t)^{-k_2} f_2(\eta), \quad \eta = x(T-t)^{-l_2}
\end{aligned}
$$

is a subsolution of (1.1), (1.2).

Proof. It is easy to see from (1.12), (1.13) that

$$
\begin{aligned}
k_1 + 1 &= mk_1 + (m+1)l_1, & k_2 + 1 &= nk_2 + (n+1)l_2, \\
m(k_1 + l_1) &= k_1\alpha + k_2 p, & n(k_2 + l_2) &= k_1 q + k_2\beta.
\end{aligned}
$$

After some computation, we obtain

$$
\begin{aligned}
\underline{u}_t &= (T-t)^{-(k_1+1)}[k_1 f_1(\xi) + l_1 f_1'(\xi)\xi],\\
|\underline{u}_x|^{m-1}\underline{u}_x &= (T-t)^{-m(k_1+l_1)}|f_1'(\xi)|^{m-1} f_1'(\xi),\\
(|\underline{u}_x|^{m-1}\underline{u}_x)_x &= m(T-t)^{-mk_1-(m+1)l_1}|f_1'(\xi)|^{m-1} f_1''(\xi),\\
\underline{v}_t &= (T-t)^{-(k_2+1)}[k_2 f_2(\eta) + l_2 f_2'(\eta)\eta],\\
|\underline{v}_x|^{n-1}\underline{v}_x &= (T-t)^{-n(k_2+l_2)}|f_2'(\eta)|^{n-1} f_2'(\eta),\\
(|\underline{v}_x|^{n-1}\underline{v}_x)_x &= n(T-t)^{-nk_2-(n+1)l_2}|f_2'(\eta)|^{n-1} f_2''(\eta),\\
|\underline{u}_x|^{m-1}\underline{u}_x(0,t) &= (T-t)^{-m(k_1+l_1)}|f_1'(0)|^{m-1} f_1'(0),\\
|\underline{v}_x|^{n-1}\underline{v}_x(0,t) &= (T-t)^{-n(k_2+l_2)}|f_2'(0)|^{n-1} f_2'(0),\\
\underline{u}^{\alpha}(0,t)\underline{v}^{p}(0,t) &= (T-t)^{-(k_1\alpha+k_2 p)} f_1^{\alpha}(0) f_2^{p}(0),\\
\underline{u}^{q}(0,t)\underline{v}^{\beta}(0,t) &= (T-t)^{-(k_1 q+k_2\beta)} f_1^{q}(0) f_2^{\beta}(0).
\end{aligned}
$$

To satisfy (2.1) we need

$$
\left.\begin{aligned}
m|f_1'(\xi)|^{m-1} f_1''(\xi) &\geqslant k_1 f_1(\xi) + l_1 f_1'(\xi)\xi,\\
n|f_2'(\eta)|^{n-1} f_2''(\eta) &\geqslant k_2 f_2(\eta) + l_2 f_2'(\eta)\eta,
\end{aligned}\right\} \tag{3.1}
$$

$$
\left.\begin{aligned}
-|f_1'(0)|^{m-1} f_1'(0) &\leqslant f_1^{\alpha}(0) f_2^{p}(0),\\
-|f_2'(0)|^{n-1} f_2'(0) &\leqslant f_1^{q}(0) f_2^{\beta}(0).
\end{aligned}\right\} \tag{3.2}
$$

We choose

$$
f_1(\xi) = A_1(C_1-\xi)_+^{m/(m-1)} \quad \text{and} \quad f_2(\eta) = A_2(C_2-\eta)_+^{n/(n-1)},
$$

where

$$
C_1 = \frac{1}{k_1}\left(\frac{m}{m-1}\right)^{m+1} A_1^{m-1}, \qquad C_2 = \frac{1}{k_2}\left(\frac{n}{n-1}\right)^{n+1} A_2^{n-1}
$$

and A_1 and A_2 will be determined later. Inserting them in (3.1), we get

$$
\begin{aligned}
&m|f_1'(\xi)|^{m-1} f_1''(\xi) - k_1 f_1(\xi) - l_1 f_1'(\xi)\xi\\
&\qquad = (C_1-\xi)_+^{1/(m-1)}\left[A_1^m\left(\frac{m}{m-1}\right)^{m+1} - k_1 A_1 (C_1-\xi)_+ + l_1 A_1 \frac{m}{m-1}\xi\right] \geqslant 0,\\
&n|f_2'(\eta)|^{n-1} f_2''(\eta) - k_2 f_2(\eta) - l_2 f_2'(\eta)\eta\\
&\qquad = (C_2-\eta)_+^{1/(n-1)}\left[A_2^n\left(\frac{n}{n-1}\right)^{n+1} - k_2 A_2 (C_2-\eta)_+ + l_2 A_2 \frac{n}{n-1}\eta\right] \geqslant 0.
\end{aligned}
$$

The assumption that

$$
pq > \left(\frac{2m}{m+1}-\alpha\right)\left(\frac{2n}{n+1}-\beta\right)
$$

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implies that

$$\frac{(n+1)p}{2m-m\alpha-\alpha} > \frac{2n-n\beta-\beta}{(m+1)q}.$$

Therefore, for any positive constants λ_1 and λ_2, there exist positive constants A_1 and A_2 sufficiently large that

$$\lambda_1 A_2^{(2n-n\beta-\beta)/(m+1)q} < A_1 < \lambda_2 A_2^{((n+1)p)/(2m-m\alpha-\alpha)}.$$

