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Proofs of an Inequality.

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§ 1. The inequality of the Arithmetic and Geometric Means of n positive quantities has been proved by many different methods; of which a classified summary has been given in the *Mathematical Gazette* (Vol. II., p. 283). The present article may be looked on as supplementary to that summary. It deals with proofs that belong to a general type, of which the proof given in the *Tutorial Algebra*, § 205, and that given by Mr G. E. Crawford in our *Proceedings*, Vol. XVIII., p. 2, are very special limiting cases. Proofs of the type in question consist of a finite number of steps, by which, starting from the n given quantities, and changing two at a time according to some law, we reach a new set of quantities whose arithmetic mean is not greater, and whose geometric mean is not less than the corresponding means of the given quantities.

Mr Crawford's remark that his proof and that of the *Tutorial Algebra* are the only possible ones on the same lines is perhaps justifiable if the phrase "on the same lines" is very narrowly interpreted. But, as will be seen in what follows, there are an infinite number of possible proofs which have the general character above stated, and which share with the two referred to the "logical advantage that the number of mental steps in the process is finite." It is true that those two are the simplest of the kind, but I have thought it of interest to show that the others exist, though I do not suggest that they are so good.

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§ 2. In order to simplify the exposition, I shall in § 2 and § 3 take no notice of special cases that may arise in which inequality reduces to equality. The proofs could be made more general by using the symbols $\not>$ and $\not<$ in place of $<$ and $>$ respectively, but the gain in generality would be at the expense of conciseness and clearness.

Let $a, b, c, \ldots$ be n unequal positive quantities, of which A is the arithmetic, and G the geometric mean.

The following proof is only slightly more complex than that of Mr Crawford :—

Let a be the greatest, b the least of the given quantities.

Let k be defined by the equation

$$\mathrm{G}+k=a+b \qquad - \quad - \quad - \quad - \quad (1)$$

Then $\quad \mathrm{G}k-ab=\mathrm{G}(a+b-\mathrm{G})-ab=(a-\mathrm{G})(\mathrm{G}-b)>0 \quad - \quad (2)$*

Thus $\quad \mathrm{G}kcd\ldots>abcd\ldots>\mathrm{G}^n$

$$\therefore \quad kcd\ldots>\mathrm{G}^{n-1}. \qquad - \quad - \quad - \quad - \quad (3)$$

Let now $\quad \mathrm{G}_1{}^{n-1}=kcd\ldots$, so that $\mathrm{G}_1>\mathrm{G}$. $\quad - \quad - \quad - \quad (4)$

Next treat the $n-1$ quantities $k, c, d, \ldots$ as we treated $a, b, c, d, \ldots$, *i.e.*, Let $\mathrm{G}_1+k_1=$ the sum of the greatest and least of them, say, k, c. It will follow, as before, that

$$\mathrm{G}_1k_1def\ldots>kcdef\ldots$$
$$>\mathrm{G}_1{}^{n-1}$$

$\therefore \quad k_1def\ldots>\mathrm{G}_1{}^{n-2}$, so that if $\mathrm{G}_2{}^{n-2}\equiv k_1def\ldots$

we have $\quad \mathrm{G}_2>\mathrm{G}_1. \quad - \quad - \quad - \quad - \quad - \quad (5)$

Proceeding in this way, we finally get $\mathrm{G}, \mathrm{G}_1, \mathrm{G}_2, \ldots \mathrm{G}_{n-1}$, a set of quantities with the same sum as $a, b, c, d, e, \ldots$.

But $\mathrm{G}<\mathrm{G}_1<\mathrm{G}_2\ldots<\mathrm{G}_{n-1}$. Hence $n\mathrm{G}<a+b+c+d+\ldots \quad (6)$

$$\therefore \quad \mathrm{G}<\mathrm{A}.$$

* Here we assume that G lies between a and b, a fact which is easily proved.

Again, if we take $Ak = ab$, so that

$$A + k - a - b = A + \frac{ab}{A} - a - b = \frac{1}{A}(A - a)(A - b) < 0 \quad - \quad (7)$$

$$\therefore \quad A + k < a + b \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad (8)$$

$$\therefore \quad A + k + c + d + \ldots < a + b + c + d + \ldots < nA$$

$$\therefore \quad k + c + d + \ldots < (n-1)A. \quad - \quad - \quad - \quad (9)$$

Hence if $\quad (n-1)A_1 \equiv k + c + d + \ldots,$ then $A_1 < A$. $\quad - \quad - \quad (10)$

Proceeding in this manner, we get quantities $A, A_1, A_2, \ldots A_{n-1}$, whose product is $= abcd \ldots$, such that

$$A > A_1 > A_2 \ldots > A_{n-1}. \quad - \quad - \quad - \quad (11)$$

Hence $\quad A^n > abcd \ldots$

$$> G^n \quad - \quad - \quad - \quad - \quad - \quad - \quad (12)$$

$$\therefore \quad A > G.$$

§ 3. But we may generalise the method, by choosing k so that A and k may have in common with a and b not the value of their geometric or arithmetic mean, but the value of some mean which by its nature lies *between* the arithmetic and the geometric mean of two quantities.

