Compositio Mathematica **132:** 121–135, 2002.

Euler Characteristics of Local Systems on $\mathcal{M}_2$

E. GETZLER
Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, U.S.A. e-mail: getzler@math.nwu.edu

(Received: 17 February 2000; accepted in final form: 11 May 2001)

Abstract. We calculate the Euler characteristics of the local systems $S^k\mathbb{V}\otimes S^\ell\Lambda^2\mathbb{V}$ on the moduli space $\mathcal{M}_2$ of curves of genus 2, where $\mathbb{V}$ is the rank 4 local system $R^1\pi_*\mathbb{C}$.

Mathematics Subject Classifications (2000). 14J15, 20J06.

Key words. Hyperelliptic curves, moduli spaces.

1. Introduction

Let $\pi_g\colon \mathcal{M}_{g,1}\to\mathcal{M}_g$ be the universal curve of genus g, $g\geqslant 2$. The local system $\mathbb{V}=R^1\pi_{g*}\mathbb{C}$ is symplectic of rank $2g$. Given a sequence $(k_1,\dots,k_g)$ of nonnegative integers, there is an associated local system

$$\mathbb{W}(1^{k_1}\dots g^{k_g})=S^{k_1}(\mathbb{V})\otimes S^{k_2}(\Lambda^2\mathbb{V})\otimes\cdots\otimes S^{k_g}(\Lambda^g\mathbb{V})$$

on $\mathcal{M}_g$, with Euler characteristic $e_g(1^{k_1}\dots g^{k_g})=e(\mathcal{M}_g,\mathbb{W}(1^{k_1}\dots g^{k_g}))$, and generating function

$$f_g(u_1,\dots,u_g)=\sum_{k_1,\dots,k_g=0}^{\infty}u_1^{k_1}\dots u_g^{k_g}e_g(1^{k_1}\dots g^{k_g}).$$

In this paper, we calculate $f_2(u_1,u_2)$.

Let $\Gamma_g=\pi_1(\mathcal{M}_g)$ be the genus g mapping class group. There is a homomorphism ρ_g from Γ_g to the symplectic group $\mathrm{Sp}(2g,\mathbb{C})$, obtained by composition of the quotient map to $\mathrm{Sp}(2g,\mathbb{Z})$ with the inclusion $\mathrm{Sp}(2g,\mathbb{Z})\hookrightarrow\mathrm{Sp}(2g,\mathbb{C})$, and the Euler characteristic $e_g(1^{k_1}\dots g^{k_g})$ may also be realized as the Euler characteristic in group cohomology

$$\sum_i(-1)^i\dim H^i(\Gamma_g,\rho^*(S^{k_1}(\mathbb{C}^{2g})\otimes S^{k_2}(\Lambda^2\mathbb{C}^{2g})\otimes\cdots\otimes S^{k_g}(\Lambda^g\mathbb{C}^{2g}))).$$

We may illustrate our method by considering the analogous problem in genus 1; here, we must replace the universal curve by the fibration $\pi_1\colon\mathcal{M}_{1,2}\to\mathcal{M}_{1,1}$. The generating function in this case equals

$$f_1(u)=\sum_{k=0}^{\infty}u^k e(\mathcal{M}_{1,1},S^k\mathbb{V})=\sum_{k=0}^{\infty}u^k e(\mathrm{SL}(2,\mathbb{Z}),S^k\mathbb{C}^2).$$

COMP4710 Pe: 354399 LE/CP/DISK (LATEX)
05-31-2002 14:51 1st Proof

To calculate f_1, we stratify the coarse moduli space $|\mathcal{M}_{1,1}|$ of $\mathcal{M}_{1,1}$, in other words the j-line, according to the automorphism group of the elliptic curve $E(j)$; let $\mathcal{M}_{1,1}(\Gamma)$ be the subvariety of $|\mathcal{M}_{1,1}|$ where $E(j)$ has automorphism group isomorphic to Γ. There are three strata:

$$\mathcal{M}_{1,1}(\mathsf{C}_2) = \mathbb{C} \setminus \{0, 1728\},$$
$$\mathcal{M}_{1,1}(\mathsf{C}_4) = \{j = 1728\},$$
$$\mathcal{M}_{1,1}(\mathsf{C}_6) = \{j = 0\}.$$

Denote the projection from the stack $\mathcal{M}_{1,1}$ to $|\mathcal{M}_{1,1}|$ by μ. If $\mathbb{W}$ is a local system on $\mathcal{M}_{1,1}$, we have

$$e(\mathcal{M}_{1,1}, \mathbb{W}) = e(|\mathcal{M}_{1,1}|, \mu_*\mathbb{W}),$$

since $R^i\mu_*\mathbb{W} = 0$ for $i > 0$. The Euler characteristic of a local system on a stratified space is the sum of the Euler characteristics over the strata:

$$e(|\mathcal{M}_{1,1}|, \mu_*\mathbb{W}) = e(\mathcal{M}_{1,1}(\mathsf{C}_2), \mu_*\mathbb{W}) + e(\mathcal{M}_{1,1}(\mathsf{C}_4), \mu_*\mathbb{W}) + e(\mathcal{M}_{1,1}(\mathsf{C}_6), \mu_*\mathbb{W}).$$

The restriction of the constructible sheaf $\mu_*\mathbb{W}$ to a stratum $\mathcal{M}_{1,1}(\Gamma)$ is a local system, and hence its Euler characteristic on this stratum is equal to the product of the Euler characteristic of $\mathcal{M}_{1,1}(\Gamma)$ and the rank of $\mu_*\mathbb{W}$ restricted to $\mathcal{M}_{1,1}(\Gamma)$. (It is the failure of the analogous property for stacks which necessitates the descent to the coarse moduli space $|\mathcal{M}_{1,1}|$.) We conclude that

$$\begin{aligned} f_1(u) &= \sum_{k=0}^{\infty} u^k \{e(\mathcal{M}_{1,1}(\mathsf{C}_2), S^k\mathbb{V}) + e(\mathcal{M}_{1,1}(\mathsf{C}_4), S^k\mathbb{V}) + e(\mathcal{M}_{1,1}(\mathsf{C}_6), S^k\mathbb{V})\} \\ &= \sum_{k=0}^{\infty} u^k \{e(\mathcal{M}_{1,1}(\mathsf{C}_2)) \dim(S^k\mathbb{C}^2)^{\mathsf{C}_2} + e(\mathcal{M}_{1,1}(\mathsf{C}_4)) \dim(S^k\mathbb{C}^2)^{\mathsf{C}_4} + \\ &\qquad + e(\mathcal{M}_{1,1}(\mathsf{C}_6)) \dim(S^k\mathbb{C}^2)^{\mathsf{C}_6}\}. \end{aligned}$$