By taking suitable λ_1, λ_2, we have

$$\left(\frac{m}{m-1}\right)^m \left(\frac{m^{m+1}}{k_1(m-1)^{m+1}}\right)^{(m-m\alpha)/(m-1)} A_1^{2m-m\alpha-\alpha} \leqslant \left(\frac{n^{n+1}}{k_2(n-1)^{n+1}}\right)^{np/(n-1)} A_2^{(n+1)p},$$

$$\left(\frac{n}{n-1}\right)^n \left(\frac{n^{n+1}}{k_2(n-1)^{n+1}}\right)^{(n-n\beta)/(n-1)} A_2^{2n-n\beta-\beta} \leqslant \left(\frac{m^{m+1}}{k_1(m-1)^{m+1}}\right)^{mq/(m-1)} A_1^{(m+1)q},$$

which means that (3.2) is also true for large A_1, A_2. The proof of Lemma 3.1 is complete. □

Corollary 3.2. *Let*

$$\alpha \leqslant \frac{2m}{m+1}, \quad \beta \leqslant \frac{2n}{n+1} \quad \text{and} \quad pq > \left(\frac{2m}{m+1}-\alpha\right)\left(\frac{2n}{n+1}-\beta\right).$$

If $l_i > k_i$, $i = 1, 2$, then the solutions of (1.1)–(1.3) blow up in finite time provided that either the altitudes or the supports of $u_0(x)$, $v_0(x)$ are large enough.

Proof. Assume that $u_0(x) \geqslant G_1 > 0$ in $[0, x_1]$ and $v_0(x) \geqslant G_2 > 0$ in $[0, x_2]$. We claim that $\underline{u}(x, 0) \leqslant u_0(x)$, $\underline{v}(x, 0) \leqslant v_0(x)$ in $\mathbb{R}^+$ provided that either G_i, $i = 1, 2$ (the altitudes of $u_0(x)$, $v_0(x)$), or x_i, $i = 1, 2$ (the supports of $u_0(x)$, $v_0(x)$), are large enough.

In fact, for any $x_1, x_2 > 0$, we can choose $T > 0$ sufficiently small that

$$\frac{1}{k_1}\left(\frac{m}{m-1}\right)^{m+1} A_1^{m-1} \leqslant \frac{x_1}{T^{l_1}}, \qquad \frac{1}{k_2}\left(\frac{n}{n-1}\right)^{n+1} A_2^{n-1} \leqslant \frac{x_2}{T^{l_2}}, \tag{3.3}$$

with $l_i > k_i > 0$, $i = 1, 2$. For such fixed small $T > 0$, by taking G_1 and G_2 large enough, we have

$$T^{-k_1} A_1^{m+1} \left(\frac{m^{m+1}}{k_1(m-1)^{m+1}}\right)^{m/(m-1)} \leqslant G_1, \qquad T^{-k_2} A_2^{n+1} \left(\frac{n^{n+1}}{k_2(n-1)^{n+1}}\right)^{n/(n-1)} \leqslant G_2. \tag{3.4}$$

Analogously, (3.4) is true for any $G_1, G_2 > 0$ by taking $T > 0$ sufficiently large. For such large $T > 0$, (3.3) also holds whenever x_1 and x_2 are large enough.

It follows from (3.3) that the support of $\underline{u}(x,0)$ (or $\underline{v}(x,0)$) is smaller than that of u_0 (or v_0). Moreover, $\|\underline{u}(\cdot\,,0)\|_\infty \leqslant G_1$, $\|\underline{v}(\cdot\,,0)\|_\infty \leqslant G_2$ due to (3.4). So we know from (3.3) and (3.4) that $\underline{u}(x,0) \leqslant u_0(x)$, $\underline{v}(x,0) \leqslant v_0(x)$ in $\mathbb{R}^+$.

Combining this result with Lemma 3.1, we have shown that $(\underline{u},\underline{v})$ is a subsolution of (1.1)–(1.3) and blows up in finite time. The proof of Corollary 3.2 is complete. □

Lemma 3.3. *Let*

$$\alpha \leqslant \frac{2m}{m+1}, \quad \beta \leqslant \frac{2n}{n+1} \quad \text{and} \quad pq > \left(\frac{2m}{m+1} - \alpha\right)\left(\frac{2n}{n+1} - \beta\right).$$

If $l_i > k_i$, $i = 1,2$, then the solutions of (1.1)–(1.3) are global, provided that both the altitudes and the supports of $u_0(x), v_0(x)$ are small enough.

