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CAUCHY POINTS OF METRIC LOCALES

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A natural approach to topology which emphasizes its geometric essence independent of the notion of points is given by the concept of frame (for instance [**4**], [**8**]). We consider this a good formalization of the intuitive perception of a space as given by the "places" of non-trivial extent with appropriate geometric relations between them. Viewed from this position, points are artefacts determined by collections of places which may in some sense by considered as collapsing or contracting; the precise meaning of the latter as well as possible notions of equivalence being largely arbitrary, one may indeed have different notions of point on the same "space". Of course, the well-known notion of a point as a homomorphism into **2** evidently fits into this pattern by the familiar correspondence between these and the completely prime filters. For frames equipped with a diameter as considered in this paper, we introduce a natural alternative, the Cauchy points. These are the obvious counterparts, for metric locales, of equivalence classes of Cauchy sequences familiar from the classical description of completion of metric spaces: indeed they are decreasing sequences for which the diameters tend to zero, identified by a natural equivalence relation.

The basic concept considered in this article is that of metric locale, that is, a locale enriched by a diameter naturally generalizing the usual notion of a diameter of subsets of metric space. This concept then turns out to be strong enough to permit the extension of the classical definitions of uniform, Lipschitz and contractive maps between metric spaces, providing us with three different categories and the expected inclusions between them. These categories are related to the corresponding classical categories of metric spaces by appropriately enriched versions of the usual spectrum (Σ) and open-sets (Ω) functors between locales and topological spaces, again adjoint as in the non-enriched case.

Now, the Cauchy points of a metric locale A carry a natural metric induced by the diameter, providing a complete metric space called the Cauchy spectrum ΨA of A. This construction determines functors Ψ from the above categories into the respective categories of complete metric spaces. Our main result then is that this functor Ψ is adjoint on the left to the corresponding (restriction of) Ω. This, for instance, explains the behaviour of Ω on complete metric (or completely metrizable) spaces: being a right adjoint it preserves products and in the uniform case, product is given by the product of the underlying locales. Moreover, we obtain new limit preservation laws in the other cases, where the product is different [**13**].

A further aspect of the Cauchy spectrum is that it gives a new description of the completion of metric frames. It is worth noting that this is done by essentially

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the same device used classically for metric spaces. Further, the completion of metric spaces can be obtained just by combining our $\Psi - \Omega$ adjunction with the $\Omega - \Sigma$ adjuction.

Finally, using an appropriate notion of regular ideal we give an alternative, purely algebraic direct description of the completion of totally bounded metric locales which is in fact a compactification.

For general facts concerning locales we refer to [7], and for category theory to [**10**].

1. Preliminaries. 1.1. A *diameter* on a frame A (see, for instance, [**12**], [**13**]) is a mapping $d : A \longrightarrow \mathbf{R}_+$ (non-negative reals plus $+\infty$) such that

(1) $d(0) = 0$,

(2) $a \leqq b \Rightarrow d(a) \leqq d(b)$,

(3) $a \wedge b \neq 0 \Rightarrow d(a \vee b) \leqq d(a) + d(b)$,

(4) for each $\epsilon > 0$, $U_\epsilon = \{a \mid d(a) < \epsilon\}$ is a cover.

A *star-diameter* satisfies, moreover,

($*$) if $a \in A$ and $S \subseteq A$ are such that $s \wedge a \neq 0$ for each $s \in S$, then

$$d\left(a \vee \bigvee S\right) \leqq d(a) + \sup\{d(b) + d(c) | b, c \in S, b \neq c\}.$$

A *metric diameter* satisfies

(M) for each $a \in A$ and $\epsilon > 0$ there are $x, y \leqq a$ such that

$$d(x), d(y) < \epsilon \quad \text{and} \quad d(x \vee y) > d(a) - \epsilon.$$

It is easy to check that (M) implies ($*$).

Sometimes the condition (4) will be dropped. Then we speak about *prediameters*.

In the sequel, we will confine ourselves to metric diameters. According to Remark in 1.4 below, this is not a serious restriction. Most of the statements can be, however, extended to star-diameters.

1.2. Recall the notation

$$Ua = \bigvee\{x | x \in U, x \wedge a \neq 0\}$$

for subsets U of A and $a \in A$. One has $U \bigvee a_i = \bigvee Ua_i$ and hence there is $\alpha_U : A \longrightarrow A$ satisfying

$$Ua \leqq b \text{ if and only if } a \leqq \alpha_U(b).$$

We write α_ϵ instead of α_{U_ϵ}.

Observations. 1. $a \leqq \alpha_\epsilon(U_\epsilon a), U_\epsilon \alpha_\epsilon(a) \leqq a$.

2. $U_\epsilon U_\delta a \leqq U_{\epsilon+\delta} a, \alpha_{\epsilon+\delta}(a) \leqq \alpha_\epsilon(\alpha_\delta(a))$.

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3. For star-diameters, $d(U_\epsilon a) \leqq d(a) + 2\epsilon$.

1.3. *Observation.* From (4) one immediately obtains that for each a and for each $\epsilon > 0$,

$$a = \bigvee \{x \mid x \leqq a, d(x) < \epsilon\}.$$

1.4. A prediameter d on a frame A is said to be *compatible* if for each a,

$$a \leqq \bigvee_{\epsilon > 0} \alpha_\epsilon a.$$

Of course, for a diameter, this condition amounts to the identity

$$a = \bigvee_{\epsilon > 0} \alpha_\epsilon a.$$

A *diametric frame* is a frame together with a compatible diameter; if the latter is metric we speak of a *metric frame*. $A, B, \ldots$ shall denote diametric frames, $d_A, d_B, \ldots$ (or just d) will indicate the respective diameters and we shall notationally confuse a diametric frame with its underlying frame.

Remark. By [**14**], the following statements are equivalent for a frame:
it is metrizable in the sense of Isbell [**6**],
it has a compatible diameter,
it has a compatible metric diameter.

1.5. For metric frames A and B, a frame homomorphism $f : A \longrightarrow B$ is called
uniform if, for each $\epsilon > 0$ there exists $\delta > 0$ such that, for any $b \in B$ with $d(b) < \delta$, there exists $a \in A$ for which $b \leqq f(a)$ and $d(a) \leqq \epsilon$;
Lipschitz if there exists k such that, for any $\epsilon > 0$ and $b \in B$ with $d(B) < \epsilon$, there exists $a \in A$ for which $b \leqq f(a)$ and $d(a) \leqq k\epsilon$;
contractive if, for any $\epsilon > 0$ and $b \in B$ with $d(b) < \epsilon$, there exists $a \in A$ for which $b \leqq f(a)$ and $d(a) < \epsilon$.

Note that contractive implies Lipschitz which in turn implies uniform.

The resulting categories will be denoted by

UnifMFrm, LipMFrm, MFrm, resp.

Their duals, the categories of *metric locales* will be denoted by

(1) UnifMLoc, LipMLoc, MLoc.

The categories of metric spaces with uniformly continuous resp. Lipschitz resp. contractive mappings are denoted by

(2) UnifMetr, LipMetr, Metr

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and their subcategories of complete metric spaces by

(3) CUnifMetr, CLipMetr, CMetr.

In the sequel we will formulate statements concerning the categories above by writing $M\mathcal{L}$ for any of the categories in (1) and $\mathcal{M}$ resp. $C\mathcal{M}$ for the corresponding category in (2) resp. (3).

1.6. The usual open-set functor $\Omega : \text{Top} \longrightarrow \text{Loc}$ modifies in the obvious way to

$$\Omega : \mathcal{M} \longrightarrow M\mathcal{L}.$$

We shall use the same notation for the restriction of Ω to $C\mathcal{M}$.