The cyclic group C_n is conjugate to the subgroup $\left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \middle| z^n = 1 \right\}$ of $SL(2, \mathbb{C})$; it follows that

$$\sum_{k=0}^{\infty} u^k e(\mathsf{C}_n, S^k\mathbb{C}^2) = \sum_{k=0}^{\infty} u^k \dim(S^k\mathbb{C}^2)^{\mathsf{C}_n} = \frac{1+u^n}{(1-u^2)(1-u^n)}.$$

It follows that

$$\begin{aligned} f_1(u) &= -\frac{1+u^2}{(1-u^2)^2} + \frac{1+u^4}{(1-u^2)(1-u^4)} + \frac{1+u^6}{(1-u^2)(1-u^6)} \\ &= \frac{1-u^2-2u^4-u^6+u^8}{(1-u^4)(1-u^6)}. \end{aligned}$$

Our calculation in genus 2 proceeds analogously: we use Bolza's stratification of the coarse moduli space $|\mathcal{M}_2|$ by the automorphism group of the corresponding curve [2]. Denote the stratum associated to the automorphism group Γ by $\mathcal{M}_2(\Gamma)$. The contribution of each stratum to $e(\mathcal{M}_2, \mathbb{W}(1^k 2^\ell))$ must be calculated using the character theory of Γ; since Γ is a finite subgroup of $SL(2, \mathbb{C})$, the McKay correspondence allows this to be done in terms of the associated Dynkin diagram.

The only tricky point is the calculation of the Euler characteristics of $\mathcal{M}_2(\Gamma)$. The hardest case is the affine surface $\mathcal{M}_2(\mathsf{C}_4)$; we prove in Section 3 that $e(\mathcal{M}_2(\mathsf{C}_4)) = 3$.

The original motivation for this work was the desire to calculate the $\mathbb{S}_n$-equivariant Euler characteristics of the moduli spaces $\mathcal{M}_{2,n}$. We explain how this may be done in Section 5.

Throughout this paper, $\varepsilon_n = \exp(2\pi i/n)$. All varieties we consider are defined over the field of complex numbers $\mathbb{C}$.

2. Finite Subgroups of SL(2, ℂ) and the McKay Correspondence

Given positive integers $p \geqslant q \geqslant r \geqslant 2$, let $\langle p, q, r\rangle$ be the group with presentation

$$\langle S, T, U \mid S^p = T^q = U^r = STU\rangle.$$

The element STU is a central involution, which we denote by $-I$.

If $p^{-1} + q^{-1} + r^{-1} > 1$, the group $\langle p, q, r\rangle$ is finite, and its order equals

$$4/(p^{-1} + q^{-1} + r^{-1} - 1).$$

Table I lists all of these groups.

According to Klein [7], the non-Abelian finite subgroups Γ of $\mathrm{SL}(2, \mathbb{C})$ are isomorphic to the finite groups $\langle p, q, r\rangle$; any such subgroup of $\mathrm{SL}(2, \mathbb{C})$ is conjugate to the subgroup generated by the element S of $\mathrm{SL}(2, \mathbb{C})$ listed in the table, and the element $U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The Abelian subgroups of $\mathrm{SL}(2, \mathbb{C})$ are all cyclic, and any such subgroup of $\mathrm{SL}(2, \mathbb{C})$ is conjugate to the subgroup generated by the element $T = \begin{pmatrix} \varepsilon_n & 0 \\ 0 & \varepsilon_n^{-1} \end{pmatrix}$ of $\mathrm{SL}(2, \mathbb{C})$

We refer to the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ as the Kleinian groups. If Γ is such a group, let V be the two-dimensional fundamental representation of Γ induced by the embedding of Γ in $\mathrm{SL}(2, \mathbb{C})$, and let V_k be the kth symmetric power $S^k\mathsf{V}$ of V (isomorphic to the space of binary forms of degree k). If Γ is a Kleinian group containing $-I$ and W is an irreducible representation of Γ, we call W even if $-I$ acts by $+1$ and odd if its acts by -1; by Schur's lemma, these are the only possibilities. For example, the fundamental representation V is odd.

Table I.

(p, q, r)	$\langle p, q, r\rangle$	Order	Name	S
$(n, 2, 2)$	Q_{4n}	$4n$	quaternionic, $n \geq 2$	$\begin{pmatrix} \varepsilon_{2n} & 0 \\ 0 & \varepsilon_{2n}^{-1} \end{pmatrix}$
$(3, 3, 2)$	T	24	binary tetrahedral	$\frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon_8^{-1} & \varepsilon_8^3 \\ \varepsilon_8 & \varepsilon_8 \end{pmatrix}$
$(4, 3, 2)$	O	48	binary octahedral	$-\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \varepsilon_8 \\ \varepsilon_8^3 & 1 \end{pmatrix}$
$(5, 3, 2)$	I	120	binary icosahedral	$\frac{1}{\sqrt{5}}\begin{pmatrix} \varepsilon_5^4 - 1 & \varepsilon_5^3 - \varepsilon_5 \\ \varepsilon_5^4 - \varepsilon_5^2 & \varepsilon_5 - 1 \end{pmatrix}$

https://doi.org/10.1023/A:1015826400148 Published online by Cambridge University Press

There is a beautiful relationship between the character theory of Kleinian groups and Dynkin diagrams, known as the McKay correspondence. If Γ is a Kleinian group, consider the graph with one vertex w_i for each isomorphism class $\{\mathsf{W}_i \mid 1 \leqslant i \leqslant r\}$ of irreducible representations of Γ, and n_{ij} edges between vertices w_i and w_j, where the positive integers n_{ij} are the Clebsch–Gordon coefficients

$$n_{ij} = \dim_{\mathbb{C}} \operatorname{Hom}_{\Gamma}(\mathsf{V} \otimes \mathsf{W}_i, \mathsf{W}_j). \tag{1}$$

The resulting graph is the Dynkin diagram of an irreducible simply-laced affine Lie algebra; equivalently, the graph is connected, the numbers n_{ij} are equal to 0 or 1, and the Cartan matrix defined by $A_{ij} = 2\delta_{ij} - n_{ij}$ is positive semi-definite, with one-dimensional null-space. In fact, the null-space is spanned by the vector whose ith component is the dimension of W_i, since by (1),

$$\sum_{j=1}^{r} (2\delta_{ij} - n_{ij}) \dim(\mathsf{W}_j) = 0.$$

EXAMPLES OF THE McKAY CORRESPONDENCE

Cyclic groups. If $\Gamma = \mathsf{C}_n$ is a cyclic group, let χ be the primitive character characterized by $\chi(T) = \varepsilon_n$. The irreducible representations of the cyclic group C_n are the powers $\{\chi^i \mid 0 \leqslant i < n\}$ of χ. Since $\mathsf{V} \otimes \chi^i \cong \chi^{i+1} \oplus \chi^{i-1}$, the associated graph is a circuit with n vertices: the Dynkin diagram $\hat{A}_{n-1}$.