Proof. In a manner similar to the proof of Lemma 3.1 we construct

$$\bar{u}(x,t) = (\tau+t)^{-k_1} f(\xi), \quad \xi = x(\tau+t)^{-l_1},$$
$$\bar{v}(x,t) = (\tau+t)^{-k_2} g(\eta), \quad \eta = x(\tau+t)^{-l_2},$$

where $f(\xi)$ and $g(\eta)$ are non-negative functions to be determined which satisfy

$$\left.\begin{aligned} m|f'(\xi)|^{m-1}f''(\xi) + k_1 f(\xi) + l_1 f'(\xi)\xi \leqslant 0,\\ n|g'(\eta)|^{n-1}g''(\eta) + k_2 g(\eta) + l_2 g'(\eta)\eta \leqslant 0,\end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{aligned} -|f'(0)|^{m-1}f'(0) \geqslant f^{\alpha}(0)g^{p}(0),\\ -|g'(0)|^{n-1}g'(0) \geqslant f^{q}(0)g^{\beta}(0).\end{aligned}\right\} \tag{3.6}$$

We choose

$$\left.\begin{aligned} f(\xi) &= A[(d_1a_1)^{(m+1)/m} - (\xi + a_1)^{(m+1)/m}]_+^{m/(m-1)},\\ g(\eta) &= B[(d_2a_2)^{(n+1)/n} - (\eta + a_2)^{(n+1)/n}]_+^{n/(n-1)}.\end{aligned}\right\} \tag{3.7}$$

Let us show that such $f(\xi)$, $g(\eta)$ defined in (3.7) with suitable constants A, B, a_i, d_i, $i=1,2$, satisfy (3.5) and (3.6).

Since $l_1 > k_1$, $l_2 > k_2$, we can choose A, B such that

$$\left.\begin{aligned} \left[k_1\left(\frac{m-1}{m+1}\right)^m\right]^{1/(m-1)} < A < \left[l_1\left(\frac{m-1}{m+1}\right)^m\right]^{1/(m-1)},\\ \left[k_2\left(\frac{n-1}{n+1}\right)^n\right]^{1/(n-1)} < B < \left[l_2\left(\frac{n-1}{n+1}\right)^n\right]^{1/(n-1)}.\end{aligned}\right\} \tag{3.8}$$

The assumption $(m+1)(n+1)pq > (2n - n\beta - \beta)(2m - m\alpha - \alpha)$ implies that

$$\frac{2m - m\alpha - \alpha}{(n+1)p} < \frac{(m+1)q}{2n - n\beta - \beta}.$$

Therefore, for any positive constants μ_1, μ_2, there exist positive constants a_1, a_2 small enough $(0 < a_1, a_2 < 1)$ that

$$\mu_1 a_1^{(m+1)q/(2n-n\beta-\beta)} < a_2^{(m-1)/(n-1)} < \mu_2 a_1^{(2m-m\alpha-\alpha)/(n+1)p}.$$

Thus,

$$\left.\begin{aligned}\left(\frac{m+1}{m-1}\right)^m A^{m-\alpha}a_1^{2m/(m-1)}(d_1^{(m+1)/m}-1)^{(m-m\alpha)/(m-1)}&\\ \geqslant a_1^{(m+1)\alpha/(m-1)}B^p a_2^{(n+1)p/(n-1)}(d_2^{(n+1)/n}-1)^{np/(n-1)},&\\ \left(\frac{n+1}{n-1}\right)^n B^{n-\beta}a_2^{2n/(n-1)}(d_2^{(n+1)/n}-1)^{(n-n\beta)/(n-1)}&\\ \geqslant a_2^{(n+1)\beta/(n-1)}A^q a_1^{(m+1)q/(m-1)}(d_1^{(m+1)/m}-1)^{mq/(m-1)}&\end{aligned}\right\} \tag{3.9}$$

hold for constants a_1, a_2 small enough and d_1, d_2 large enough.

From (3.7)–(3.9) it is easy to check that $f(\xi)$ and $g(\eta)$ defined in (3.7) satisfy (3.5) and (3.6). Together with (1.12) and (1.13), we know that $\bar{u}_t \geqslant (|\bar{u}_x|^{m-1}\bar{u}_x)_x$ and $\bar{v}_t \geqslant (|\bar{v}_x|^{n-1}\bar{v}_x)_x$ in $\mathbb{R}^+ \times \mathbb{R}^+$ and $-|\bar{u}_x|^{m-1}\bar{u}_x(0,t) \geqslant \bar{u}^\alpha(0,t)\bar{v}^p(0,t)$, $-|\bar{v}_x|^{n-1}\bar{v}_x(0,t) \geqslant \bar{u}^q(0,t)\bar{v}^\beta(0,t)$ for $t > 0$. Moreover, it is easy to see from (3.7) that $\bar{u}(x,0) \geqslant u_0(x)$, $\bar{v}(x,0) \geqslant v_0(x)$ for $x > 0$, provided that both the altitudes and the supports of the initial data are sufficiently small. Thus, $(\bar{u},\bar{v})$ is a global supersolution of (1.1)–(1.3), which implies the global existence of solutions to (1.1)–(1.3) with small initial data. The proof of Lemma 3.3 is complete. □

Lemma 3.4. *Let*

$$\alpha \leqslant \frac{2m}{m+1}, \quad \beta \leqslant \frac{2n}{n+1} \quad \text{and} \quad pq > \left(\frac{2m}{m+1}-\alpha\right)\left(\frac{2n}{n+1}-\beta\right).$$

If $l_1 < k_1$ or $l_2 < k_2$, then every non-negative, non-trivial solution of (1.1)–(1.3) blows up in finite time.

