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LETTERS TO THE EDITOR

AN ALTERNATIVE PROOF OF LORDEN'S RENEWAL INEQUALITY

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Let $X_1, X_2, \cdots$ be i.i.d. non-negative random variables with distribution function F, mean $\mu = E[X_i]$ and finite second moment $\lambda^2 = E[X_i^2]$. Denote by H the renewal function

$$H(t) = E\left[\#\left\{n \geqq 0; \sum_{i=1}^{n} X_i \leqq t\right\}\right],$$

for convenience defined for any t. In [3] Lorden proved the inequality

$$(1) \qquad t/\mu \leqq H(t) \leqq t/\mu + \lambda^2/\mu^2,$$

to be valid for all $t \geqq 0$. For extensions to random walks where the X_i's may be negative and for consequences for boundary excess distributions cf. Daley [1] and Lorden [3].

Lorden in his proof of (1) uses that H is subadditive, $H(t+s) \leqq H(t) + H(s)$, a fact that we shall combine with well-known properties of the stationary renewal sequence to give an alternative proof of (1).

For simplicity assume that $\mu = 1$. Let Y_1 and Y_2 be independent with densities $1 - F(t)$ for $t \geqq 0$, i.e. with the stationary delay distribution of the renewal process. Then,

$$(2) \qquad E[H(t - Y_i)] = t, \qquad t \geqq 0, \qquad i = 1, 2,$$

see Feller [2], pp. 368–369. Certainly (2) and the fact that H is non-decreasing proves the left inequality in (1). Here is the trick to prove the right one: by subadditivity

$$H(t) = E[H(t)] = E[H(t + Y_1 - Y_2 + Y_2 - Y_1)] \leqq E[H(t + Y_1 - Y_2)] + E[H(Y_2 - Y_1)].$$

Together with (2) this implies

$$\begin{aligned} H(t) &\leqq E[E[H(t + Y_1 - Y_2) \mid Y_1]] + E[E[H(Y_2 - Y_1) \mid Y_2]] \\ &= E[t + Y_1] + E[Y_2] = t + 2E[Y_1] = t + \lambda^2, \end{aligned}$$

as desired.

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