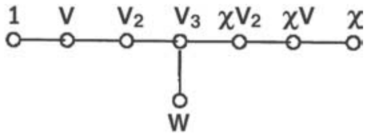
Quaternionic Groups. The McKay correspondence associates to the quaternionic group Q_{4n} the Dynkin diagram of $\hat{D}_{n-1}$:

The irreducible representations $\mathsf{V}_{[i]}$, $1 \leqslant i \leqslant n-1$, are two-dimensional, and $\mathsf{V} \cong \mathsf{V}_{[1]}$. The group Q_{4n} has four one-dimensional characters, as follows:

ρ	$\rho(S)$	$\rho(T)$	$\rho(U)$
1	1	1	1
χ_0	1	-1	-1
χ_+	-1	i^n	$-i^n$
χ_-	-1	$-i^n$	i^n

Note that $\chi_+\mathsf{V}_{[i]} \cong \chi_-\mathsf{V}_{[i]} \cong \mathsf{V}_{[n-i]}$ and that $\chi_0\mathsf{V}_{[i]} \cong \mathsf{V}_{[i]}$.

The binary octahedral group. The cases T, O and I of the McKay correspondence correspond respectively to the affine Dynkin diagrams $\hat{E}_6$, $\hat{E}_7$ and $\hat{E}_8$. Of these, we only need the case of O in this paper; its Dynkin diagram is as follows:



The unique nontrivial one-dimensional character χ is characterized by $\chi(S) = \chi(U) = -1$. Note that $\chi\mathsf{V}_3 \cong \mathsf{V}_3$ and that $\chi\mathsf{W} \cong \mathsf{W}$.

3. The Automorphism Group of a Hyperelliptic Curve

Denote by $\mathcal{I}_{2g+2}$ the affine variety of polynomials of degree $2g+2$ with non-vanishing discriminant. We identify a polynomial $f \in \mathcal{I}_{2g+2}$ with the binary form $y^{2g+2}f(x/y)$. If $f \in \mathcal{I}_{2g+2}$, consider the affine varieties $V(z^2 - f(x)) \subset \operatorname{Spec} \mathbb{C}[x, z]$ and $V(\tilde{z}^2 - \tilde{x}^{2g+2}f(1/\tilde{x})) \subset \operatorname{Spec} \mathbb{C}[\tilde{x}, \tilde{z}]$. The hyperelliptic curve C_f associated to f is the smooth curve defined by gluing $V(z^2 - f(x))$ and $V(\tilde{z}^2 - \tilde{x}^{2g+2}f(1/\tilde{x}))$ by the identification $(\tilde{x}, \tilde{z}) = (1/x, z/x^{g+1})$.

The involution $\sigma\colon C_f \to C_f$ defined by $\sigma(x, z) = (x, -z)$ is called the hyperelliptic involution of C_f; it acts on $H^0(C_f, \Omega)$ by $-I$ and is in the centre of the automorphism group $\operatorname{Aut}(C_f)$ of C_f.

LEMMA 1. *The curve C_f has genus g, and $H^0(C_f, \Omega)$ has basis $\omega_i = x^i\,\mathrm{d}x/z$, $0 \leqslant i < g$.*

Proof. The fixed points of the hyperelliptic involution are the $2g+2$ Weierstrass points of C_f. The projection $(x, z) \mapsto x$ exhibits C_f as a double cover of $\mathbb{P}^1$, ramified at the roots of f; thus, its genus equals g.

On the affine variety $V(z^2 - f(x))$, the differentials $2z\,\mathrm{d}z$ and $f'(x)\,\mathrm{d}x$ are equal; thus $\omega_i = 2\,x^i\,\mathrm{d}z/f'(x)$. Since the functions z and $f'(x)$ have no common zeroes (the polynomial $f(x)$ has no multiple roots), we conclude that the differentials ω_i are regular on $V(z^2 - f(x))$, so long as $i \geqslant 0$.

Let $\tilde{f}(x) = x^{2g+2}f(1/x)$. On the affine variety

$$V(\tilde{z}^2 - \tilde{x}^{2g+2}f(1/\tilde{x})) = V(\tilde{z}^2 - \tilde{f}(\tilde{x})),$$

the differential forms $2\tilde{z}\,\mathrm{d}\tilde{z}$ and $\tilde{f}'(\tilde{x})\,\mathrm{d}\tilde{x}$ are equal. Since

$$\omega_i = -\tilde{x}^{g-i-1}\,\mathrm{d}\tilde{x}/\tilde{z} = -2\,\tilde{x}^{g-i-1}\,\mathrm{d}\tilde{z}/\tilde{f}'(\tilde{x}),$$

the differentials ω_i are regular on $V(\tilde{z}^2 - \tilde{x}^{2g+2}f(1/\tilde{x}))$ so long as $i < g$.

https://doi.org/10.1023/A:1015826400148 Published online by Cambridge University Press

We have exhibited g linearly independent algebraic one-forms on C_f; since C_f has genus g, they form a basis of $H^0(C_f, \Omega)$. $\square$

The group $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^\times$ acts by rational transformations on $\operatorname{Spec} \mathbb{C}[x,z]$ by the formula

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, u\right) \cdot (x,z) = \left(\frac{ax+b}{cx+d}, \frac{uz}{(cx+d)^{g+1}}\right).$$

If $f \in \mathcal{I}_{2g+2}$, the subgroup of elements of $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^\times$ which preserve the subvariety $V(z^2 - f(x))$ is a group of the form

$$\Gamma(\rho) = \{(\gamma, u) \mid \gamma \in \Gamma, u^2 = \rho(\gamma)\} \subset \mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^\times,$$

where $\Gamma = \Gamma_f$ is the finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ consisting of elements whose action on $\mathbb{P}^1$ preserves the set of roots of f, and $\rho = \rho_f$ is an even character of Γ_f. We have the short exact sequence

$$0 \to \langle(-I, (-1)^{g+1})\rangle \to \Gamma(\rho) \to \mathrm{Aut}(C_f) \to 0.$$

Given a Γ-module W and an integer n, let $\mathsf{W}(n)$ be the $\Gamma(\rho)$-module with underlying vector space W on which the element $(\gamma, u) \in \Gamma(\rho)$ acts by $(\gamma, u) \cdot w = u^n(\gamma \cdot w)$. We have the isomorphisms $\mathsf{W}(n+2) \cong \rho \otimes \mathsf{W}(n)$ and $\mathsf{W}(n)^\vee \cong \mathsf{W}(-n)$. With this notation, the irreducible representations of $\mathrm{Aut}(C_f)$ have the form $\mathsf{W}(n)$, where W is an isomorphism class of irreducible representations of Γ_f, and $n \equiv g (\mathrm{mod}\ 2)$ (respectively $n \equiv g+1 (\mathrm{mod}\ 2)$) if W is even (respectively, odd).

PROPOSITION 2. *As a representation of* $\mathrm{Aut}(C_f)$, $H^0(C_f, \Omega) \cong \mathsf{V}_{g-1}(-1)$.