Proof. In the spirit of [**7**], we construct a self-similar solution to (1.1) in the form of a Zel'dovich–Kompaneetz–Barenblatt profile [**9**]:

$$\begin{aligned} u_B(x,t) &= (\tau+t)^{-1/2m}h_1(\xi),\\ v_B(x,t) &= (\tau+t)^{-1/2n}h_2(\eta),\\ \xi &= x(\tau+t)^{-1/2m},\\ \eta &= x(\tau+t)^{-1/2n},\\ h_1(\xi) &= C_m(C^{(m+1)/m}-\xi^{(m+1)/m})_+^{m/(m-1)},\\ h_2(\eta) &= C_n(C^{(n+1)/n}-\eta^{(n+1)/n})_+^{n/(n-1)}.\end{aligned}$$

By taking

$$C_m = \left[\frac{1}{2m}\left(\frac{m-1}{m+1}\right)^m\right]^{1/(m-1)}, \qquad C_n = \left[\frac{1}{2n}\left(\frac{n-1}{n+1}\right)^n\right]^{1/(n-1)},$$

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it is easy to check that h_1, h_2 satisfy

$$(|h_1'|^{m-1}h_1')'(\xi) + \frac{\xi}{2m}h_1'(\xi) + \frac{1}{2m}h_1(\xi) = 0, \quad h_1'(0) = 0,$$
$$(|h_2'|^{n-1}h_2')'(\eta) + \frac{\eta}{2n}h_2'(\eta) + \frac{1}{2n}h_2(\eta) = 0, \quad h_2'(0) = 0.$$

It follows from $h_1'(0) = h_2'(0) = 0$ that the self-similar solution $(u_B(x,t), v_B(x,t))$ satisfies $(u_B)_x(0,t) = (v_B)_x(0,t) = 0$ on the boundary.

By using well-known properties of weak solutions of problem (1.1)–(1.3) (see [**9**]), we deduce that $u(0,t_0), v(0,t_0) \geqslant 0$ for some $t_0 \geqslant 0$ and $u(x,t_0)$, $v(x,t_0)$ are continuous. So, there exists $\tau > 0$ large enough and $C > 0$ small enough such that

$$u(x,t_0) \geqslant u_B(x,t_0), \quad v(x,t_0) \geqslant v_B(x,t_0) \quad \text{for } x > 0.$$

Thus, the self-similar solution $(u_B(x,t), v_B(x,t))$ is a subsolution to (1.1)–(1.3) in $\mathbb{R}^+ \times (t_0, T)$ and, hence,

$$u(x,t) \geqslant u_B(x,t), \quad v(x,t) \geqslant v_B(x,t) \quad \text{for } x > 0,\ t \geqslant t_0.$$

Without loss of generality, we assume that $l_1 < k_1$. Then $T^{l_1} \ll T^{k_1}$ for large T. So there exists $t^* \geqslant t_0$ such that

$$T^{l_1} \ll (\tau + t^*)^{(2m+1)/(4m^2+2m-2)} \ll T^{k_1}. \tag{3.10}$$

Let $\underline{u}(x,t)$ be as defined in Lemma 3.1. The inequality (3.10) implies that $\underline{u}(x,0) \leqslant u_B(x,t^*)$ for $x > 0$. Observing that (3.10) holds for general non-trivial $u_0(x)$, we know that every non-negative, non-trivial solution of (1.1)–(1.3) blows up in finite time. The proof of Lemma 3.4 is complete. □

Lemma 3.5. *Let*

$$\alpha \leqslant \frac{2m}{m+1}, \quad \beta \leqslant \frac{2n}{n+1} \quad \text{and} \quad pq > \left(\frac{2m}{m+1} - \alpha\right)\left(\frac{2n}{n+1} - \beta\right).$$

If $l_i = k_i$, $i = 1, 2$, then every non-negative, non-trivial solution of (1.1)–(1.3) blows up in finite time.

Proof. Assume that there exists a global non-negative non-trivial solution (u,v) of (1.1)–(1.3). We make the following change of variables:

$$\varphi(\xi,\tau) = (1+t)^{k_1}u(\xi(1+t)^{l_1}, t),$$
$$\psi(\eta,\tau) = (1+t)^{k_2}v(\eta(1+t)^{l_2}, t),$$
$$\tau = \log(1+t).$$

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These functions satisfy

$$\left.\begin{aligned}\varphi_\tau &= (|\varphi_\xi|^{m-1}\varphi_\xi)_\xi + l_1\xi\varphi_\xi + k_1\varphi,\\ \psi_\tau &= (|\psi_\eta|^{n-1}\psi_\eta)_\eta + l_2\eta\psi_\eta + k_2\psi,\end{aligned}\right\} \tag{3.11}$$

$$\left.\begin{aligned}-|\varphi_\xi|^{m-1}\varphi_\xi(0,\tau) &= \varphi^\alpha(0,\tau)\psi^p(0,\tau),\\ -|\psi_\eta|^{n-1}\psi_\eta(0,\tau) &= \varphi^q(0,\tau)\psi^\beta(0,\tau).\end{aligned}\right\} \tag{3.12}$$

As (u,v) is, by hypothesis, global, so is (φ,ψ). On the other hand, we will construct $(\underline{\varphi},\underline{\psi})$ to system (3.11), (3.12) increasing in time, with initial data $(\underline{\varphi}^0,\underline{\psi}^0)$ such that $\underline{\varphi}^0(\xi)\leqslant u(\xi,0)$, $\underline{\psi}^0(\eta)\leqslant v(\eta,0)$. We will prove that $(\underline{\varphi},\underline{\psi})$ cannot exist globally, thus contradicting the global existence of (u,v). In order to achieve our goal, we use an adaptation for systems of the general monotonicity approach for single quasilinear equations described in [**6**].

