Introduction: Why the pinch technique?

Non-Abelian gauge theories (NAGTs) have dominated the world of experimentally accessible particle physics for more than three decades in the form of the standard model with its $SU(2) \times U(1)$ (electroweak theory) and $SU(3)$ (quantum chromodynamics: QCD) components. NAGTs are also the ingredients of grand unified theories and technicolor theories and play critical roles in supersymmetry and string theory. It is no wonder that thousands of papers have been written on them. But many of these papers violate the principle of gauge invariance, resulting in calculations of propagators, vertices, and other off-shell form factors that are valid only in the particular gauge chosen. Until these are combined into a gauge-invariant expression, they have very limited, if any, physical meaning. The reason for such violation of gauge invariance is that standard and widely used Feynman graph techniques generate gauge-dependent Green's functions (proper self-energies, three-point vertices, etc.) for the gauge bosons.

Of course, there is one combination of off-shell Green's functions – the on-shell S-matrix – that is gauge invariant no matter which gauge is used for the propagators and vertices that go into it. Thus, authors who (correctly) insist on calculating only gauge-invariant quantities often restrict themselves to dealing only with the S-matrix. This is fine as long as the question at hand can be answered with perturbation theory, but to calculate gauge invariantly a nonperturbative feature using only S-matrix elements is not easy. Some of the processes involving off-shell Green's functions include confinement and chiral symmetry breakdown in QCD; higher-order radiative corrections to gauge-boson decay widths in electroweak theory; nonperturbative effects in the magnetic sector of high-temperature QCD and electroweak theory; and physical quantities embedded in the electroweak S-matrix such as the magnetic dipole and electric quadrupole moments of the W-boson, the top-quark magnetic moment, and the neutrino electric charge radius. These are physical, measurable phenomena, and to approximate them with gauge-dependent calculations really makes no sense.

The Abelian predecessor of NAGTs, quantum electrodynamics (QED), is also a theory with local gauge invariance. But in QED, it is much easier than in an NAGT to enforce this crucial property. For example, the gauge-boson (photon) propagator in QED is gauge invariant for any momentum and any choice of gauge for internal lines, whereas the gauge-boson propagator in an NAGT, if calculated with standard Feynman graph techniques, is not. Enforcing gauge invariance on off-shell Green's functions for an NAGT is difficult even in perturbation theory, let alone in attempts to understand nonperturbative effects such as confinement in QCD. For NAGTs in covariant gauges, one also has to deal with Faddeev–Popov ghosts, which are absent from QED.

Long ago, a new method for enforcing gauge invariance in off-shell Green's functions, and the Schwinger–Dyson equations that couple them, emerged; this was later termed the *pinch technique*. This is a technique to disassemble and reassemble the Feynman graphs of a gauge-invariant object, such as the S-matrix, into new off-shell Green's functions such as gauge-boson proper self-energies that are strictly gauge invariant, just as in QED. It works by systematically identifying parts of n-point Feynman graphs in that S-matrix element that actually belong to Green's functions with fewer legs through the use of elementary Ward identities. Moreover, requiring that the new pinch technique proper self-energy have the right analytic properties in momentum as well as other physically reasonable properties uniquely specified this new self-energy. Although the actual calculation was not carried out until considerably later, it was clear that any kind of proper gauge-boson vertex, such as the three- or four-point vertex, could also be made completely gauge invariant through application of the same ideas. Uniqueness of the new self-energy and vertices was assured by requiring them to obey Ward identities having an essentially Abelian structure, with no explicit ghost contributions, rather than the much more complex Slavnov–Taylor identities of conventional NAGTs. Another technique, called the *gauge technique* (described later), allows for the construction of nonperturbative proper three-point vertices for the gauge bosons that are approximately valid for small gauge-boson momentum but *exactly* gauge invariant and that satisfy the correct pinch technique Ward identities.

This is simple enough to do at the one-loop level, but at higher orders, a more systematic approach is needed. The surprising and far from obvious outcome is that the simple prescription for calculating gauge-invariant pinch technique Green's functions to any order is to use the conventional Feynman graphs in the background-field Feynman gauge.

The pinch technique – or its algorithmic equivalent, graphs in the background-field Feynman gauge – even shows up in string theory. Usually we think of string theory as yielding only on-shell amplitudes, but a consistent extrapolation of string amplitudes off shell in the field-theory (or zero Regge slope) limit shows that the resulting off-shell gauge-boson amplitudes are automatically presented in the background-field Feynman gauge.

The purpose of this book is to describe the pinch technique and its evolution from simple one-loop beginnings to a systematic method at all orders of perturbation theory and then to fully gauge-invariant Schwinger–Dyson equations, leading to the many applications of the pinch technique that have been developed over the years. The pinch technique has led to the clarification of a number of problems in NAGTs that were essentially unsolvable by techniques depending on conventional (Feynman–graph–derived) off-shell Green's functions because these are gauge dependent.