The usual spectrum functor $\Sigma : \text{Loc} \longrightarrow \text{Top}$ modifies to

$$\Sigma : M\mathcal{L} \longrightarrow \mathcal{M}$$

by defining

$$\rho(\xi, \eta) = \inf\{d(a) | \xi(a) = \eta(a) = 1\}.$$

One can easily prove ([**13**]) that, again, Σ is right adjoint to Ω.

1.7. Let A be a diametric frame and $f : A \longrightarrow B$ a surjective frame homomorphism. In order to make B into a diametric frame we define

$$d_B(b) = \inf\{d(a) | f(a) \geqq b\}.$$

It is easy to see that
d_B is indeed a compatible diameter, and
$f : A \longrightarrow B$ is contractive.
The resulting diametric frame will be called a diametric sub-locale (or: diametric quotient frame). Also, it is not hard to prove that a diametric sublocale of a metric one is metric.

1.8. The following is straightforward:

LEMMA. *Let $f : A \longrightarrow B$ be uniform, resp. Lipschitz, resp. contractive. Then for each $\epsilon > 0$ there is a $\delta > 0$ such that $\alpha_\epsilon \leqq \alpha_\delta f$, resp.*
there is a k such that, for any $\epsilon > 0, f\alpha_{k\epsilon} \leqq \alpha_\epsilon f$, resp.
for any $\epsilon > 0, f\alpha_\epsilon \leqq \alpha_\epsilon f$.
The inequalities $f\alpha_\epsilon \leqq \alpha_\delta f$ in these formulas can be replaced by $U_\delta f \leqq fU_\epsilon$.

Note. For metric frames, the converse of the third statement is also true, whereas it fails for the general case ([**13**], [**15**]). The corresponding question concerning the other two statements is open.

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2. Cauchy points and Cauchy spectrum. 2.1. Let A be a metric locale. An *end* in A is a decreasing sequence $(a_n)_{n\in\omega}$ of non-zero elements of A such that

$$\lim d(a_n) = 0.$$

Ends $(a_n)_{n\in\omega}, (b_n)_{n\in\omega}$ are said to be *equivalent* (write $(a_n)_{n\in\omega} \sim (b_n)_{n\in\omega}$) if also $(a_n \wedge b_n)_{n\in\omega}$ is an end. The equivalence classes are called the *Cauchy points* of A. For Cauchy points ξ, η put

$$\rho(\xi, \eta) = \lim d(x_n \vee y_n)$$

where $(x_n)_{n\in\omega} \in \xi$ and $(y_n)_{n\in\omega} \in \eta$. Obviously this does not depend on the choice of representatives and constitutes a metric on the set of all Cauchy points. The resulting metric space is called the *Cauchy spectrum* of A and denoted by ΨA.

We emphasize that ΨA is never empty for non-trivial A: Pick any non-zero $a_1 \in U_1$ and if non-zero $a_1 \in U_1, \dots, a_k \in U_{1/k}$ are chosen such that $a_1 \geqq \cdots \geqq a_k$, then there exists (recall 1.3) a non-zero $a_{k+1} \leqq a_k$ such that

$$a_{k+1} \in U_{1/(k+1)}.$$

Since $d(a_n) < 1/n$, this process determines an end.

Notes. 1. The term "end" appears in various places in the topological literature; it is used for notions more or less related to ours whose purpose is to define specific types of extension spaces ([**3**], [**5**], [**9**]). An analogous concept is that of Cauchy approximation in the context of normed algebras ([**11**]).

2. In the Introduction, we have outlined a certain general pattern of defining a "point" in a locale, including the arbitrariness in the choice of the equivalence. In view of this, we note that the equivalence chosen here is in a way canonical. The above formula for ρ evidently defines a pseudometric on the set of ends; once one agrees this definition is natural, our equivalence appears very particular, being the only one under which this pseudometric becomes a metric. Also, it is the smallest equivalence containing the relation R with

$$(a_n)R(b_n) \quad \text{if and only if for all } n, \quad a_n \wedge b_n \neq 0.$$

However, another choice of equivalence of ends of interest is, for instance, the smallest one containing the relation of being a subsequence. We intend to present a more detailed study of the arbitrariness aspect in the fabrication of "points" in "pointless topology" in a further article.

PROPOSITION 2.2. *ΨA is complete.*

Proof. Let ξ_n be a Cauchy sequence in ΨA. Choose representatives $(x_{ni})_{i\in\omega} \in \xi$ so that $d(x_{ni}) < 1/i$. We have

$$\forall\epsilon > 0\, \exists n, \forall k, l \geqq n \Rightarrow \rho(\xi_k, \xi_l) < \epsilon.$$

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Thus,

$$\forall \epsilon > 0 \, \exists n, \forall k, l \geqq n \, \exists r = r(k,l), s \geqq r \Rightarrow d(x_{ks} \vee x_{ls}) < \epsilon.$$

Put

$$x_n = \bigvee_{k \geqq n} x_{kk}.$$

Take $\epsilon > 0$ and take the n from the formula above. Moreover, choose n sufficiently large so that $1/n < \epsilon$. Then, for $k, l \geqq n$,

$$d(x_{kk} \vee x_{ll}) \leqq d(x_{kk} \vee x_{ks} \vee x_{ls} \vee x_{ll}) < 3\epsilon$$

and hence, by the star property, $d(x_n) < 7\epsilon$. Let ξ be the class of (x_n). We have

$$\rho(\xi_n, \xi) \leqq d(x_{nn} \vee x_n) = d(x_n)$$

and hence ξ_n converges to ξ.

2.3. Let $f : A \longrightarrow B$ be uniform. We easily see that for an end (b_n) in B there is an end (a_n) in A such that $f(a_n) \geqq b_n$ for each n. Furthermore, if $f(a_n) \geqq b_n$, $f(a'_n) \geqq b'_n$ and $(b_n) \sim (b'_n)$ then $(a_n) \sim (a'_n)$. (Indeed, there is an end (c_n) such that $f(c_n) \geqq b_n \vee b'_n$. Consequently,

$$c_n \wedge a_n \neq 0 \neq c_n \wedge a'_n$$

and hence

$$d(a_n \vee a'_n) \leqq d(a_n \vee c_n \vee a'_n) \leqq d(a_n) + d(c_n) + d(a'_n) \longrightarrow 0.)$$

Thus, we can define a mapping

$$\Psi f : \Psi B \longrightarrow \Psi A$$

by the formula

$$\Psi f(\xi) \ni (a_n) \text{ such that } f(a_n) \geqq b_n \text{ for some } (b_n) \in \xi.$$

PROPOSITION. *Thus defined Ψ constitutes a functor*

$$\Psi : M\mathcal{L} \longrightarrow C\mathcal{M}.$$

Proof. Let $\forall \epsilon > 0 \; \exists \delta > 0$ such that

$$d(b) < \delta \Rightarrow \exists a, d(a) < \epsilon, f(a) \geqq b$$

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(resp. $\exists k$, $\delta = k^{-1}\epsilon$; resp. $\delta = \epsilon$). Let

$$\rho(\xi, \eta) < \delta, (x_n) \in \xi, (y_n) \in \eta.$$

Thus, for sufficiently large n, $d(x_n \vee y_n) < \delta$ and hence there is an a with $d(a) < \epsilon$ and $f(a) \geqq x_n \vee y_n$. Let $f(u_n) \geqq x_n, f(v_n) \geqq y_n$. Evidently

$$u_n \wedge a \neq 0 \neq v_n \wedge a \text{ and } f(u_n \wedge a) \geqq x_n, f(v_n \wedge a) \geqq y_n.$$

Thus,

$$\rho(\Psi f(\xi), \Psi f(\eta)) \leqq d(a) < \epsilon.$$

LEMMA 2.4. *For any point $\xi : A \longrightarrow \mathbf{2}$ there exists an end (a_n) such that $\xi(a_n) = 1$ for all n; moreover, any two such ends are equivalent.*

Proof by induction. Since $\xi(\bigvee U_1) = 1$ there is an a_1 with $d(a_1) < 1$ such that $\xi(a_1) = 1$. Let us have $a_1 \geqq \cdots \geqq a_{n-1}$ with $\xi(a_k) = 1$ and $d(a_k) < 1/k$. By 1.3 there is an $a_n \leqq a_{n-1}$ such that $\xi(a_n) = 1$ and $d(a_n) < 1/n$.