Proof. Under the action of $(\gamma, u) \in \Gamma(\rho)$, ω_i transforms into

$$\left(\frac{ax+b}{cx+d}\right)^i \left(\frac{uz}{(cx+d)^{g+1}}\right)^{-1} \frac{\mathrm{d}x}{(cx+d)^2} = u^{-1}(ax+b)^i(cx+d)^{g-i-1}\, \mathrm{d}x/z.$$

Expanding the right-hand side in terms of the basis ω_i, we recover the action of $\mathrm{Aut}(C_f)$ on $\mathsf{V}_{g-1}(-1)$. $\square$

COROLLARY 3. *As a representation of* $\mathrm{Aut}(C_f)$, $H^1(C_f, \mathbb{C}) \cong \mathsf{V}_{g-1}(1) \oplus \mathsf{V}_{g-1}(-1)$.

Proof. $H^1(C_f, \mathbb{C}) \cong H^0(C_f, \Omega) \oplus H^1(C_f, \mathcal{O}) \cong H^0(C_f, \Omega) \oplus H^0(C_f, \Omega)^\vee$ $\square$

Given a Kleinian group Γ and a character ρ, let $\mathcal{I}_{2g+2}(\Gamma, \rho)$ be the subvariety of $\mathcal{I}_{2g+2}$ consisting of polynomials such that the pair (Γ_f, ρ_f) is conjugate to (Γ, ρ).

The quotient $\mathcal{H}_g$ of $\mathcal{I}_{2g+2}$ by the group $(\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^\times)/\langle(-I, (-1)^{g+1})\rangle$ is the moduli space of hyperelliptic curves of genus g. It is a complex orbifold of dimension $2g-1$, stratified by the images $\mathcal{H}_g(\Gamma, \rho)$ of the subvarieties $\mathcal{I}_{2g+2}(\Gamma, \rho)$.

It carries a local system $\mathbb{V}$ whose fibre at $[f]$ is isomorphic to $H^1(C_f, \mathbb{C})$; by Corollary 3, this local system has underlying vector bundle

$$\mathcal{I}_{2g+2} \times_{(\mathrm{SL}(2,\mathbb{C})\times\mathbb{C}^\times)/\langle(-I,(-1)^{g+1})\rangle} [\mathsf{V}_{g-1}(1) \oplus \mathsf{V}_{g-1}(-1)].$$

The same argument which was used in the introduction to calculate $e(\mathcal{M}_{1,1}, S^k\mathbb{V})$ proves the following result. This proposition will be used in Section 4 to calculate the Euler characteristics $e_2(1^k 2^\ell)$.

PROPOSITION 4.

$$e(\mathcal{H}_g, \mathbb{W}(1^{k_1} \cdots g^{k_g})) = \sum_{(\Gamma,\rho)} e(\mathcal{H}_g(\Gamma, \rho)) \cdot \dim\left(\bigotimes_{i=1}^{g} S^{k_i}(\Lambda^i(\mathsf{V}_{g-1}(1) \oplus \mathsf{V}_{g-1}(-1))) \right)^{\Gamma(\rho)}.$$

4. The Stratification of $\mathcal{H}_2$

We now specialize to genus 2; this case is special, in that $\mathcal{H}_2$ is identical with the moduli space $\mathcal{M}_2$ of smooth projective curves of genus 2.

Bolza [2] has shown that the stratification $\mathcal{H}_2 = \coprod \mathcal{H}_2(\Gamma, \rho)$ has seven strata. In Figure 1, we give a diagram showing these strata, as well as two more pieces of data which we will need: the Euler characteristics $e(\mathcal{H}_2(\Gamma, \rho))$ of the strata, and a normal form for polynomials in $\mathcal{I}_{2g+2}(\Gamma, \rho)$. Since no two distinct strata have the same isotropy group Γ, we may, without ambiguity, denote the stratum $\mathcal{H}_2(\Gamma, \rho)$ by $\mathcal{H}_2(\Gamma)$.

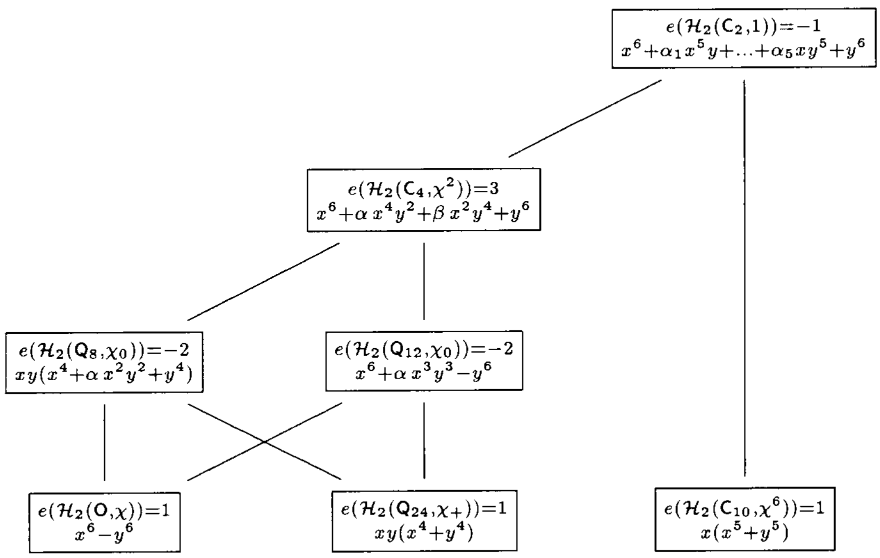


Figure 1. The Bolza stratification of $\mathcal{H}_2$.

https://doi.org/10.1023/A:1015826400148 Published online by Cambridge University Press

In this section, we calculate the Euler characteristics of these strata. Since $\mathcal{H}_2(\mathsf{C}_{10})$, $\mathcal{H}_2(\mathsf{Q}_{24})$ and $\mathcal{H}_2(\mathsf{O})$ each consists of precisely one point, it is clear that they have Euler characteristic 1. Since $\mathcal{H}_2$ is contractible, it also has Euler characteristic 1; thus the Euler characteristics of all of the strata add up to 1. It remains to calculate the Euler characteristics $e(\mathcal{H}_2(\mathsf{C}_4))$, $e(\mathcal{H}_2(\mathsf{Q}_8))$ and $e(\mathcal{H}_2(\mathsf{Q}_{12}))$. Of these, the first is the most difficult; in calculating it, we use Clebsch's classification of the covariants of binary sextics.

The Euler characteristic of $\mathcal{H}_2(\mathsf{C}_4)$. If f and g are binary forms of degree k and ℓ respectively, define their pth Ueberschiebung $(f,g)_p$ by the formula

$$s(f,g)_p = \frac{(k+\ell-p)!}{(k+\ell)!}\left(\frac{\partial^2}{\partial_x\partial_\eta} - \frac{\partial^2}{\partial_y\partial_\xi}\right)^p f(x,y)g(\xi,\eta)\Big|_{x=\xi,y=\eta}.$$

The binary form $(f,g)_p$ is a joint covariant of f and g.