We take initial data $(\underline{\varphi}^0,\underline{\psi}^0)$ satisfying

$$\left.\begin{aligned}(|\underline{\varphi}^0_\xi|^{m-1}\underline{\varphi}^0_\xi)_\xi + l_1\xi\underline{\varphi}^0_\xi + k_1\underline{\varphi}^0 \geqslant 0,\\ (|\underline{\psi}^0_\eta|^{n-1}\underline{\psi}^0_\eta)_\eta + l_2\eta\underline{\psi}^0_\eta + k_2\underline{\psi}^0 \geqslant 0,\end{aligned}\right\} \tag{3.13}$$

and the compatibility condition

$$\left.\begin{aligned}-|\underline{\varphi}^0_\xi|^{m-1}\underline{\varphi}^0_\xi(0) &= (\underline{\varphi}^0)^\alpha(0)(\underline{\psi}^0)^p(0),\\ -|\underline{\psi}^0_\eta|^{n-1}\underline{\psi}^0_\eta(0) &= (\underline{\varphi}^0)^q(0)(\underline{\psi}^0)^\beta(0).\end{aligned}\right\} \tag{3.14}$$

Following an idea for scalar equations from [**7**], we get

$$\left.\begin{aligned}\underline{\varphi}^0(\xi) &= h_1(\xi+b_1) = C_m[C^{(m+1)/m} - (\xi+b_1)^{(m+1)/m}]_+^{m/(m-1)},\\ \underline{\psi}^0(\eta) &= h_2(\eta+b_2) = C_n[C^{(n+1)/n} - (\eta+b_2)^{(n+1)/n}]_+^{n/(n-1)},\end{aligned}\right\} \tag{3.15}$$

where C_m, C_n are defined in Lemma 3.4. Since $l_1 = k_1$, $l_2 = k_2$, we know from (1.13) that $l_1 = k_1 = 1/2m$ and it is easy to check that $(\underline{\varphi}^0,\underline{\psi}^0)$ satisfies (3.13), (3.14) for suitable $b_1, b_2 \in (0,C)$.

Since $u_0(0)>0$, $v_0(0)>0$ with the continuity of $u_0(x)$ and $v_0(x)$, it follows from (3.11) and (3.15) that

$$u_0(x) = \varphi(\xi,0) \geqslant h_1(\xi+b_1) = \underline{\varphi}^0(\xi),$$
$$v_0(x) = \psi(\eta,0) \geqslant h_2(\eta+b_2) = \underline{\psi}^0(\eta),$$

on $\mathbb{R}^+$ provided that $C>0$ is sufficiently small. Denote by $(\underline{\varphi}(\xi,\tau),\underline{\psi}(\eta,\tau))$ the solution of (3.11), (3.12) with the initial data $(\underline{\varphi}^0(\xi),\underline{\psi}^0(\eta))$.

Since $|\underline{\varphi}_\xi|^{m-1}\underline{\varphi}_\xi \leqslant 0$ on the boundary and $\underline{\varphi}^0_\xi \leqslant 0$, we know that $\underline{\varphi}(\xi,\tau)$ is nonincreasing in ξ. Moreover, we can show that $\underline{\varphi}(\xi,\tau)$ is non-decreasing in τ on $\mathbb{R}^+\times\mathbb{R}^+$. The proof is similar to the proof of [**7**, Proposition 3.1] and we omit it.

Next we claim that there exists a non-trivial function $\Phi(\xi)$, such that

$$+\infty > \lim_{\tau\to+\infty} \underline{\varphi}(\xi,\tau) = \Phi(\xi) \quad \text{for any } \xi > 0.$$

In fact, if the claim is not true, we assume that $\lim_{\tau\to+\infty} \underline{\varphi}(\xi,\tau) = +\infty$ uniformly on $[0,\xi_0]$. Since $\underline{\varphi}$ is nonincreasing in ξ, for any $G > 0$, there is a positive τ_0 such that $\underline{\varphi}(\xi,\tau_0) > G$ on $[0,\xi_0]$. In other words, at the time $t_0 = \mathrm{e}^{\tau_0} - 1$, the profile $\underline{\varphi}(\xi,\tau)$ in the original variable satisfies $u(x,t_0) \geqslant (1+t_0)^{-k_1}G$ for $x \in [0,\xi_0(1+t_0)^{l_1}]$. Let $\underline{u}(x,t)$ be defined in Lemma 3.1. Observing $k_1 = l_1$, we know that