Let (b_n) be another end with $\xi(b_n) = 1$. Then

$$\xi(a_n \wedge b_n) = 1$$

and hence $a_n \wedge b_n \neq 0$.

2.5. By 2.4 we have mappings

$$\psi_A : \Sigma A \longrightarrow \Psi A$$

defined by the condition:

there exists $(a_n) \in \psi_A(\xi)$ such that $\xi(a_n) = 1$ for all n.

PROPOSITION. *ψ_A are isometric imbeddings and constitute a natural transformation $\psi : \Sigma \longrightarrow \Psi$.*

Proof. Let $\rho(\xi, \eta) < \epsilon$. Thus, there is an a with $d(a) < \epsilon$, $\xi(a) = \eta(a) = 1$. Let $\xi(a_n) = \eta(b_n) = 1$. Then $a_n \wedge a \neq 0 \neq b_n \wedge a$, and $(a_n \wedge a) \in \psi(\xi)$, $(b_n \wedge a) \in \psi(\eta)$. Thus,

$$\rho(\psi(\xi), \psi(\eta)) < \epsilon.$$

On the other hand, let

$$\rho(\psi(\xi), \psi(\eta)) < \epsilon.$$

There are $(a_n), (b_n)$ such that $\xi(a_n) = 1 = \eta(b_n)$ and, for sufficiently large k,

$$d(a_k \vee b_k) < \epsilon.$$

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Since

$$\xi(a_k \vee b_k) = \eta(a_k \vee b_k) = 1,$$

we have $\rho(\xi, \eta) < \epsilon$.

Now let $f : A \longrightarrow B$ be uniform. We have an end

$$(a_n) \in \Psi f(\psi(\xi))$$

such that $f(a_n) \geqq b_n$ for some end such that $\xi(b_n) = 1$. Then

$$(\Sigma f(\xi))(a_n) = \xi(f(a_n)) \geqq \xi(b_n) = 1.$$

Thus, $\psi \circ \Sigma f = \Psi f \circ \psi$.

Remark. It should be noted that ψ_A may be trivial, that is, the embedding of the empty metric space. For instance, if $\mathbf{R}$ is the real line with its usual metric, and A the metric quotient of double pseudocomplementation, then $\Sigma A = \emptyset$ since the underlying frame of A is the Boolean algebra of regular-open sets in $\mathbf{R}$ which has no atoms and hence no points. Note that by way of contrast, $\Psi A \cong \mathbf{R}$. In particular, of course, ΨA should not be mistaken for the completion of ΣA.

LEMMA 2.6. *Let $a \nleqq b$. Then there is an end (a_n) such that $a_1 \leqq a$ and, for all n, $a_n \nleqq b$. In particular, if $a \neq 0$ then there is an end (a_n) with $a_n \leqq a$.*

Proof by induction. By 1.3 there is an a_1 such that $d(a_1) < 1$ and $a_1 \nleqq b$. Let $a_1 \geqq \cdots \geqq a_{n-1}$ be such that $a_{n-1} \nleqq b, d(a_k) < 1/k$. By 1.3 again, there is an a_n such that $d(a_n) < 1/n, a_n \leqq a_{n-1}$ and $a_n \nleqq b$.

2.7. Write

$$a \in \mathcal{N}_+(\xi)$$

if there is $(x_n) \in \xi$ such that $x_k \leqq a$ for some k. Further, write

$$a \in \mathcal{N}(\xi)$$

if there is $\epsilon > 0$ such that $\alpha_\epsilon a \in \mathcal{N}_+(\xi)$. Define subsets $\theta(a), \theta_+(a)$ of ΨA by putting

$$\theta(a) = \{\xi | a \in \mathcal{N}(\xi)\}, \quad \theta_+(a) = \{\xi | a \in \mathcal{N}_+(\xi)\}.$$

LEMMA 1. *Each $\theta(a)$ is open in ΨA. Thus, θ is a monotone mapping $A \longrightarrow \Omega\Psi A$.*

2. diam $\theta(a) \leqq d(a)$. *Consequently, for each $\xi \in U \in \Omega\Psi A$ there is an a such that $\xi \in \theta(a) \subseteq U$. Hence $\{\theta(a) | a \in A\}$ is a basis of $\Omega\Psi A$.*

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3. *We have the implication*

$$\theta(x)\cap\theta_+(a)\neq 0\Rightarrow\theta(x)\cap\theta(a)\neq 0.$$

Consequently, $\mathrm{Cl}\,\theta_+(a)=\mathrm{Cl}\,\theta(a)$.

Proof. Let $\xi\ni(x_n), x_k\leqq\alpha_{2\epsilon}a$ and $\rho(\xi,\eta)<\epsilon$. Thus we have a $(y_n)\in\eta$ and an $l\geqq k$ such that

$$d(y_1\vee x_1)<\epsilon.$$

Since $x_k\leqq\alpha_{2\epsilon}a\leqq\alpha_\epsilon\alpha_\epsilon a$, we have $y_1\leqq\alpha_\epsilon a$ and hence $\eta\in\theta(a)$.

2. This immediately follows from the definition of distance in ΨA.

3. Let for some $(x_n)\sim(y_n)$, k and $\epsilon>0$

$$x_k\leqq\alpha_\epsilon x\quad\text{and}\quad y_k\leqq a.$$

Since, for sufficiently large $r\geqq k$,

$$d(x_r\vee y_r)<\epsilon,$$

we have $y_r\leqq x$ and hence $0\neq y_r\leqq x\wedge a$. Since

$$x=\bigvee_{\epsilon>0}\alpha_\epsilon x\text{ and }a=\bigvee_{\epsilon>0}\alpha_\epsilon a,$$

we have, be distributivity, $\alpha_\epsilon x\wedge\alpha_\epsilon a\neq 0$ for some $\epsilon>0$ and hence $\theta(x)\cap\theta(a)\neq 0$ by 2.6. The consequence follows from 2.

LEMMA 2.8. *Let* $a\nleqq b$. *Then* $\theta(a)\backslash\theta(b)\neq\emptyset$.

Proof. Since $a\nleqq b$, there is an $\epsilon>0$ such that $\alpha_\epsilon a\nleqq b$. By 2.6 there is an end (a_n) such that $a_1\leqq\alpha_\epsilon a$ and $a_n\nleqq b$ for all n. Let ξ be the class containing (a_n). Thus, $\xi\in\theta(a)$. If $(b_n)\in\xi$ is such that $b_k<\alpha_\delta b$, we have, for sufficiently large k,

$$d(a_k\vee b_k)<\delta$$

and hence a contradiction $a_k\leqq b$.

LEMMA 2.9. *Let* $f:A\longrightarrow B$ *be a uniform homomorphism and* $a\in A$. *Then*

$$\Omega\Psi f(\theta(a))\subseteq\theta(f(a))\quad\textit{and}\quad\Psi f(\theta(f(a)))\subseteq\theta_+(a).$$

Proof. Let

$$\xi\in\Omega\Psi f(\theta(a))=(\Psi f)^{-1}(\theta(a)).$$

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Then $\Psi f(\xi) \in \theta(a)$ and we have $(a_n) \in \xi$ and $(b_n) \in \Psi f(\xi)$ such that $f(b_n) \geqq a_n$ and, for a suitable $\epsilon > 0$ and k, $b_k \leqq \alpha_\epsilon a$. Thus, by 1.8 we have for some $\delta > 0$

$$a_k \leqq f(b_k) \leqq f(\alpha_\epsilon a) \leqq \alpha_\delta f(a)$$

and hence $\xi \in \theta(f(a))$.