For a proof of the following result, see §130 of [5].

LEMMA 5. *If l and m are a pair of quadratic forms, let $C(l,m)$ be the joint invariant*

$$C(l,m) = \begin{vmatrix} (l,l)_2 & (l,m)_2 \\ (m,l)_2 & (m,m)_2 \end{vmatrix}.$$

The quadratic forms l and m may be simultaneously diagonalized (i.e. there are coordinates ξ and η such that $l = a\xi^2 + b\eta^2$ and $m = c\xi^2 + d\eta^2$) if and only if $C(l,m) \neq 0$.

Given a binary sextic f, define a quartic covariant $i = (f,f)_4$ of degree 2, and quadratic covariants

$$l = (i,f)_4, \quad m = (i,l)_2, \quad n = (i,m)_2$$

of degrees 3, 5 and 7 respectively. Let $R = -2\,((l,m)_1, n)_2$. Using his symbol calculus for covariants of binary forms, Clebsch has shown ([3], § 113) that

(1) if $R \neq 0$, f is a cubic polynomial in the quadratic forms l, m and n;
(2) if $R = 0$ and $C(l,m) \neq 0$, f is a cubic polynomial in the quadratic forms l and m.

In each case, the coefficients of the representation are explicit rational invariants of f. For an exposition of the proofs, see §29 of [5].

We are interested in the second case above. By the condition $C(l,m) \neq 0$ and Lemma 5, we see that there are coordinates ξ and η such that $f \in \mathbb{C}[\xi^2, \eta^2]$. Furthermore, provided the discriminant of f is nonzero, we can rescale the coordinates ξ and η in such a way that the coefficients of ξ^6 and η^6 equal 1.

In conclusion, a binary sextic f with nonvanishing discriminant such that $R = 0$ and $C(l,m) \neq 0$ is equivalent to a sextic in the normal form

$$f(x, y) = x^6 + \alpha x^4 y^2 + \beta x^2 y^4 + y^6. \tag{2}$$

By §I of Bolza [2], these are precisely the sextics whose image in $\mathcal{H}_2$ lies in $\mathcal{H}_2(\mathsf{C}_4)$.

Let $X = \alpha^3 + \beta^3$ and $Y = \alpha\beta$. From these functions, we may recover the normal form f of (2) thus, they give global coordinates on the stratum $\mathcal{H}_2(\mathsf{C}_4)$. To see this, observe that

$$(\alpha^3 - \beta^3)^2 = X^2 - 4Y^3.$$

Thus, we may recover from X and Y the coefficients α and β, up to the action of the dihedral group generated by the transformations $(\alpha, \beta) \mapsto (e^{2\pi i/3}\alpha, e^{-2\pi i/3}\beta)$ sand $(\alpha, \beta) \mapsto (\beta, \alpha)$. However, this dihedral group is precisely the symmetry group of the normal form f.

The discriminant of the normal form f of (2) equals

$$-64(4X - Y^2 - 18Y + 27)^2,$$

while

$$C(l, m) = -\frac{2^{10}}{3^{10}5^{12}}(4X - Y^2 + 110Y - 1125)^2(X^2 - 4Y^3).$$

Denoting the divisors $(4X - Y^2 - 18Y + 27)$, $(4X - Y^2 + 110Y - 1125)$ and $(X^2 - 4Y^3)$ in $\mathbb{C}^2$ by Δ_0, Δ_1 and Δ_2, we conclude that

$$\mathcal{H}_2(\mathsf{C}_4) \cong \mathbb{C}^2 \backslash \Delta_0 \cup \Delta_1 \cup \Delta_2.$$

In fact, Bolza shows that Δ_1 corresponds to the stratum $\mathcal{H}_2(\mathsf{Q}_8)$, while Δ_2 corresponds to the stratum $\mathcal{H}_2(\mathsf{Q}_{12})$.

It is now quite easy to calculate $e(\mathcal{H}_2(\mathsf{C}_4))$. Since Δ_0 and Δ_1 are graphs in $\mathbb{C}^2$, they have Euler characteristic 1. The projection $(X, Y) \mapsto Y$ displays Δ_2 as a double cover of $\mathbb{C}$ ramified at 0, showing that it too has Euler characteristic 1. As for the intersections of the three divisors, we have

$$\Delta_0 \cap \Delta_1 = \{(54, 9)\}, \quad \Delta_0 \cap \Delta_2 = \{(54, 9), (-2, 1)\},$$
$$\Delta_1 \cap \Delta_2 = \{(54, 9), (-250, 25), (6750, 225)\}.$$

Combining these results, we conclude that $e(\mathcal{H}_2(\mathsf{C}_4)) = 3$.

The Euler characteristic of $\mathcal{H}_2(\mathsf{Q}_8)$. Two binary sextics

$$f(x, y) = xy(x^4 + \alpha_i x^2 y^2 + y^4), \quad i = 1, 2,$$

are equivalent if and only if $\alpha_1^2 = \alpha_2^2$; thus, we may parametrize the stratum $\mathcal{H}_2(\mathsf{Q}_8)$ by α^2. Three values of α^2 are excluded:

https://doi.org/10.1023/A:1015826400148 Published online by Cambridge University Press

(1) when $\alpha^2 = 4$, the sextic has vanishing discriminant;
(2) when $\alpha^2 = 0$, the sextic has automorphism group O;
(3) when $\alpha^2 = 100/9$, the sextic has automorphism group Q_{24}.

It follows that the parameter α^2 identifies the stratum $\mathcal{H}_2(\mathsf{Q}_8)$ with $\mathbb{C}\backslash\{0, 4, \frac{100}{9}\}$, and hence, that $e(\mathcal{H}_2(\mathsf{Q}_8)) = -2$.

The Euler characteristic of $\mathcal{H}_2(\mathsf{Q}_{12})$. Two binary sextics

$$f(x, y) = x^6 + \alpha_i x^3 y^3 - y^6, \quad i = 1, 2,$$

are equivalent if and only if $\alpha_1^2 = \alpha_2^2$; thus, we may parametrize the stratum $\mathcal{H}_2(\mathsf{Q}_{12})$ by α^2. Three values are excluded:

(1) when $\alpha^2 = -4$, the sextic has vanishing discriminant;
(2) when $\alpha^2 = 0$, the sextic has automorphism group Q_{24};
(3) when $\alpha^2 = 50$, the sextic has automorphism group O.

It follows that the parameter α^2 identifies the stratum $\mathcal{H}_2(\mathsf{Q}_{12})$ with $\mathbb{C}\backslash\{0, -4, 50\}$, and hence, that $e(\mathcal{H}_2(\mathsf{Q}_{12})) = -2$.