For example, let $\quad \sqrt{A} + \sqrt{k} = \sqrt{a} + \sqrt{b} \quad - \quad - \quad - \quad (13)$

$$\therefore \quad \sqrt{A}\sqrt{k} - \sqrt{a}\sqrt{b}$$

$$= \sqrt{A}(\sqrt{a} + \sqrt{b} - \sqrt{A}) - \sqrt{a}\sqrt{b} = (\sqrt{a} - \sqrt{A})(\sqrt{A} - \sqrt{b}) > 0$$

$$\therefore \quad Ak > ab. \quad - \quad - \quad - \quad - \quad (14)$$

But by (13), $\quad A + k + 2\sqrt{A}\sqrt{k} = a + b + 2\sqrt{a}\sqrt{b}$

$$\therefore \quad A + k < a + b \quad - \quad - \quad - \quad - \quad (15)$$

Thus $A, k, c, d, \ldots$ have a smaller sum and a greater product than $a, b, c, d, \ldots$.

Hence if A_1 is the arithmetic mean of $k, c, d, \ldots$, then $A_1 < A$.

Dealing with $k, c, d, \ldots$ as before with $a, b, c, d, \ldots$ and repeating the process, each time with one quantity fewer, we get

$$abcd \ldots\ldots < Akcd \ldots\ldots < AA_1k_1 \ldots\ldots < AA_1A_2 \ldots\ldots A_{n-1} < A^n$$

$$\therefore \quad G < A.$$

§ 4. The general type of such proofs may be indicated thus :—

Of n positive quantities $a, b, c, d, \ldots$ let a be one which is not less than any of the others - - - - - - - - (21)

and let A denote the arithmetic mean and G the geometric mean of $a, b, c, d, \ldots$.

Let us find M and k such that

$$Mk \nless ab \qquad (22)$$

$$M+k \ngtr a+b \qquad (23)$$

Let us now re-arrange the quantities $k, c, d, \ldots$ and denote them by $a', b', c', \ldots$ where a' is one which is not less than any of the others, and again find M', k' such that

$$M'k' \nless a'b' \qquad (24)$$

$$M'+k' \ngtr a'+b' \qquad (25)$$

and proceed to deal with $k', c', d', \ldots$ as previously with $k, c, d, \ldots$; and continue the process till the last M, say N, and the last k, say l, are found.

Then
$$MM'M'' \ldots \ldots Nl \nless abcde \ldots \ldots \qquad (26)$$
and
$$M+M'+M''+\ldots+N+l \ngtr a+b+c+d+\ldots \qquad (27)$$
are deduced at once from (22), (24), (23), (25) and the other inequalities like them.

Now suppose further that the M's and k's have been chosen so that
$$M \nless M' \nless M'' \ldots \nless N \nless l \ldots \qquad (28)$$
and we shall have, by (26) and (28), $M^n \nless abcd \ldots$

$$\therefore \quad M \nless G. \qquad (29)$$

On the other hand, if we arrange so that

$$M \ngtr M' \ngtr M'' \ldots \ngtr N \ngtr l \qquad (30)$$

then by (27) we shall have $nM \ngtr a+b+c \ldots$

$$\therefore \quad M \ngtr A. \qquad (31)$$

§ 5. Now in order to prove that $A \nless G$ in the case of (28) we need to secure that $M \ngtr A$, which would be most simply attained by putting $M = A$.

We must then choose the auxiliary quantity k in such a manner as to satisfy (22) and (23); and for this it is in general sufficient to choose k by assuming

$$\phi(M, k) = \phi(a, b) \qquad (32)$$

where $\phi(a, b)$ is some mean which by its nature is intermediate between $\sqrt{ab}$ and $(a+b)/2$.

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As an example, let us take $\phi(a, b) \equiv \left\{\frac{a^2+pab+b^2}{p+2}\right\}^{\frac{1}{2}}$ where $p > 2$, so that

$$M^2 + pMk + k^2 \quad = a^2 + pab + b^2. \qquad (33)$$

Then $$(M+k)^2 - (a+b)^2 = (p-2)(ab - Mk). \qquad (34)$$

Thus $M + k - (a+b)$ and $ab - Mk$ have the same sign.