Let $\xi \in \theta(f(a))$. Choose $(a_n) \in \xi$ and such that $f(b_n) \geqq a_n$. Thus, $f(b_n \wedge a) \geqq a_n$ and hence $b_n \wedge a \neq 0$. We have $(b_n \wedge a) \in \Psi f(\xi)$ and hence $\Psi f(\xi) \in \theta(a)$.

3. $\Psi - \Omega$ adjunction. 3.1. In this section, Ω will be understood as defined on $C\mathcal{M}$. By 2.7 we have monotone mappings

$$\theta = \theta_A : A \longrightarrow \Omega\Psi A.$$

Define

$$\tau = \tau_A : \Omega\Psi A \longrightarrow A$$

by putting

$$\tau(U) = \bigvee\{x | \theta(x) \subseteq U\}.$$

PROPOSITION. *τ is a left adjoint to θ; moreover, $\tau\theta(a) = a$ for each a.*

Proof. By 2.8 we have $x \leqq a$ if and only if $\theta(x) \leqq \theta(a)$. Hence,

$$\tau\theta(a) = \bigvee\{x | \theta(x) \leqq \theta(a)\} = \bigvee\{x | x \leqq a\} = a.$$

Let $\xi \in U$. By 2.7.1 there is an a such that $\xi \in \theta(a) \subseteq U$. Since $\theta(a) \subseteq U$ we have $a \leqq \tau(U)$ and hence finally

$$\xi \in \theta(a) \subseteq \theta\tau(U).$$

Thus, also $U \subseteq \theta\tau(U)$.

Remark. As a result of this adjointness, we also have

$$\tau(U) = \bigwedge\{x | U \subseteq \theta(x)\},$$

apart from the original definition.

LEMMA 3.2. *We have*

$$\tau(U) = \bigvee\{x | \operatorname{Cl}\theta(x) \subseteq U\} = \bigvee\{x | \theta_+(x) \subseteq U\}.$$

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Proof. By 3.1, τ preserves joins. Thus,

$$\begin{aligned}\tau(U) &= \bigvee_{\epsilon>0} \tau(\alpha_\epsilon U)\\ &= \bigvee_{\epsilon>0} \bigvee\{x|\theta(x) \leqq \alpha_\epsilon U\}\\ &\leqq \bigvee\{x|\,\mathrm{Cl}\,\theta(x) \subseteq U\} \leqq \tau(U).\end{aligned}$$

Now, use 2.7.3.

PROPOSITION 3.3. *The mappings τ_A are metric quotient frame homomorphisms $\Omega\Psi A \longrightarrow A$.*

Proof. By 3.1, τ preserves joins (and θ preserves meets). Obviously $\tau(1) = 1$. By distributivity

$$\begin{aligned}\tau(U) \wedge \tau(W) &= \bigvee\{x \wedge y|\theta(x) \subseteq U, \theta(y) \subseteq W\}\\ &\leqq \bigvee\{x \wedge y|\theta(x \wedge y) \subseteq U \cap W\}\\ &\leqq \tau(U \cap W) \leqq \tau(U) \wedge \tau(W).\end{aligned}$$

Thus, τ is a homomorphism. Since $\tau\theta(a) = a$ and

$$\operatorname{diam}\theta(a) \leqq d_A(a),$$

τ is contractive. Thus, to prove that d_A is induced by the diameter of $\Omega\Psi A$ as in 1.7 it suffices to prove that one cannot have $a \leqq \tau(U)$ and $d(U) < d_A(a)$. Suppose this has occurred. Choose $\epsilon > 0$ so that $d(U) < d_A(a) - 4\epsilon$. Since d_A is metric, there are non-zero $b, c \leqq a$ such that $d(b), d(c) < \epsilon$ and

$$d(a) < d(b \vee c) + \epsilon.$$

Since

$$a \leqq \bigvee\{z|\theta(z) \subseteq U\},$$

we have x' and y' such that

$$\theta(x'), \theta(y') \subseteq U \quad \text{and } x = x' \wedge b \neq 0 \neq y' \wedge c = y.$$

Since θ is injective, we have $\theta(x), \theta(y) \neq 0$. Choose $\xi \in \theta(x), \eta \in \theta(y), (x_n) \in \xi, (y_n) \in \eta$ and k such that $x_k \leqq x, y_k \leqq y$. Then, for sufficiently large k, we obtain the contradiction

$$\begin{aligned}d(x_k \vee y_k) &< \rho(\xi, \eta) + \epsilon \leqq d(U) + \epsilon < d(a) - 3\epsilon\\ &< d(b \vee c) - 2\epsilon \leqq d(x_k \vee y_k).\end{aligned}$$

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3.4. As metric quotient frame homomorphisms, the $\tau_A : \Omega\Psi A \longrightarrow A$ are contractive and hence belong to the category MFrm. Moreover, we have

PROPOSITION. *The* τ_A *constitute a natural transformation* $\tau : \Omega\Psi \longrightarrow \mathrm{id}$.

Proof. Let $f : A \longrightarrow B$ be uniform. By 3.2 and 2.9,

$$\begin{aligned} f(\tau(U)) &= \bigvee\{f(x)|\theta_+(x) \subseteq U\} \\ &\leqq \bigvee\{f(x)|\Psi f(\theta(f(x))) \subseteq U\} \\ &\leqq \bigvee\{y|\Psi f(\theta(y)) \subseteq U\} \\ &= \bigvee\{y|\theta(y) \subseteq (\Psi f)^{-1}(U)\} \\ &= \bigvee\{y|\theta(y) \subseteq \Omega\Psi f(U)\} \\ &= \tau(\Omega\Psi f(U)). \end{aligned}$$

On the other hand, by 2.9 and 3.1,

$$\begin{aligned} \tau(\Omega\Psi f(U) &\leqq \tau(\Omega\Psi f(\theta\tau(U))) \\ &\leqq \tau\theta(f(\tau(U))) = f(\tau(U)). \end{aligned}$$

3.5. Let (X, ρ) be a complete metric space and let (U_n) be an end in $\Omega(X, \rho)$. Then $\bigcap \mathrm{Cl}\, U_n$ is a one-point set and, obviously, if $(U_n) \sim (W_n)$ then $\bigcap \mathrm{Cl}\, W_n = \bigcap \mathrm{Cl}\, U_n$. Thus, the formula

$$\mu(\xi) = \mu_{(X,\rho)}(\xi) \in \bigcap \mathrm{Cl}\, U_n \quad \text{where } (U_n) \in \xi$$

defines a mapping

$$\mu_{(X,\rho)} : \Psi\Omega(X, \rho) \longrightarrow (X, \rho).$$

PROPOSITION. *Each* $\mu_{(X,\rho)}$ *is an isometry and the system*

$$\mu = (\mu_{(X,\rho)})_{(X,\rho)}$$

constitutes a natural equivalence

$$\mu : \Psi\Omega \cong \mathrm{id}.$$

Proof. Obviously μ is onto. Let d be the usual diameter in $\Omega(X, \rho)$. If $x \in \bigcap \mathrm{Cl}\, U_n$, $y \in \bigcap \mathrm{Cl}\, W_n$ where $(U_n) \in \xi$ and $(W_n) \in \eta$, we have

$$\rho(\xi, \eta) = \lim d(U_n \cup W_n) = \lim d(\mathrm{Cl}\, U_n \cup \mathrm{Cl}\, W_n) = \rho(x, y).$$