5. The calculation of $f_2(u,v)$

We have seen in Proposition 4 that

$$f_2(u, v) = \sum_{(\Gamma,\rho)} e(\mathcal{H}_2(\Gamma, \rho)) \cdot \sum_{k,\ell=0}^{\infty} u^k v^\ell \dim(S^k(\mathbb{V}) \otimes S^\ell(\wedge^2\mathbb{V}))^{\Gamma(\rho)}.$$

where by Corollary 3, $\mathbb{V} = \mathsf{V}(1) \oplus \mathsf{V}(-1)$ and $\Lambda^2\mathbb{V} = \mathsf{V}_2 \oplus \rho \oplus 1 \oplus \rho^{-1}$.

PROPOSITION 6.

$$\sum_{k,\ell=0}^{\infty} u^k v^\ell \dim\left(S^k\mathbb{V} \otimes S^\ell\Lambda^2\mathbb{V}\right)^{\Gamma(\rho)}$$
$$= \dim\left(\frac{1 + u^2(\mathsf{V}^2 + \rho + \rho^{-1}) + u^4}{(1 - u^2(\mathsf{V}^2 - 2)\rho + u^4\rho^2)(1 - u^2(\mathsf{V}^2 - 2)\rho^{-1} + u^4\rho^{-2})} \times \right.$$
$$\left. \times \frac{1}{(1 - v(\mathsf{V}^2 - 2) + v^2)(1 - v\rho)(1 - v)^2(1 - v\rho^{-1})}\right)^{\Gamma(\rho)}.$$

Proof. All of the following calculations are performed in the ring $R(\Gamma)(u, v)$. Using the formulas $\sigma_t(A \oplus B) = \sigma_t(A) \cdot \sigma_t(B)$ and $\sigma_t(A) = \lambda_{-t}(A)^{-1}$, we see that

$$\begin{aligned}
\sum_{k,\ell=0}^{\infty} & u^k v^\ell S^k\mathbb{V}\otimes S^\ell\Lambda^2\mathbb{V} = \sigma_u(\mathbb{V})\cdot\sigma_v(\Lambda^2\mathbb{V}) \\
&= \sigma_u(\mathsf{V}(1))\cdot\sigma_u(\mathsf{V}(-1))\cdot\sigma_v(\mathsf{V}_2)\cdot\sigma_v(\rho)\cdot\sigma_v(1)\cdot\sigma_v(\rho^{-1}) \\
&= \lambda_{-u}(\mathsf{V}(1))^{-1}\cdot\lambda_{-u}(\mathsf{V}(-1))^{-1}\cdot\lambda_{-v}(\mathsf{V}_2)^{-1}\cdot\lambda_{-v}(\rho)^{-1}\cdot \\
&\quad\cdot\lambda_{-v}(1)^{-1}\cdot\lambda_{-v}(\rho^{-1})^{-1} \\
&= (1-u\mathsf{V}(1)+u^2\rho)^{-1}\cdot(1-u\mathsf{V}(-1)+u^2\rho^{-1})^{-1}\cdot \\
&\quad\cdot(1-v(\mathsf{V}^2-1)+v^2(\mathsf{V}^2-1)-v^3)^{-1}\cdot(1-v\rho)^{-1}. \\
&\quad\cdot(1-v)^{-1}\cdot(1-v\rho^{-1})^{-1}.
\end{aligned}$$

The third factor of the denominator may be simplified by the factorization

$$(1-v(\mathsf{V}^2-1)+v^2(\mathsf{V}^2-1)-v^3)^{-1} = (1-v(\mathsf{V}^2-2)+v^2)^{-1}(1-v)^{-1}.$$

To simplify the factors involving the variable u, we use the formulas

$$\begin{aligned}
1-u\mathsf{V}(1)+u^2\rho &= \frac{\big(1-u\mathsf{V}(1)+u^2\rho\big)\big(1+u\mathsf{V}(1)+u^2\rho\big)}{1+u\mathsf{V}(1)+u^2\rho} \\
&= \frac{1-u^2(\mathsf{V}^2-2)\rho+u^4\rho^2}{1+u\mathsf{V}(1)+u^2\rho}
\end{aligned}$$

and, similarly,

$$1-u\mathsf{V}(-1)+u^2\rho^{-1} = \frac{1-u^2(\mathsf{V}^2-2)\rho^{-1}+u^4\rho^{-2}}{1+u\mathsf{V}(-1)+u^2\rho^{-1}}.$$

Hence,

$$\begin{aligned}
&(1-u\mathsf{V}(1)+u^2\rho)^{-1}\cdot(1-u\mathsf{V}(-1)+u^2\rho^{-1})^{-1} \\
&\quad= \frac{(1+u\mathsf{V}(1)+u^2\rho)(1+u\mathsf{V}(-1)+u^2\rho^{-1})}{(1-u^2(\mathsf{V}^2-2)\rho+u^4\rho^2)(1-u^2(\mathsf{V}^2-2)\rho^{-1}+u^4\rho^{-2})} \\
&\quad= \frac{(1+u^2(\mathsf{V}^2+\rho+\rho^{-1})+u^4)+(u+u^3)(\mathsf{V}(1)\oplus\mathsf{V}(-1))}{(1-u^2(\mathsf{V}^2-2)\rho+u^4\rho^2)(1-u^2(\mathsf{V}^2-2)\rho^{-1}+u^4\rho^{-2})}.
\end{aligned}$$

No representation of $\Gamma(\rho)$ of the form $\mathsf{W}(n)$ with n odd can have a nontrivial space of invariants; we conclude that we may discard the terms which are odd in u before taking the space of invariants under the group $\Gamma(\rho)$. □

To apply this formula, we substitute for V the matrix (n_{ij}) with entries 0 and 1 associated to the Dynkin diagram corresponding to Γ in the McKay correspondence, and for ρ the permutation matrix (p_{ij}) with entries

$$p_{ij} = \dim_{\mathbb{C}} \mathrm{Hom}_\Gamma(\rho\otimes\mathsf{W}_i, \mathsf{W}_j).$$

https://doi.org/10.1023/A:1015826400148 Published online by Cambridge University Press

In this way, we obtain a matrix with entries in $\mathbb{Z}(u, v)$; the desired power series is the diagonal entry corresponding to the trivial representation. We list the results for the seven cases of (Γ, ρ) in Table II, expressed as a sum of terms of the form $r(u, v)/s(u)t(v)$. We have also listed the sum of the contributions for all strata other than $\mathcal{H}_2(\mathsf{C}_{10})$; to obtain the formula for $f_2(u, v)$, we simply add this total to the contribution for C_{10}.