$$\begin{aligned} G^{-1}(1+t_0)^{l_1}A_1^{m+1}\left[\frac{1}{k_1}\left(\frac{m}{m-1}\right)^{m+1}\right]^{m/(m-1)} &\leqslant T^{k_1} = T^{l_1} \\ &\leqslant \xi_0(1+t_0)^{k_1}\left[\frac{1}{k_1}\left(\frac{m}{m-1}\right)^{m+1}A_1^{m-1}\right]^{-1} \end{aligned}$$

for suitable T, provided that $G > 0$ is large enough, which means that the first parts of (3.3) and (3.4) hold with $x_1 = \xi_0(1+t_0)^{l_1}$ and $G_1 = G(1+t_0)^{-k_1}, k_1 = l_1$. Thus, $u(x,t_0) \geqslant \underline{u}(x,0)$ for $x > 0$. This implies that $\underline{u}(x,t)$ will blow up in finite time. However, u was assumed to be global. This contradiction shows that the function $\Phi(\xi)$ is well defined.

Finally, we will complete the proof. In view of the regularity of bounded solutions of the degenerate equations (see [**9**]), by using the standard argument (see [**10**]), we can pass to the limit in the first equation in (3.11) to get

$$(|\Phi_\xi|^{m-1}\Phi_\xi)_\xi + l_1\xi\Phi_\xi + k_1\Phi = 0. \tag{3.16}$$

We know that $0 < \Phi(0) < C$. Because of the regularity of $\underline{\varphi}$ in the region where $\Phi > 0$ [**9**], we can pass to the limit in the boundary condition in (3.12) to obtain

$$-|\Phi_\xi|^{m-1}\Phi_\xi(0) = \Phi^\alpha(0)\Psi^p(0) \neq 0, \tag{3.17}$$

where non-trivial $\Psi(\eta) = \lim_{\tau\to+\infty} \underline{\psi}(\eta,\tau)$. However, such a non-trivial compactly supported function dees not exist. In fact, by integrating (3.16) on $(0,+\infty)$, we have

$$\begin{aligned} 0 &= \int_0^{+\infty} (|\Phi_\xi|^{m-1}\Phi_\xi)_\xi + l_1\xi\Phi_\xi + k_1\Phi \,\mathrm{d}\xi \\ &= (|\Phi_\xi|^{m-1}\Phi_\xi + l_1\xi\Phi)|_0^{+\infty} + \int_0^{+\infty} (-l_1+k_1)\Phi \,\mathrm{d}\xi \\ &= -|\Phi_\xi|^{m-1}\Phi_\xi(0), \end{aligned}$$

which contradicts (3.17). The proof of Lemma 3.5 is complete. □

Proof of Theorem 1.3. Lemmas 3.1 and 3.3–3.5 show that Fujita exponents for (1.1)–(1.3) are described by $l_i = k_i$, $i = 1, 2$, and Theorem 1.3 is proved. □

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4. Blow-up rate estimate

Proof of Theorem 1.4. Since $u_t \geqslant 0$, we thus have that $(|u_x|^{m-1}u_x)_x \geqslant 0$ and $\|u(\cdot\,,t)\|_\infty = u(0,t)$. In the same way, we obtain $\|v(\cdot\,,t)\|_\infty = v(0,t)$.

Now let us define

$$M(t) = u(0,t) = \max u(\cdot\,,t) \quad \text{and} \quad N(t) = v(0,t) = \max v(\cdot\,,t).$$

Following ideas from [**8**], we set

$$\left.\begin{aligned} \varphi_M(y,s) &= \frac{1}{M(t)}u(ay,bs+t), \quad y>0, \quad -\frac{t}{b}<s<0, \quad t<T, \\ \psi_N(y,s) &= \frac{1}{N(t)}v(cy,ds+t), \quad y>0, \quad -\frac{t}{d}<s<0, \quad t<T. \end{aligned}\right\} \tag{4.1}$$

This pair of functions (φ_M, ψ_N) satisfies

$$0 \leqslant \varphi_M, \psi_N \leqslant 1, \quad \varphi_M(0,0) = \psi_N(0,0) = 1, \quad (\varphi_M)_s, (\psi_N)_s \geqslant 0.$$

Choosing

$$a = \left(\frac{M^{m-\alpha}}{N^p}\right)^{1/m}, \quad b = \frac{M^{(2m-m\alpha-\alpha)/m}}{N^{(m+1)p/m}}, \quad c = \left(\frac{N^{n-\beta}}{M^q}\right)^{1/n}, \quad d = \frac{N^{(2n-n\beta-\beta)/n}}{M^{(n+1)q/n}},$$

we have that φ_M and ψ_N are solutions of

$$\begin{aligned} (\varphi_M)_s &= (|(\varphi_M)_y|^{m-1}(\varphi_M)_y)_y, \\ (\psi_N)_s &= (|(\psi_N)_y|^{n-1}(\psi_N)_y)_y, \\ -|(\varphi_M)_y|^{m-1}(\varphi_M)_y(0,s) &= (\varphi_M)^\alpha(0,s)(\psi_N)^p(0,s), \\ -|(\psi_N)_y|^{n-1}(\psi_N)_y(0,s) &= (\varphi_M)^q(0,s)(\psi_N)^\beta(0,s). \end{aligned}$$