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Let $f : X \longrightarrow Y$ be uniform, $(U_n) \in \xi \in \Psi\Omega(X)$. We have

$$(W_n) \in \Psi\Omega(f)(\xi)$$

such that $f^{-1}(W_n) \supseteq U_n$ for each n. Since f is continuous, we obtain

$$f^{-1}(\mathrm{Cl}\, W_n) \supseteq \mathrm{Cl} f^{-1}(W_n) \supseteq \mathrm{Cl}\, U_n$$

and hence

$$f^{-1}\left(\bigcap \mathrm{Cl}\, W_n\right) = \bigcap f^{-1}(\mathrm{Cl}\, W_n) \supseteq \bigcap \mathrm{Cl}\, U_n \ni \mu(\xi)$$

so that $f(\mu(\xi)) \in \bigcap \mathrm{Cl}\, W_n$ and hence

$$f\mu(\xi) = \mu(\Psi\Omega f(\xi)).$$

LEMMA 3.6. *For each* $U \in \Omega\Psi A$, $\mathrm{Cl}\,\theta\tau(U) = \mathrm{Cl}\, U$.

Proof. Recall 2.7.2. Let $\xi \in \mathrm{Cl}\,\theta\tau(U)$. If $\xi \in \theta(a)$ then

$$0 \neq \theta(a) \cap \theta\tau(U) = \theta(a \wedge \tau(U))$$

and hence $a \wedge \tau(U) \neq 0$. Thus, there is an x such that $a \wedge x \neq 0$ and $\theta(x) \subseteq U$. By 3.1, θ is one-one and hence

$$0 \neq \theta(a \wedge x) = \theta(a) \cap \theta(x) \subseteq \theta(a) \cap U.$$

Thus, $\xi \in \mathrm{Cl}\, U$.

LEMMA 3.7. *For* $U, W \in \Omega(X, \rho)$ *one has*

$$\theta(W) \subseteq \mu^{-1}(U) \textit{ if and only if } W \subseteq U.$$

Proof. Let $\theta(W) \subseteq \mu^{-1}(U)$ and let $x \in W$. Choose a decreasing system $Z_1 \supseteq Z_2 \supseteq \ldots$ of neighbourhoods of x such that $Z_1 \subseteq \alpha_\epsilon W$ and $d(Z_n) \longrightarrow 0$, and denote by $\bar{x}$ the Cauchy point containing (Z_n). Then $\bar{x} \in \theta(W)$ and $x = \mu(\bar{x}) \in U$.

On the other hand, if $\xi \in \theta(U)$ we have (Z_n), k and ϵ such that $Z_k \subseteq \alpha_\epsilon U$. Hence $\mathrm{Cl}\, Z_k \subseteq U$ so that $\mu(\xi) \in Z_k$. Thus, $\theta(U) \subseteq \mu^{-1}(U)$.

THEOREM 3.8. *The functor* $\Omega : C\mathcal{M} \longrightarrow M\mathcal{L}$ *is a right adjoint to* $\Psi : M\mathcal{L} \longrightarrow C\mathcal{M}$. *The transformations* $\tau : \mathrm{id} \longrightarrow \Omega\Psi$ *(localic interpretation) and* $\mu : \Psi\Omega \cong \mathrm{id}$ *are the units of this adjunction.*

Proof. Let $(a_n) \in \xi \in \Psi A$. Hence there is a $(U_n) \in \Psi\tau(\xi)$ such that $\tau(U_n) \geqq a_n$. Thus, by 3.6,

$$\mathrm{Cl}\, U_n = \mathrm{Cl}\,\theta\tau(U_n) \supseteq \mathrm{Cl}\,\theta(a_n).$$

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By 2.7.3,

$$\xi \in \theta_{+}(a_n) \subseteq \mathrm{Cl}\,\theta(a_n)$$

and hence $\xi \in \bigcap \mathrm{Cl}\, U_n$ so that

$$\xi = \mu(\Psi\tau(\xi)).$$

Thus,

$$\mu\Psi \circ \Psi\tau = \mathrm{id}.$$

Let $U \in \Omega(X, \rho)$. By 3.7,

$$\begin{aligned}\tau(\Omega\mu(U)) = \tau(\mu^{-1}(U)) &= \bigcup\{W \mid \theta(W) \subseteq \mu^{-1}(U)\}\\ &= \bigcup\{W \mid W \subseteq U\} = U.\end{aligned}$$

Thus,

$$\tau\Omega \circ \Omega\mu = \mathrm{id}.$$

3.9. *Remarks.* 1. Recall 1.6. The adjunctions

$$C\mathcal{M} \underset{\Psi}{\overset{\Omega}{\rightleftarrows}} M\mathcal{L} \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} \mathcal{M}$$

compose to

$$O\mathcal{M} \underset{\Psi\Omega}{\overset{\subset}{\rightleftarrows}} \mathcal{M},$$

the reflection of $C\mathcal{M}$ in $\mathcal{M}$ (the classical completion).

2. In analogy with corresponding facts for metric spaces, a metric frame is called complete whenever it has no proper dense extension in MLoc, or, equivalently, there is no proper dense metric quotient homomorphism to it. It follows from the results of Isbell ([**6**]) that, for any complete metric space X, the metric frame ΩX is complete. Since $\tau_A : \Omega\Psi A \longrightarrow A$ is a dense metric quotient frame homomorphism (3.3) and ΨA is complete (2.2), this makes $\tau_A : \Omega\Psi A \longrightarrow A$ the *completion* of A. Note that this makes any metric frame a dense quotient of a spatial one.

3. By 3.8, in particular, Ω, which preserves colimits in $\mathcal{M}$ preserves also limits in $C\mathcal{M}$. Realize that the case $\mathcal{M}$ = UnifMetr implies also the preservation of limits of completely metrizable frames under Ω : Top $\longrightarrow$ Loc. On the other hand, in the case $\mathcal{M}$ = Metr we obtain preservation of different constructions (see [**13**]).

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4. The mappings $\theta : A \longrightarrow \Omega\Psi A$ are not, in general, frame homomorphism. In the composition with $\Omega\psi$

$$A \xrightarrow{\theta} \Omega\Psi A \xrightarrow{\Omega\psi} \Omega\Sigma A$$

(recall 2.5), however, they yield the unit of the $\Omega - \Sigma$ adjunction (sending a to Σ_a).

5. Theorem 3.8 holds also for the corresponding categories of star-diametric locales. The property (M) has been used only in proving that τ is a sublocale mapping (3.3); τ is contractive anyway. On the other hand, in the $\Omega - \Sigma$ adjunction, (M) may be essential ([**13**]).

PROPOSITION 3.10. 1. *For any A, every Cauchy point of A is a point if and only if A is complete spatial.*

2. *The embedding ψ_A is dense if and only if the adjunction unit $\eta_A : A \longrightarrow \Omega\Sigma A$ is dense..*

Proof. 1. Let each Cauchy point of A be a point, that is, let the embedding ψ_A from 2.5 be an isomorphism. Thus, since $\tau\theta = \mathrm{id}$, we have by 3.9.4, $\eta_A = \Omega\psi_A \circ \theta_A$ one-one. As η_A is onto anyway, it is an isomorphism and hence A is spatial. Furthermore we have

$$\theta = (\Omega\psi)^{-1} \circ \Omega\psi \circ \theta = (\Omega\psi)^{-1} \circ \eta$$

so that in this case θ is a frame homomorphism, onto, and since it is anyway one-one, it is an isomorphism. Thus, A is complete, since ΨA is.