A quick outline of the book

This book begins with two introductory chapters (Chapters 1 and 2) that derive two- and three-gluon pinch technique Green's functions at the one-loop level, showing how a dynamical mass is demanded by infrared slavery for QCD-like gauge theories and outlining the full scope of the pinch technique program. Then Chapters 3 through 6 go into considerable detail to derive the pinch technique Green's functions to all orders; show that they are the same as the background-field Feynman-gauge Green's functions; derive the Schwinger–Dyson equations for the pinch technique in QCD-like theories; apply a special gauge-invariant method called the *gauge technique* to truncate these equations; and show how the pinch technique Schwinger–Dyson equations give rise to a dynamical gluon mass in these QCD-like theories. Chapters 7, 8, and 9 then cover applications of these results to QCD-like theories. Chapter 10 develops the pinch technique for electroweak theory to all orders, and Chapter 11 describes several applications of the pinch technique to electroweak physics, thermal gauge theories, and supersymmetric gauge theories.

Once we get past one loop, it is hard slogging through all the details, so the reader who wants a tour d'horizon of the pinch technique can find it in Chapters 1 and 2. There we hint at the nonperturbative ideas used in later chapters, making some remarks on the Schwinger–Dyson equation for the pinch technique propagator at the one-dressed-loop level. The reader interested in the inner workings of the pinch technique should read the next four chapters. The remaining chapters are on applications, so readers who are more interested in applications of the pinch technique than in its derivation can spend their time on Chapter 7 and onward after reading introductory Chapters 1 and 2. Although the book is not intended to be a textbook, it should be accessible, with perhaps a little extra work on such techniques as the background-field method and the Batalin–Vilkovisky formalism, to advanced graduate students as well as to researchers in gauge theories.

Some uses of the pinch technique

In QCD-like theories, the most important uses of the pinch technique come from the not-unexpected discovery in solving the Schwinger–Dyson equations that *asymptotic freedom*, or more accurately,[1] *infrared slavery*, requires a dynamically generated gluon mass to resolve the otherwise intractable infrared singularities of infrared slavery. From this simple result – repeatedly confirmed by lattice simulations – follows a cascade of topological quantum solitons that plausibly explain confinement and other nonperturbative phenomena of QCD-like theories and that are also seen in lattice simulations, as we briefly discuss at the end of this section and in detail in Chapters 7 and 8.

The pinch technique for NAGTs has considerable physical interest in three dimensions as well as four. One reason is that $d = 3$ NAGTs carry all of the critical nonperturbative phenomena, stemming from infrared slavery, associated with the sector of finite-temperature NAGTs having zero Matsubara frequency and therefore (perturbatively) massless magnetic gluons. There is such a sector for QCD-like theories and also for electroweak theory above the crossover temperature at which the standard-model Higgs field has zero vacuum expectation value. Another reason is the instructive differences in the conditions and techniques used in $d = 3$; for example, there is no asymptotic freedom, but there is infrared slavery that gives worse infrared divergences in $d = 3$ than in $d = 4$. We cover $d = 3$ NAGTs in Chapter 9 and a few other applications of the pinch technique to finite-temperature gauge theories in Chapter 11.

The pinch technique has also been developed to all orders in electroweak theory as the prime example of symmetry breaking in NAGTs, as we recount in Chapter 10. Here the pinch technique makes possible many applications given in Chapter 11, including gauge-invariant, non-Abelian, off-shell charges and form factors; the neutrino charge radius; and gauge-invariant constructions of resonance widths that are gauge dependent in the usual Feynman-graph formalism.

Finally, applications to NAGTs are even embedded in supersymmetric theories in which the contributions of scalars and fermions to the off-shell gluonic Green's

[1] An NAGT in $d = 3$ cannot be asymptotically free, but it can show infrared slavery, that is, severe infrared singularities coming from wrong-sign phenomena; in general, this book applies both to $d = 3$ and $d = 4$ NAGTs.

functions of the pinch technique confirm, by explicit calculation, well-known supersymmetry relations that would not hold for conventional Feynman-graph definitions of the Green's functions. These, too, are discussed in Chapter 11.

Constructing pinch technique Green's functions and equations

Because it is far from obvious, we will spend considerable time on the demonstration that the systematic all-orders construction of pinch technique Green's functions gives the same Green's functions as would be calculated in the background field method exhibited in the Feynman gauge. The first suggestions that the pinch technique and the background-field Feynman gauge are related came from a one-loop calculation showing that the pinch technique gives the same results as the background-field method in the Feynman gauge; we give our version of these one-loop results in Chapters 1 and 2.

This gauge, then, transcends its usual meaning as just another gauge because it incorporates – for kinematic reasons that we will explain – quite a different algorithm: that of the gauge-invariant pinch technique. In the pinch technique (as well as in the background-field method), the Ward identities relating a Green's function of n gauge potentials and no ghosts to those of fewer than n legs are not the standard Slavnov–Taylor identities, which involve ghosts, but rather elementary Ward identities of QED type, making no reference to ghosts. There are, as is surely necessary in any covariant gauge-invariant description of an NAGT, ghost contributions and Green's functions with ghost legs, but in a pinch technique Green's function involving only gauge potentials, the internal ghost loops are not explicitly distinguishable from gluons in the final product. We describe these developments in detail in Chapters 3 and 4.