Again $(Mk - ab)(Mk + ab + pM^2) = M^2k^2 - a^2b^2 + pM^2(Mk - ab)$

$$= M^2(k^2 + pMk + M^2) - a^2b^2 - M^4 - pabM^2$$

$$= M^2(a^2 + pab + b^2) - a^2b^2 - M^4 - pabM^2$$

$$= (a^2 - M^2)(M^2 - b^2). \qquad (35)$$

Hence if we choose b (as we may) to be the least of the quantities $a, b, \ldots\ldots$, then $a \not< M \not< b$, and (35) is $\not< 0$;

$$\therefore \quad Mk \not< ab; \qquad (36)$$

$$\therefore \quad M + k \not> a + b. \qquad (37)$$

Thus the conditions (22) and (23) are attained.

Again, $$nM = a + b + c \ldots$$

$$\not< M + k + c + d + \ldots, \quad \text{by (37)},$$

$$\therefore \quad \overline{n-1}M \not< k + c + d + \ldots$$

$$\not< a' + b' + c' + \ldots$$

$$\not< \overline{n-1}M',$$

since M' is the arithmetic mean of $a', b', c', \ldots$. Hence $M \not< M'$, *i.e.*, the condition (28) is satisfied.

§ 6. It is now clear that the proposition $A \not< G$ is susceptible of proof in an infinite number of ways, each belonging to the type we are considering. This statement is already justified by the fact that each value of p which we may choose, provided it is < 2, gives a separate variety of the type. It would perhaps be difficult to prove that the method of choosing k indicated in connection with (32) will *always* satisfy the conditions required. But there is no difficulty in getting a large variety of forms of ϕ for which the proof is easily completed, still keeping $M \equiv A$.

As another example, take $\phi(a, b) \equiv \left(a^{\frac{1}{r}} + b^{\frac{1}{r}}\right)/2$, so that

$$\mathrm{M}^{\frac{1}{r}} + k^{\frac{1}{r}} = a^{\frac{1}{r}} + b^{\frac{1}{r}}\,; \qquad (38)$$

$$\therefore \quad \mathrm{M}^{\frac{1}{r}} - b^{\frac{1}{r}} = a^{\frac{1}{r}} - k^{\frac{1}{r}} \qquad (39)$$

and

$$\frac{\mathrm{M} - b}{a - k} = \frac{\mathrm{M}^{\frac{1}{r}} - b^{\frac{1}{r}}}{a^{\frac{1}{r}} - k^{\frac{1}{r}}} \cdot \frac{\mathrm{M}^{\frac{r-1}{r}} + b^{\frac{1}{r}}\mathrm{M}^{\frac{r-2}{r}} + \ldots\ldots + b^{\frac{r-1}{r}}}{a^{\frac{r-1}{r}} + k^{\frac{1}{r}}a^{\frac{r-2}{r}} + \ldots\ldots + k^{\frac{r-1}{r}}}. \qquad (40)$$

Now (38) shows $k \not< b$ since $\mathrm{M} \not> a$.

Hence each term of the numerator of the last fraction in (40) is $\not>$ the corresponding term in the denominator, while the fraction $\dfrac{\mathrm{M}^{\frac{1}{r}} - b^{\frac{1}{r}}}{a^{\frac{1}{r}} - k^{\frac{1}{r}}} = 1$ in virtue of (39).

Hence
$$\frac{\mathrm{M} - b}{a - k} \not> 1\,; \qquad \therefore \quad \mathrm{M} - b \not> a - k\,;$$

$$\therefore \quad \mathrm{M} + k \not> a + b. \qquad (41)$$

Again
$$\mathrm{M}^{\frac{1}{r}}k^{\frac{1}{r}} - a^{\frac{1}{r}}b^{\frac{1}{r}} = \mathrm{M}^{\frac{1}{r}}\left(a^{\frac{1}{r}} + b^{\frac{1}{r}} - \mathrm{M}^{\frac{1}{r}}\right) - a^{\frac{1}{r}}b^{\frac{1}{r}}$$

$$= \left(a^{\frac{1}{r}} - \mathrm{M}^{\frac{1}{r}}\right)\left(\mathrm{M}^{\frac{1}{r}} - b^{\frac{1}{r}}\right) \not< 0\,;$$

$$\therefore \quad \mathrm{M}k \not< ab. \qquad (42)$$

Thus the conditions (22) and (23) are fulfilled, and it is easy to prove, as before, that the condition (28) is also fulfilled.

Here again, r being any positive integer, we have an infinite number of possible methods of proof.

§7. Going back now to the case (30) which is an alternative to (28), we should have to modify the method of proof. The proof requires that $\mathrm{M} \not< \mathrm{G}$, and the simplest way of securing this is to make $\mathrm{M} \equiv \mathrm{G}$. Then k would have to be chosen so as to satisfy conditions (22), (23), and (30). This can be done as before, with the aid of a relation of the form (32).

§8. Forms of proof of the same type, but still more general, might be given by choosing M as well as k to be some mean lying *between* the arithmetic and the geometric mean of the quantities with reference to which it is defined.