On the other hand, let A be complete spatial. Thus,

(a) η_A is an isomorphism, and

(b) $\tau_A : \Omega\Psi A \longrightarrow A$ is an isomorphism.

Since $\tau\theta = \mathrm{id}$ we have by (b) that θ is an isomorphism and, by (a) and 3.9.4, we see that $\Omega\psi_A$ is an isomorphism. Since, for sober spaces, Ω reflects isomorphism, ψ_A is one.

2. If ψ_A is dense, we obviously have $\eta_A = \Omega\psi_A \circ \theta_A$ dense. On the other hand, let η_A be dense and let $U \in \Omega\Psi A$ be non-void. By 2.7.2, there is an $a \neq 0$ such that $\theta(a) \subseteq U$. Thus,

$$\Omega\psi(U) \geqq \Omega\psi(\theta(a)) = \eta(a) \neq 0.$$

COROLLARY. *For any compact A, every Cauchy point is a point.*

Proof. Recall that for arbitrary frames, compact regular implies spatial.

Remark. Note that, even for metric A, $\eta_A : A \longrightarrow \Omega\Sigma A$ may be dense without A being spatial.

4. Compactification of totally bounded metric locales. In this section, we present an alternative description of the completion in a special case, that of

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totally bounded diameters, defined by the condition that each of the covers U_ϵ (1.1.(4)) has a finite subcover. For this we shall use the ideal frame $\mathfrak{J}A$ of A in a manner analogous to one of the descriptions of the Stone-Čech compactification of frames in [**1**].

4.1. Let A be a diametric frame. An ideal $J \in \mathfrak{J}A$ is said to be *regular* if, for each $x \in J$, we have also $U_\epsilon x \in J$ for some $\epsilon > 0$. The system of all regular ideals will be denoted by $\mathfrak{R}A$.

PROPOSITION. *$\mathfrak{R}A$ is a subframe of $\mathfrak{J}A$.*

Proof. Trivially 0 and A are in $\mathfrak{R}A$, and obviously $\mathfrak{R}A$ is closed under intersections and up-directed unions. Thus, it suffices to prove that it is closed under binary unions. let $J, K \in \mathfrak{R}A$ and $x = a \vee b \in J \vee K$ $(a \in J, b \in K)$. We have $U_\epsilon a \in J$, $U_\delta b \in K$ for some $\epsilon, \delta > 0$. Let, say, $\epsilon \leqq \delta$. Then

$$U_\epsilon x \leqq U_\epsilon(a \vee b) = U_\epsilon a \vee U_\epsilon b \leqq U_\epsilon a \vee U_\delta b \in J \vee K.$$

Thus, $J \vee K$ is regular.

4.2. In the following, we let UnifDFrm be the category of diametric frames and uniform homomorphisms. Further, for $f : A \longrightarrow B$ a map in this category and $J \in \mathfrak{R}A$, put

$$\mathfrak{R}f(J) = \{x | x \leqq f(a) \quad \text{for some } a \in J\}.$$

Finally, let KFrm be the category of compact frames.

PROPOSITION. *The correspondences $A \longmapsto \mathfrak{R}A$, $f \longmapsto \mathfrak{R}f$ define a functor $\mathfrak{R}$:* UnifDFrm $\longrightarrow$KFrm.

Proof. Since $\mathfrak{R}A$ is compact as subframe of $\mathfrak{J}A$, it suffices to prove that

$$\{x | \exists a \in J, x \leqq f(a)\},$$

which is obviously an ideal, is regular. Let $x \leqq f(a)$, $a \in J$. There is an $\epsilon > 0$ such that $U_\epsilon a \in J$. By 1.8 we have a $\delta > 0$ such that

$$U_\delta x \leqq U_\delta f(a) \leqq f(U_\epsilon a).$$

4.3. Let a be an element of a diametric frame A. Put

$$\sigma(a) = \{x | \exists \epsilon > 0, U_\epsilon x \leqq a\}.$$

By 1.2.2, $\sigma(a) \in \mathfrak{R}A$. Since d is compatible we have

$$\bigvee \sigma(a) = a.$$

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Further we have

$$J \subseteq \sigma\left(\bigvee J\right)$$

(indeed, if $x \in J$, we have $U_\epsilon x \in J$ for some $\epsilon > 0$, and hence $U_\epsilon x \leqq \bigvee J$). Thus we have an adjunction

$$\bigvee J \leqq a \text{ if and only if } J \subseteq \sigma(a)$$

between the monotone mappings $\bigvee : \Re A \to A$ and $\sigma : A \to \Re A$.

PROPOSITION. $\bigvee$ *is a dense frame homomorphism.*

Proof. Obviously, $\bigvee$ preserves joins. Further, we have

$$\begin{aligned}\bigvee J \wedge \bigvee K &= \bigvee\{x \wedge y \mid x \in J, y \in K\}\\ &\leqq \bigvee\{z \mid z \in J \cap K\} = \bigvee(J \cap K) \leqq \bigvee J \wedge \bigvee K.\end{aligned}$$

The density is obvious.

4.4. For $J \in \Re A$ puts $\partial(J) = d(\vee J)$.

PROPOSITION. ∂ *is a prediameter on* $\Re A$. *If* d *satisfies* $(*)$ *or* (M), *so does* ∂.

Proof. Obviously ∂ is monotone and $\partial(O) = 0$. If $J \cap K \neq 0$ we have an $a \neq 0$ in $J \cap K$ and hence $\bigvee K \wedge \bigvee K \neq 0$. Thus,

$$\begin{aligned}\partial(J \vee K) &= d\left(\bigvee J \vee \bigvee K\right)\\ &\leqq d\left(\bigvee J\right) + d\left(\bigvee K\right) = \partial(J) + \partial(K).\end{aligned}$$

The star property is checked quite analogously as the subadditivity.

Now let d be metric. Take $0 \neq J \in \Re A$ and $\epsilon > 0$. There exist $0 \neq x, y \leqq \bigvee J$ such that

$$d(x), d(y) < \epsilon \quad \text{and} \quad d(x \vee y) \geqq d\left(\bigvee J\right) - \epsilon.$$

Since $x \wedge \bigvee J \neq 0$, there is a u, $0 \neq u \leqq x$, such that $\sigma(u) \leqq J$ (there is an $a \in J$, $x \wedge a \neq 0$; consider a non-zero $\alpha_\epsilon(x \wedge a)$). Similarly there is a $0 \neq v \leqq y$ with $\sigma(v) \leqq J$. We have

$$\partial(\sigma(u)), \partial(\sigma(v))(= d(u), d(v)) < \epsilon \quad \text{and}$$

$$d(x \vee y) \leqq d(u \vee v) + 2\epsilon,$$

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and hence

$$\partial(\sigma(u) \vee \sigma(v)) = d\left(\bigvee \sigma(u) \vee \bigvee \sigma(v)\right) = d(u \vee v)$$
$$\leqq \left(d \bigvee J\right) - 3\epsilon = \partial(J) - 3\epsilon.$$

PROPOSITION 4.5. *Let d be a star diameter. Then ∂ is a diameter if and only if A is totally bounded.*

Proof. Let A be totally bounded. Take an $\epsilon > 0$. We have $u_1, \ldots, u_n$ such that

$$d(u_i) < \epsilon \quad \text{and} \quad \bigvee_{i=1}^{n} u_i = 1.$$

Then

$$\partial(\sigma(U_\epsilon u_i)) = d(U_\epsilon u_i) \leqq 3\epsilon \quad \text{and}$$
$$\bigvee_{i=1}^{n} \sigma(U_\epsilon u_i) = 1.$$