We may use our formula for $f_2(u, v)$ to calculate the Euler characteristics $e_{a,b}$ of irreducible local systems $\mathbb{V}_{\langle a,b\rangle}$ on $\mathcal{M}_2$. Let $\chi_{a,b}$ denote the character of $\mathbb{V}_{a,b}$. If g is an element of $\mathrm{Sp}(4, \mathbb{C})$ with eigenvalues $\{x, y, x^{-1}, y^{-1}\}$, the Weyl character formula says that

$$\chi_{a,b}(g) = \frac{\begin{vmatrix} x^{a+2} - x^{-a-2} & x^{b+1} - x^{-b-1} \\ y^{a+2} - y^{-a-2} & y^{b+1} - y^{-b-1} \end{vmatrix}}{\begin{vmatrix} x^2 - x^{-2} & x - x^{-1} \\ y^2 - y^{-2} & y - y^{-1} \end{vmatrix}}.$$

Multiplying by $u^{a-b}v^b$ and summing over a and b, we see that

$$\sum_{k,\ell \geqslant 0} u^k v^\ell \mathbb{V}_{\langle k+\ell,\ell\rangle} = (1-v)^2((1+v)(1+u^2v) - uv\,\mathbb{V}) \sum_{k,\ell \geqslant 0} u^k v^\ell\, S^k(\mathbb{V}) \otimes S^\ell(\Lambda^2\mathbb{V}).$$

6. The Equivariant Euler Characteristic of $\mathcal{M}_{2,n}$

The following Euler characteristics are immediate from Table II:

k	0	2	0	4	2	0	6	4	2	0
ℓ	0	0	1	0	1	2	0	1	2	3
$e_2(1^k2^\ell)$	1	0	0	0	−1	0	−1	−1	−1	−3

One might imagine from these data that all of the Euler characteristics $e_2(1^k2^\ell)$ are negative: however, $e_2(1^{10}) = 1$.

Using the Leray–Serre spectral sequence, the $\mathbb{S}_n$-equivariant Euler characteristics of the moduli spaces $\mathcal{M}_{g,n}$ may be expressed in terms of the Euler characteristics $e_g(1^{k_1} \dots g^{k_g})$: by [4], we have

$$\sum_{n=0}^{\infty} e_{\mathbb{S}_n}(\mathcal{M}_{g,n}) = e\Big((1+p_1)^2 \prod_{k=1}^{\infty} (1+p_k)^{-\frac{1}{k}\sum_{d|k} d!\,\mu(k/d)\mathrm{ch}_d(\mathbb{V})}\Big). \quad (3)$$

In this formula, we identify the virtual representation ring $R(\mathbb{S}_n)$ of the symmetric group $\mathbb{S}_n$ with the space of symmetric functions of degree n, which, when tensored with $\mathbb{Q}$, is in turn isomorphic to the algebra of polynomials of the power sums p_k.

Table II. Contributions of the strata $\mathcal{H}_2(\Gamma, \rho)$ to $f_2(u, v)$

(Γ, ρ)	$r(u, v)$	$s(u)$	$t(v)$
$(\mathsf{C}_2, 1)$	$u^4 + 6u^2 + 1$	$(1-u^2)^4$	$(1-v)^6$
(C_4, χ^2)	$(u^2+1)^2(v^4+6v^2+1) + 16u^2(v^3+v)$	$(1-u^2)^4$	$(1-v)^2(1-v^2)^4$
(Q_8, χ_0)	$(u^8+u^6+4u^4+u^2+1)(v^4+1) + (4u^8+14u^6+12u^4+14u^2+4)v^2 -$ $-(u^4-10u^2+1)(v^3+v)$	$(1-u^2)^2(1-u^4)^2(1-u^2)^4$	$(1-v)^2(1-v^2)^4$
$(\mathsf{Q}_{12}, \chi_0)$	$(u^{12}+u^{10}+u^8+6u^6+u^4+u^2+1)(v^6+1) +$ $+(3u^{12}+15u^{10}+31u^8+34u^6+31u^4+15u^2+3)(v^4+v^2) +$ $+2u^2((3u^8+5u^6+14u^4+5u^2+3)(v^5+v) + 2(u^4+1)(5u^4+11u^2+5)v^3)$	$(1-u^2)^2(1-u^6)^2$	$(1-v)(1-v^2)^4(1-v^3)$
$(\mathsf{Q}_{24}, \chi_+)$	$(u^{16}+u^{14}+2u^{12}+4u^{10}+8u^8+4u^6+2u^4+u^2+1)(v^8+1) +$ $+(2u^{16}+11u^{14}+24u^{12}+32u^{10}+30u^8+32u^6+24u^4+11u^2+2)(v^6+v^2) -$	$(1-u^4)^2(1-u^6)^2$	$(1-v)^2(1-v^2)^3(1-v^6)$
	$-(u^{12}-3u^{10}-4u^8-12u^6-4u^4-3u^2+1)(v^7+v) -$ $-(2u^{12}-5u^{10}-12u^8-18u^6-12u^4-5u^2+2)(v^5+v^3) +$ $+2(2u^{12}+2u^{10}+5u^8+6u^6+5u^4+2u^2+2)v^4$	$(1-u^2)^2(1-u^6)^2$	
(O, χ)	$(u^{20}+u^{18}+2u^{14}+6u^{12}+4u^{10}+6u^8+2u^6+u^2+1)(v^8+1) +$ $+u^2(u^{16}+3u^{14}+15u^{12}+28u^{10}+26u^8+28u^6+15u^4+3u^2+1)(v^7+v) -$ $-(u^{20}-5u^{18}-27u^{16}-57u^{14}-87u^{12}-106u^{10}-87u^8-57u^6-$ $-27u^4-5u^2+1)(v^5+v^3) + 2(u^{20}+6u^{18}+18u^{16}+33u^{14}+46u^{12}+56u^{10}+$ $+46u^8+33u^6+18u^4+6u^2+1)v^4$	$(1-u^4)(1-u^6)^2(1-u^8)$	$(1-v)(1-v^2)^3(1-v^3)(1-v^4)$
	$(u^8+3u^6+3u^2+1)(v^6+v^2)$	$(1-u^2)^4(1+u^4)$	
$(\mathsf{C}_{10}, \chi^6)$	$(u^{12}-u^{10}+4u^8+4u^4-u^2+1)(v^4+1) -$ $-(3u^{12}-11u^{10}+8u^8-8u^6+8u^4-11u^2+3)(v^3+v) +$ $+(5u^{12}-13u^{10}+16u^8-8u^6+16u^4-13u^2+5)v^2$	$(1-u^2)^3(1-u^{10})$	$(1-v)^5(1-v^5)$