We observe that there exists a number s_* such that φ_M and ψ_N are well defined for every $(y,s) \in A = \{y>0,\ s_*<s<0\}$ with M and N sufficiently large. Indeed, we assume $-t/b \to 0$ otherwise. Hence, φ_M is a solution of the equation $(\varphi_M)_s = (|(\varphi_M)_y|^{m-1}(\varphi_M)_y)_y$, defined in a small interval of time $(-t/b,0)$. The flux is bounded by

$$-|(\varphi_M)_y|^{m-1}(\varphi_M)_y(0,s) = (\varphi_M)^\alpha(0,s)(\psi_N)^p(0,s) \leqslant 1,$$

and the initial data are small ($\varphi_M(y,-t/b) = u_0(ay)/M(t) \leqslant \varepsilon$) if M is large enough. But this contradicts the fact that $\varphi_M(0,0) = 1$.

Next we claim that, under the assumption of Theorem 1.4, there exist constants c and C for sufficiently large M and N such that

$$c \leqslant (\varphi_M)_s(0,0) \leqslant C, \qquad c \leqslant (\psi_N)_s(0,0) \leqslant C. \tag{4.2}$$

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First we will prove $(\varphi_M)_s(0,0) \leqslant C$ and $(\psi_N)_s(0,0) \leqslant C$. From the results for bounded solutions of degenerate equations in [**8**] we find that every sequence $(\varphi_{Mj}, \psi_{Nj})$ is equicontinuous, where

$$\varphi_{Mj}(y,s) = \frac{1}{M(t_{j_1})} u(ay, bs + t_{j_1}), \qquad \psi_{Nj}(y,s) = \frac{1}{N(t_{j_2})} v(cy, ds + t_{j_2})$$

and $-t_{j_1}/b \to s_*$ as $j_1 \to +\infty$, $-t_{j_2}/d \to s_*$ as $j_2 \to +\infty$. Therefore, passing to a subsequence if necessary, we have that $\varphi_{Mj} \to \varphi$, $\psi_{Nj} \to \psi$ uniformly on a compact set of $\{y \geqslant 0,\ s_* \leqslant s \leqslant 0\}$. These functions φ, ψ are continuous and satisfy $\varphi(0,0) = \psi(0,0) = 1$. Hence, there exists a neighbourhood U of $(0,0)$ and $U \subset A$, such that $\varphi, \psi > \frac{1}{2}$ in U. As we have uniform convergence over $\bar{U}$ (we can assume $\bar{U}$ is compact), for sufficiently large j we have that $\frac{1}{4} \leqslant \varphi_{Mj}, \psi_{Nj} \leqslant 1$. Therefore, φ_{Mj} and ψ_{Nj} are solutions of uniformly parabolic equations in $\bar{U}$ (see [**18**]). By using the Schauder estimate (see [**11**]), we have

$$\|\varphi_{Mj}\|_{C^{2+\alpha,1+\alpha/2}} \leqslant C, \quad \|\psi_{Nj}\|_{C^{2+\alpha,1+\alpha/2}} \leqslant C \quad \text{in } \bar{U}.$$

For sufficiently large M and N, we conclude that $(\varphi_M)_s(0,0) \leqslant C$ and $(\psi_N)_s(0,0) \leqslant C$. The first half of the claim is proved.

It remains to prove that $c \leqslant (\varphi_M)_s(0,0)$ and $c \leqslant (\psi_N)_s(0,0)$. Otherwise, there exists a sequence $\{M_j\} \to 0$ such that $(\varphi_{Mj})_s(0,0) \to 0$. Just as before, we need to obtain that $\varphi_{Mj} \to \varphi$ and $\psi_{Nj} \to \psi$ and that $\|\varphi_{Mj}\|_{C^{2+\alpha,1+\alpha/2}} \leqslant C$ and $\|\psi_{Nj}\|_{C^{2+\alpha,1+\alpha/2}} \leqslant C$ in $\bar{U}$. Since $C^{2+\alpha,1+\alpha/2}$ is compactly included in $C^{2+\beta,1+\beta/2}$, $\beta < \alpha$, we can conclude, refining the sequence if necessary, that $\varphi_s(0,0) = 0$.

However, we observe that φ is a weak solution of

$$\begin{aligned} (\varphi)_s &= (|(\varphi)_y|^{m-1}(\varphi)_y)_y \qquad \text{in } \mathbb{R}^+ \times (s_*, 0), \\ -|(\varphi)_y|^{m-1}(\varphi)_y(0,s) &= (\varphi)^{\alpha}(0,s)(\psi)^p(0,s) \quad \text{for } s \in (s_*, 0). \end{aligned}$$

Then $W = \varphi_s \geqslant 0$ satisfies

$$\begin{aligned} W_s &= m(|\varphi_y|^{m-1}W_y)_y \quad \text{in } \mathbb{R}^+ \times (s_*, 0), \\ -m|\varphi_y|^{m-1}W_y &= p\psi_s\psi^{p-1}(0,s)\varphi^{\alpha}(0,s) + \alpha W\varphi^{\alpha-1}(0,s)\psi^p(0,s) \geqslant 0 \quad \text{for } s \in (s_*, 0). \end{aligned}$$

Hence, W has a minimum at $(0,0)$. By Hopf's lemma, which can be applied whenever $\varphi > 0$, we can conclude that $W \equiv 0$ (see [**18**]), that is φ does not depend on s. Hence, $\varphi = \varphi(y)$ is a solution of

$$\begin{aligned} 0 &= (|(\varphi)_y|^{m-1}(\varphi)_y)_y, \\ -|(\varphi)_y|^{m-1}(\varphi)_y(0) &= 1. \end{aligned}$$

So φ is unbounded. This contradicts $0 \leqslant \varphi \leqslant 1$. The second part of the claim is proved.