On the other hand, let ∂ be a diameter and let $\epsilon > 0$. Since

$$\bigvee\{J | \partial(J) < \epsilon\} = A,$$

we have $J_1, \ldots, J_n$ such that $\partial(J_i) < \epsilon$, and $x_i \in J_i$ such that

$$\bigvee_{i=1}^{n} x_i = 1.$$

Now

$$d(x_i) \leqq d(\bigvee J_i) = \partial(J_i) < \epsilon.$$

LEMMA 4.6.

$$\sigma(a) = \bigvee_{\epsilon > 0} \sigma(\alpha_\epsilon a).$$

Proof. If $x \in \sigma(a)$ we have $x \leqq \alpha_{2\epsilon}(a)$ for some $\epsilon > 0$. Since $\alpha_{2\epsilon} a \leqq \alpha_\epsilon \alpha_\epsilon a$, $x \in \sigma(\alpha_\epsilon a)$ for this ϵ. Thus,

$$\sigma(a) \subseteq \bigvee_{\epsilon > 0} \sigma(\alpha_\epsilon a).$$

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Since

$$a = \bigvee_{\epsilon>0} \alpha_\epsilon a,$$

we have

$$\sigma(a) \supseteq \bigvee_{\epsilon>0} \sigma(\alpha_\epsilon a).$$

LEMMA 4.7. *For each J and $\epsilon > 0$ we have*

$$U_\epsilon J = \bigvee\{\sigma(x)|d(x) < \epsilon, \sigma(x) \cap J \neq 0\}.$$

Proof. We have $\sigma(\bigvee K) \supseteq J$ and

$$\partial\left(\sigma\left(\bigvee J\right)\right) = d\left(\bigvee \sigma \bigvee J\right) = d\left(\bigvee J\right) = \partial(J).$$

LEMMA 4.8. *We have $U_\epsilon\sigma(a) \leqq \sigma(U_\epsilon a)$ and consequently*

$$\sigma(\alpha_\epsilon a) \leqq \alpha_\epsilon\sigma(a).$$

Proof. By 4.7,

$$\begin{aligned} U_\epsilon\sigma(a) &= \bigvee\{\sigma(x)|d(x) < \epsilon, \sigma(x \wedge a) \neq 0\} \\ &\leqq \bigvee\{\sigma(x)|d(x) < \epsilon, x \wedge a \neq 0\} \leqq \sigma(U_\epsilon a). \end{aligned}$$

PROPOSITION 4.9. *∂ is a compatible pre-diameter on $\mathfrak{R}A$.*

Proof. By 4.6 and 4.8,

$$\sigma(a) = \bigvee_{\epsilon>0} \sigma(\alpha_\epsilon a) \leqq \bigvee_{\epsilon>0} \alpha_\epsilon\sigma(a).$$

4.10. Let $\mathcal{TB}$ be the category of totally bounded metric resp. star-diametric frames with uniform resp. Lipschitz resp. contractive homomorphism, $\mathcal{C}$ the full subcategory generated by compact ones. Further, take $\mathfrak{R}A$ as diametric frame with diameter ∂.

THEOREM. *$\mathcal{C}$ is coreflective in $\mathcal{TB}$, with coreflection functor $\mathfrak{R} : \mathcal{TB} \longrightarrow \mathcal{C}$ and with the join maps $\bigvee : \mathfrak{R}A \longrightarrow A$ as coreflection maps.*

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Proof. Let $f : A \longrightarrow B$ be uniform (resp. Lipschitz, resp. contractive). Let J be in $\Re A$ and let $\partial(J) < \epsilon$. Then $d(\bigvee J) < \epsilon$ and hence there is an a with $d(a) < \delta$ (δ from the definition of uniform mappings; in the Lipschitz case, $\delta = k^{-1}\epsilon$, in the contractive one, $\delta = \epsilon$) such that $f(a) \geqq \bigvee J$. Take an $\eta > 0$ such that

$$\eta < \frac{1}{2}(\delta - d(a)).$$

Thus, $d(U_\eta a) < \delta, \partial(\sigma(U_\eta a)) < \delta$ and $a \in \sigma(U_\eta a)$ so that

$$J \subseteq \{x | x \leqq f(a)\} \subseteq \Re f(\sigma(U_\eta a)).$$

We have

$$\inf \left\{ \partial(J) \,\middle|\, \bigvee J \geqq x \right\} = \inf \left\{ d\left(\bigvee J\right) \,\middle|\, \bigvee K \geqq x \right\} = d(x)$$

and

$$\begin{aligned} \bigvee \Re f(J) &= \bigvee \{x | \exists a \in J, x \leqq f(a)\} \\ &= \bigvee \{f(a) | a \in J\} = f\left(\bigvee J\right). \end{aligned}$$

Finally, if A is compact then $\bigvee : \Re A \longrightarrow A$ is an isomorphism since $\Re A$, being diametrizable, is regular and $\bigvee$ is dense. Thus, the corresponding $\sigma : A \longrightarrow \Re A$ is the frame homomorphism inverse to $\bigvee$. Since

$$\partial(\sigma(a)) = d\left(\bigvee \sigma(a)\right) = d(a),$$

$\bigvee$ is isodiametric.

4.11. In the remainder of this section we will explicitly show the connection between the present construction and that of Section 3 for totally bounded A. Recall that, by regularity, the prime elements (meet irreducibles) of $\Re A$ are exactly the maximal elements.

4.12. Recall 2.7 and put, for a Cauchy point ξ,

$$J(\xi) = \{x | x \notin \mathcal{N}_+(\xi)\}.$$

PROPOSITION. *For totally bounded A, each $J(\xi)$ is a maximal regular ideal.*

Proof. Obviously $0 \in J(\xi)$ and $J(\xi)$ is decreasing. If $x \vee y \in \mathcal{N}_+(\xi)$, we have an $(x_n) \in \xi$ such that $x_k \leqq x \vee y$ for $k \geqq r$. Let y be in $J(\xi)$. Then in particular $x_k \nleqq y$ for all k and hence $x_k \wedge x \neq 0$ for $k \geqq r$. We have $(x_n \wedge x) \sim (x_n)$ and hence $x \in \mathcal{N}_+(\xi)$.

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Thus, $J(\xi)$ is an ideal. We will show that it is regular. Let, on the contrary, there be an $x \in J(\xi)$ such that for each $\epsilon > 0$ there is an end $(x_n(\epsilon))$ and $k(\epsilon)$ such that

$$x_{k(\epsilon)}(\epsilon) \leqq U_\epsilon x.$$

Take a general $(x_n) \in \xi$ and a $k \geqq k(\epsilon)$ such that

$$d(x_k \wedge x_{k(\epsilon/2)}(\epsilon/2)) \leqq \epsilon/2.$$

Then

$$x_k \leqq U_{\epsilon/2} U_{\epsilon/2} x \leqq U_\epsilon x.$$

Thus we have in fact for each $\epsilon > 0$ and each $(x_n) \in \xi$ a $k(\epsilon)$ such that

$$x_{k(\epsilon)} \leqq U_\epsilon x.$$

Take a fixed $(x_n) \in \xi$ and choose $k_1 < k_2 < \cdots$ so that

$$d(x_{k_n}) < 1/n \quad \text{and} \quad x_{k_n} \leqq U_{1/n} x.$$

Thus, there are y_n such that

$$d(y_n) \leqq 1/n, \quad x_{k_n} \wedge y_n \neq 0.$$

Put

$$z_n^0 = y_n \wedge x, \quad z_n = \bigvee_{k \geqq n} z_k^0.$$

Thus, $0 \neq z_n \leqq x, z_1 \geqq z_2 \geqq \cdots$ and by the star property $d(z_n) \leqq 3/n$. Moreover, we have

$$(z_n) \sim (x_{k_n}) \sim (x_n)$$

so that $x \in \mathcal{N}_+(\xi)$ which is a contradiction.