The next step is to consider the infinite tower of equations, the Schwinger–Dyson equations, that couple all the pinch technique Green's functions. For example, the pinch technique propagator depends on the three- and four-point gluon vertex functions, which in turn depend on higher-point functions – and so on. We introduce closure to this infinite tower of equations by using the *gauge technique*, in which we construct *approximate* n-point pinch technique Green's functions that depend only on fewer-point pinch technique Green's functions and that *exactly* satisfy the correct Ward identities. Gauge technique Green's functions, which differ from their exact counterparts by completely conserved quantities, more and more accurately represent the exact Green's functions as momenta on the legs become smaller (because in the presence of a mass gap, completely conserved quantities have kinematic zeroes at vanishing momenta of higher order than any in the gauge technique Green's functions). In consequence, the gauge technique Green's functions

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describe rather well the nonperturbative properties of gauge theories with infrared slavery but require correction in the ultraviolet; fortunately, such corrections are straightforward because of asymptotic freedom. We cover the gauge technique in Chapter 5, from its simple beginnings long ago in QED to considerably more complex forms for NAGTs.

The final step in moving toward applications is to derive the Schwinger–Dyson equations by considering the all-order perturbative results; this is done in Chapter 6. These Schwinger–Dyson equations include those for ghosts, although strictly speaking, these are not necessary (e.g., in the pinch technique Schwinger–Dyson equations as derived from light-cone gauge graphology). We should and do derive the Schwinger–Dyson equations without reference to any particular closure scheme, although the only practical way to use these Schwinger–Dyson equations is with the gauge technique, with ultraviolet corrections added perturbatively. Of course, the solutions to the Schwinger–Dyson equations closed with the gauge technique are only approximate, as will always be the case however NAGTs are approached, but they have the great virtue of local gauge invariance, quite unlike the usual Schwinger–Dyson equations based on conventional Feynman graphology.

By far the most interesting feature of the solutions of these equations is that the infrared slavery singularities are tamed by gluons' becoming massive, yet without any violation of local gauge invariance (cf. the $d = 2$ Schwinger model). It is critical that this be a *running* mass that vanishes (up to logarithms) like $\langle G^2\rangle q^{-2}$ at large momenta, where $\langle G^2\rangle$ is the usual vacuum expectation value of the squared field strengths.[2] This is precisely analogous to the dynamical (constituent) quark mass, vanishing like $\langle\bar{\psi}\psi\rangle q^{-2}$. If the mass did not run, it would not be a dynamical mass but a bare mass (like the current mass of quarks), and the renormalizability of an NAGT with a bare gluon mass is problematical. This vanishing of the running mass is very roughly analogous to the vanishing of a Higgs field along the axis of a soliton in symmetry-breaking theories.

The mere technical solution of the infrared-singularity problem by generation of a dynamical gluon mass is perhaps interesting but, taken alone, seems not to bear on the great issues of QCD such as confinement and chiral symmetry breaking. This is far from true because gauge-invariant mass generation in an NAGT has two important implications. First, there must be massless pure-gauge, longitudinally coupled excitations, much like Goldstone fields, and explicit in the gauge technique Green's functions. Second, these pure-gauge parts, which are indispensable parts of solitons of the massive gluon effective action, carry the long-range

[2] In conventional Feynman graphs for the propagator, there are condensate contributions to something that might have been identified with a running mass, except that there are contributions from non-gauge-invariant condensates. Only in the pinch technique propagator is there only a gauge-invariant condensate contribution.

topological properties essential for confinement and chiral symmetry breakdown. In themselves, the massless parts would have Dirac singularities (lying on closed sheets in $d = 4$ and on closed strings in $d = 3$), but every topological soliton has another contribution to the gauge potential that is massive and not pure gauge, giving rise only to short-range gauge field strengths. This massive part also has a Dirac singularity that exactly cancels that of the massless pure-gauge part, yielding a soliton that is essentially nonsingular and of finite action. There are many solitons, but the three most interesting classes are the center vortex, essentially a closed two-surface (responsible for confinement via a Bohm–Aharonov effect coming from linkage of a Wilson loop to the closed surface of the massless long-range, pure-gauge part of the center vortex); its closely related descendant, the nexus (a monopole whose world line must lie on the center-vortex surface, thereby dividing it into regions of different orientation and providing for the generation of nonintegral topological charge); and the sphaleron (rather like the electroweak sphaleron, interpretable not only as a cross section of a topology-changing configuration but also as an unstable glueball). These solitons and their consequences will be discussed in much more detail in Chapters 7 and 8, with Chapter 9 devoted to the special case of three dimensions, which has a number of interesting features not found in $d = 4$.