Next, by using (4.1) and (4.2), we have

$$c \leqslant \frac{M^{(m-m\alpha-\alpha)/m}}{N^{(m+1)p/m}} M'(t) \leqslant C, \qquad c \leqslant \frac{N^{(n-n\beta-\beta)/n}}{M^{(n+1)q/n}} N'(t) \leqslant C.$$

This is equivalent to

$$\left.\begin{aligned} cN^{(m+1)p/m} \leqslant M^{(m-m\alpha-\alpha)/m}M' \leqslant CN^{(m+1)p/m},\\ cM^{(n+1)q/n} \leqslant N^{(n-n\beta-\beta)/n}N' \leqslant CM^{(n+1)q/n}. \end{aligned}\right\} \tag{4.3}$$

Thus,

$$\begin{aligned} CN^{((m+1)p/m)+(n-n\beta-\beta)/n}N'(t) &\geqslant cCN^{(m+1)p/m}M^{(n+1)q/n}\\ &\geqslant cM^{(m-m\alpha-\alpha)/m+((n+1)q)/n}M'(t), \end{aligned} \tag{4.4}$$

which implies that

$$N^{(((m+1)p/m)+(n-n\beta-\beta)/n)+1} \geqslant C_1 M^{(m-m\alpha-\alpha)/m+((n+1)q/n)+1}. \tag{4.5}$$

For Theorem 1.4 (i), it follows from (4.5) that

$$N \geqslant C_2 M^{[m(n+1)q-n(m\alpha+\alpha-2m)]/[n(m+1)p-m(n\beta+\beta-2n)]} = C_2 M^{k_2/k_1}. \tag{4.6}$$

Combining (4.3) with (4.6), we have

$$M^{(m-m\alpha-\alpha)/m-(m+1)pk_2/mk_1}M'(t) \geqslant C_3. \tag{4.7}$$

Observation yields

$$\begin{aligned} 1 + \frac{m-m\alpha-\alpha}{m} - \frac{(m+1)pk_2}{mk_1} &= \frac{2m-m\alpha-\alpha}{m} - \frac{(m+1)p}{m}\frac{m(n+1)q-n(m\alpha+\alpha-2m)}{n(m+1)p-m(n\beta+\beta-2n)}\\ &= \frac{(2m-m\alpha-\alpha)n(m+1)p-(2m-m\alpha-\alpha)m(n\beta+\beta-2n)}{mn(m+1)p-m^2(n\beta+\beta-2n)}\\ &\qquad - \frac{m(m+1)(n+1)pq-n(m+1)p(m\alpha+\alpha-2m)}{mn(m+1)p-m^2(n\beta+\beta-2n)}\\ &= \frac{(m\alpha+\alpha-2m)(n\beta+\beta-2n)-(m+1)(n+1)pq}{n(m+1)p-m(n\beta+\beta-2n)}\\ &= -\frac{1}{k_1}. \end{aligned}$$

By integrating (4.7) on (t,T), we get

$$M(t) \leqslant C_1(T-t)^{-k_1}, \tag{4.8}$$

By (4.6), we have

$$N(t) \geqslant C_4(T-t)^{-k_2}. \tag{4.9}$$

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Similarly, we have from (4.3) that

$$cN^{(m+1)p/m+(n-n\beta-\beta)/n}N'(t) \leqslant cCN^{(m+1)p/m}M^{(n+1)q/n} \leqslant CM^{(m-m\alpha-\alpha)/m+((n+1)q)/n}M'(t), \tag{4.10}$$

$$M \geqslant C_4 N^{k_1/k_2}, \qquad N^{(n-n\beta-\beta)/n-(n+1)qk_1/nk_2}N'(t) \geqslant C_5, \tag{4.11}$$

with

$$1 - \frac{(n+1)qk_1}{nk_2} + \frac{n-n\beta-\beta}{n} = -\frac{1}{k_2}.$$

From (4.11) we obtain that

$$N(t) \leqslant C_2(T-t)^{-k_2}, \qquad M(t) \geqslant C_3(T-t)^{-k_1}. \tag{4.12}$$

For Theorem 1.4 (ii), it follows, by (4.5), that

$$M \geqslant C_2' N^{[n(m+1)p-m(n\beta+\beta-2n)]/[m(n+1)q-n(m\alpha+\alpha-2m)]} = C_2' N^{k_1/k_2}. \tag{4.13}$$

Combining (4.3) with (4.13), we have

$$N^{(n-n\beta-\beta)/n-(n+1)qk_1/nk_2}N'(t) \geqslant C_3'.$$

By using a method similar to (i), we can prove that the estimate (4.8), (4.9) and (4.12) is also true for (ii). The proof of Theorem 1.4 is complete. □

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