Table III. Calculation of $e_{\mathbb{S}_n}(\mathcal{M}_{2,n})$

n		$e_{\mathbb{S}_n}(\mathcal{M}_{2,n})$
0	e_2	1
1	$2e_2\,s_1$	$2\,s_1$
2	$(e_2+e_2(2))s_2+(e_2+e_2(1^2))s_{1^2}$	$s_2+s_{1^2}$
3	$(e_2(2)-e_2(1^2))s_3+(e_2(2)+e_2(1^2))s_{21}+2\,e_2(1^2)\,s_{1^3}$	0
4	$(e_2-e_2(1^2)-e_2(2))s_4+(-e_2-e_2(1^2)+e_2(2))s_{31}+$ $+(-e_2+e_2(1^2)+e_2(2^2))s_{2^2}+(e_2-e_2(2)+e_2(1^22))s_{21^2}+(e_2(1^2)+e_2(1^4))s_{1^4}$	s_4-s_{31} $-s_{2^2}$
5	$(2\,e_2-2\,e_2(2))s_5+(e_2-e_2(1^22)-e_2(2^2))s_{41}+$ $+(-3\,e_2+2\,e_2(1^2)+2\,e_2(2)-e_2(1^22)+e_2(2^2))s_{32}+$ $+(-e_2(1^2)+e_2(2)-e_2(1^4)-e_2(2^2))s_{31^2}+$ $+(e_2-2\,e_2(2)+e_2(1^22)+e_2(2^2))s_{2^21}+$ $+(e_2-e_2(1^2)-e_2(2)+e_2(1^4)+e_2(1^22))s_{21^3}+2\,e_2(1^4)\,s_{1^5}$	$2(s_5+s_{41})$ $-2\,s_{32}$
6	$(e_2-e_2(1^2)-2\,e_2(2)+e_2(1^22))s_6+(2\,e_2-e_2(1^2)-e_2(2)+e_2(1^4))s_{51}+$ $+(2\,e_2(2)-e_2(2^2))s_{42}+(2\,e_2(1^2)+e_2(2)-e_2(1^4)-2\,e_2(1^22)-e_2(2^2))s_{41^2}+$ $+(-2\,e_2+2\,e_2(1^2)+e_2(1^4)+2\,e_2(2^2))s_{3^2}+$ $+(-e_2-e_2(1^2)-2\,e_2(2)+e_2(1^22)+e_2(2^2))s_{321}+$ $+(-2\,e_2(1^2)+e_2(2)-e_2(1^4)-e_2(2^2))s_{31^3}+$ $+(e_2-e_2(1^2)-e_2(2)+e_2(1^4)+e_2(2^2))s_{2^3}+$ $+(e_2-e_2(1^2)-e_2(2)+e_2(1^4)+e_2(1^22^2))s_{2^21^2}+$ $+(-e_2+2\,e_2(1^2)+e_2(2)-e_2(1^22)+e_2(1^42))s_{21^4}+(e_2(1^6)+e_2(1^4))s_{1^6}$	$2\,s_{51}$ $2\,s_{41^2}$ $-2\,s_{3^2}$ $-2\,s_{321}$ $-3\,s_{2^3}$ $s_{2^21^2}$ $-(s_{21^4}+s_{1^6})$
7	$(-e_2(1^4)+e_2(1^22)-e_2)s_7+(e_2(1^4)+2\,e_2(1^22)+e_2(2^2)-3\,e_2(1^2)-3\,e_2(2))s_{61}+$ $+(-2\,e_2(2^2)-2\,e_2(1^2)+2\,e_2(2)+2\,e_2)s_{52}+$ $+(e_2(1^4)-2\,e_2(1^22)+e_2(2^2)+3\,e_2(1^2)-e_2(2))s_{51^2}+$ $+(e_2(1^22)+e_2(2^2)+e_2(1^2)+e_2(2)-e_2)s_{43}+$ $+(-e_2(1^22^2)-e_2(2^3)-2\,e_2(1^4)-e_2(1^22)-e_2(2^2)+$ $+4\,e_2(1^2)+3\,e_2(2)-1)s_{421}+$ $+(-e_2(1^42)-e_2(1^22^2)-e_2(1^4)-e_2(1^22)+3\,e_2(1^2)+e_2(2)-1)s_{41^3}+$ $+(2\,e_2(1^4)+e_2(1^22)+3\,e_2(2^2)+e_2(1^2)-3\,e_2(2)-1)s_{3^21}+$ $+(-e_2(1^22^2)+e_2(2^3)-e_2(1^4)+e_2(1^22)-e_2(1^2)-2\,e_2(2)-1)s_{32^2}+$ $+(-e_2(1^42)-e_2(2^3)+2\,e_2(1^4)+2\,e_2(1^22)-4\,e_2(1^2)-e_2(2)+2\,e_2)s_{321^2}+$ $+(-e_2(1^6)-e_2(1^22^2)-e_2(1^4)+e_2(1^22)-e_2(1^2)-e_2)s_{31^4}+$ $+(e_2(1^22^2)+e_2(2^3)-e_2(1^22)-e_2(2^2)+e_2(2)+1)s_{2^31}+$ $+(e_2(1^42)+e_2(1^22^2)-2\,e_2(1^22)+3\,e_2(1^2)+e_2(2))s_{2^21^3}+$ $+(e_2(1^6)+e_2(1^42)-e_2(1^4)-e_2(1^22)+2\,e_2(1^2)+e_2(2)-e_2)s_{21^5}+2\,e_2(1^6)s_{1^7}$	$-2(s_7+s_{61})$ $2\,s_{52}$ $2\,s_{51^2}$ $-2\,s_{43}$ $4\,s_{421}$ $2\,s_{41^3}$ $-2\,s_{3^21}$ $-4\,s_{32^2}$ $4\,s_{321^2}$ 0 $-2\,s_{2^31}$ $-2(s_{21^5}+s_{1^7})$

In applying (3) in genus 2, we may take advantage of the fact that $e_2(1^k2^\ell)$ vanishes if k is odd. We obtain the results listed in the second column of Table III. Substituting the values for $e_2(1^k2^\ell)$, we obtain the equivariant Euler characteristics of $\mathcal{M}_{2,n}$, $0 \leqslant n \leqslant 7$. The dimensions of these virtual representations of $\mathbb{S}_n$ agree with the Euler characteristics of $\mathcal{M}_{2,n}$ calculated by Bini *et al.* [1].

Acknowledgement

This paper was completed while the author was a guest of the School of Mathematical Sciences of the Australian National University. The author is partially supported under NSF grant DMS-9704320.

References

1. Bini, G. Gaiffi, G. and Polito, M: A formula for the Euler characteristic of $\overline{\mathcal{M}}_{2,n}$. math. AG/9806048.
2. Bolza, O: On binary sextics with linear transformations into themselves, *Amer. J. Math.* **10** (1888), 47–70.
3. Clebsch, A: *Theorie der binären algebraischen Formen*, Teubner, Leipzig, 1872.
4. Getzler, E: Resolving mixed Hodge modules on configuration spaces, *Duke Math. J.* **96** (1999), 175–203. alg-geom/9611003.
5. Gordan, P: *Dr. Paul Gordan's Vorlesungen über Invariantentheorie. Herausgegeben von dr. Georg Kerschensteiner. Zweiter Band: Binäre Formen*, Teubner, Leipzig, 1887, (Reprinted by Chelsea Publishing, New York, 1985).
6. Klein, F: *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Teubner, Leipzig, 1884 (Reprinted by Birkhäuser Verlag, Basel, 1993).