Finally, let $J(\xi) \subseteq J$ and let J be a regular ideal. First, we will prove that

(1) if $y \notin J(\xi)$ and if, for some $\epsilon > 0, x \wedge U_\epsilon y = 0$, then $x \in J(\xi)$.

Indeed, if $x, y \in \mathcal{N}_+(\xi)$, we have $(x_n), (y_n) \in \xi$ such that $x_k \leqq x$ and $y_k \leqq y$ for sufficiently large k. Take k so large that, moreover,

$$d(x_k \vee y_k) < \epsilon.$$

then

$$0 \neq x_k \leqq U_\epsilon y \wedge x.$$

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Now, since $U_{2\epsilon}x \geqq U_\epsilon U_\epsilon x$, we have, for any $\epsilon > 0$,

$$\bigvee\{x | x \wedge U_\epsilon y = 0, d(x) < \epsilon\} \vee U_{2\epsilon}y = 1 \tag{2}$$

for any $y \notin J(\xi)$.

Let $y \in J \backslash J(\xi)$ and let $U_{2\epsilon}y \in J$. By the total boundedness, (1) and (2), we have $x_1, \ldots, x_n \in J(\xi)$ such that

$$\bigvee x_i \vee U_{2\epsilon}y = 1.$$

Thus, $J = A$.

LEMMA 4.13. *Let J be a maximal regular ideal, $x, y \notin J$. Then, for each $\epsilon > 0$,*

$$U_\epsilon x \wedge U_\epsilon y \neq 0.$$

Proof. Since $x, y \notin J$ and hence $\sigma(U_\epsilon x), \sigma(U_\epsilon y) \nsubseteq J$, there are $a, b \in J$ such that

$$a \vee U_\epsilon x = 1 = b \vee U_\epsilon y.$$

Thus,

$$1 = (a \vee U_\epsilon x) \wedge (b \vee U_\epsilon x) = z \vee (U_\epsilon x \wedge U_\epsilon y)$$

with $x \in J$. Thus, $U_\epsilon x \wedge U_\epsilon y$ cannot be in J, let alone be equal to 0.

LEMMA 4.14. *Let J be a maximal regular ideal in a totally bounded A. Then there exists an end (x_n) such that $x_n \notin J$ for each n. Moreover, if (y_n) is another such end then $(x_n) \sim (y_n)$.*

Proof. Since $J \neq A$ and A is totally bounded, we easily see that for each $\epsilon > 0$ there is a $y(\epsilon) \notin J$ such that $d(y(\epsilon)) < \epsilon$. Put

$$y_n = U_{1/n}y(1/n), \quad x_n = \bigvee_{k \geqq n} y_k.$$

By 4.13 and the star property, $d(x_n) < 9/n$.

Now let also $y_n \notin J$. Put

$$z_n = U_{1/n}x_n \wedge U_{1/n}y_n.$$

By 4.13, (z_n) is an end and we have $(x_n) \sim (z_n) \sim (y_n)$.

4.15. By 4.14 we have a uniquely determined $\xi(J)$ associated with J by the existence of an end $(x_n) \in \xi(J)$ such that $x_n \notin J$ for all n.

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PROPOSITION. *The correspondences* $\xi \longmapsto J(\xi)$ *and* $J \longmapsto \xi(J)$ *are mutually inverse maps* $\Psi A \longrightarrow \Sigma\mathfrak{R}A$ *and* $\Sigma\mathfrak{R}A \longrightarrow \Psi A$.

Proof. Take a ξ and an $(x_n) \in \xi$. We have $x_n \in \mathcal{N}_+(\xi)$ so that $\xi(J(\xi)) = \xi$. Take a maximal regular J. Let $x \notin J(\xi(J))$. Hence, there is an (x_n) with $x_n \notin J$ and $x_k \leqq x$ for some k. Thus, $x \notin J$. Consequently, $J \subseteq J(\xi(J))$ and, by the maximality, $J = J(\xi(J))$.

4.16. For any uniform $f : A \longrightarrow B$, we have the square of maps

$$\begin{array}{ccc} \Psi B & \xrightarrow{\Psi f} & \Psi A \\ \lambda_B \downarrow & & \downarrow \lambda_A \\ \Sigma\mathfrak{R}B & \xrightarrow[\Sigma\mathfrak{R}f]{} & \Sigma\mathfrak{R}A \end{array}$$

where the isomorphisms λ_A and λ_B are given by 4.15. We now show this square commutes, that is:

PROPOSITION. *The isomorphisms* $\lambda_A : \Psi A \longrightarrow \Sigma\mathfrak{R}A$ *are natural in* A.

Proof. We view the spectrum functor Σ as given by the respective prime elements. Then, $\Sigma\mathfrak{R}f$ is induced by the right adjoint $g : \mathfrak{R}B \longrightarrow \mathfrak{R}A$ of $\mathfrak{R}f : \mathfrak{R}A \longrightarrow \mathfrak{R}B$, and we have to prove that, for any $\xi \in \Psi B$,

$$g(J(\xi)) = J(\Psi f(\xi)).$$

First note that

$$y \in \mathcal{N}_+(\Psi f(\xi)) \quad \text{if and only if } f(y) \in \mathcal{N}_+(\xi).$$

(Indeed, let $f(y) \in \mathcal{N}_+(\xi)$. Then there is $(x_n) \in \xi$ and k such that $x_k \leqq f(y)$. Now let $(y_n) \in \Psi f(\xi)$ be such that $f(y_n) \geqq x_n$. Thus, for $n \geqq k$, $f(y \wedge y_n) \geqq x_n$ and hence $(y \wedge y_n) \in \Psi f(\xi)$ so that $y \in \mathcal{N}_+(\Psi f(\xi))$. The other implication is immediate.)

Next we have $J \subseteq g(J(\xi))$ if and only if $\mathfrak{R}f(J) \subseteq J(\xi)$ if and only if $f(J) \subseteq J(\xi)$ if and only if

$$\begin{aligned} J &\subseteq f^{-1}(J(\xi)) \\ &= \{y \mid f(y) \in \mathcal{N}_+(\xi)\} \\ &= \{y \mid y \notin \mathcal{N}_+(\Psi f(\xi))\} \\ &= J(\Psi f(\xi)). \end{aligned}$$

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Note. There is another aspect of this which should be mentioned. If A is totally bounded, ΨA is complete and totally bounded, hence a compact metric space, and then $\Omega\Psi A$ belongs to $\mathcal{C}$. Moreover, it is easy to see that $\tau_A : \Omega\Psi A \longrightarrow A$ (3.1) is also the coreflection map, and hence there are unique, natural isomorphisms

$$\Re A \xrightarrow{\sim} \Omega\Psi A$$

which then induce

$$\Psi A \xrightarrow{\sim} \Psi\Omega\Psi A \xrightarrow{\sim} \Sigma\Re A.$$

This provides an alternative approach to the natural isomorphisms given by 4.15 and the present paragraph.

4.17. It is perhaps of some interest to see the mechanism of the isomorphism between $\Re A$ and A in the case of compact A. $\bigvee : \Re A \longrightarrow A$ is onto and dense for any A, and hence an isomorphism if A is compact because a dense homomorphism between compact regular frames is one-one. Regarding spectra, the points $\xi : A \longrightarrow \mathbf{2}$ correspond to the maximal elements $s \in A$, and for such s the corresponding maximal element of $\Re A$ is $\sigma(s)$ (4.3) since $\bigvee \sigma(s) = s$.

4.18. The basic notion of this section, the compact regular frame of regular ideals, can also be considered for uniform frames instead of metric ones. A detailed study of this, a small part of which covers the generalization of the present results to arbitrary totally bounded uniform frames, is presented in [**2**].

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