ABSTRACT THEORY OF PACKINGS AND COVERINGS. II

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1. Preliminaries and summary. The present paper is closely related to a paper with the same title by A. M. Macbeath [3]. We use many notions which are defined there for a measure-space; nevertheless we define them once more because we consider the slightly different case of a measure-ring.

Let $(\mathbf{S}, \mu)$ be a measure-ring with unity such that the measure μ is σ-finite (for definitions cf. [2]). We assume that there exists a countable group G of transformations which map $\mathbf{S}$ onto itself and which preserve the measure μ and the operations $\cup$, $-$, $\cap$ in $\mathbf{S}$. We denote by gA the image of $A \in \mathbf{S}$ by $g \in G$. For any subset Γ of G we write ΓA instead of $\bigcup_{g \in \Gamma} gA$. We say that an element P belonging to $\mathbf{S}$ is a *packing* (more precisely, a G-packing) of an element $A \in \mathbf{S}$ if $gP \subset A$ for every $g \in G$ and the elements gP are disjoint for different g. We call an element $C \in \mathbf{S}$ a *covering* of A if $A \subset GC$. If an element $F \in \mathbf{S}$ is simultaneously a packing and a covering of A, then F is called a *fundamental domain* for A. If, in particular, A is the ring unity 1, then we call P (or C) a packing (or covering) of $\mathbf{S}$, and F a fundamental domain for $\mathbf{S}$. In Theorem 1 we give a condition on $(\mathbf{S}, \mu)$ and G which is equivalent to the existence of a fundamental domain F for $\mathbf{S}$.

If P and C are a packing and a covering of an element $A \in \mathbf{S}$, then $\mu P \leqslant \mu C$. This result is stated in [3] (Theorem 1) for the ring $\mathbf{S}$ of all measurable subsets of a measure-space $(X, \mathbf{S}, \mu)$ and for $A = X$. However, the proof which is given there is more general and it can be applied to a measure-ring $(\mathbf{S}, \mu)$ and to an arbitrary $A \in \mathbf{S}$. We shall use this result in several parts of our proof, referring to it as to the theorem about packings and coverings.

Let p be the upper bound of all measures μP, where P are packings of $\mathbf{S}$, and let c be the lower bound of measures μC, where C are coverings of $\mathbf{S}$. These numbers exist since the zero element $0 \in \mathbf{S}$ is a packing and the ring unity 1 is a covering of $\mathbf{S}$. By the theorem about packings and coverings we have $p \leqslant c$. In Theorem 2 we give a condition on $(\mathbf{S}, \mu)$ and G which is equivalent to $p = c$.

The corollaries contain results which are analogous to Theorems 1 and 2 but concern the ring of measurable sets of a measure-space. We construct also examples which show that these theorems fail to be true if the measure is not σ-finite.

2. Results. Let $e \in G$ be the identity transformation. We denote by (π), (ρ) and (δ) the following properties:

(π) *If* $A \in \mathbf{S}$, $g \in G$, $A \neq 0$ *and* $g \neq e$, *then there exists a* $B \subset A$ *such that* $B \neq 0$ *and* $B \cap gB = 0$.

(ρ) *If* $A \in \mathbf{S}$ *has arbitrarily small coverings, then* $A = 0$.

(δ) *If for some* $A \in \mathbf{S}$ *and a certain* $g \in G$, $g \neq e$ *we have* $B \cap gB \neq 0$ *for every* $B \subset A$, $B \neq 0$, *then* A *has arbitrarily small coverings.*

THEOREM 1. *There exists a fundamental domain* F *for* $\mathbf{S}$ *if and only if both* (π) *and* (ρ) *hold.*

THEOREM 2. $p = c$ *is equivalent to* (δ).

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We shall verify in § 6 that the σ-finiteness of μ is in both theorems an indispensable assumption. Let us consider now a measure-space $(X, \mathbf{S}, \mu)$, where the measure μ is σ-finite and complete (see [**2**]). Let G be a countable group of transformations of X onto itself which preserve measurability and measure. We denote by (π_0), (π_0'), (ρ_0), (δ_0) the properties :

(π_0) *If* $A \in \mathbf{S}$, $g \in G$, $\mu A > 0$ *and* $g \neq e$, *then there exists a* $B \subset A$ *such that* $\mu B > 0$ *and* $B \cap gB = \phi$ (ϕ is the empty set).

(π_0') *No* $g \neq e$ *has fixed points* (*i.e.* $gx \neq x$ *for* $x \in X$).

(ρ_0) *If* $A \in \mathbf{S}$ *has arbitrarily small coverings, then* $\mu A = 0$.

(δ_0) *If for some* $A \in \mathbf{S}$ *and a certain* $g \in G$, $g \neq e$ *we have* $\mu(B \cap gB) > 0$ *for every* $B \subset A$ *with* $\mu B > 0$, *then* A *has arbitrarily small coverings.*

Applying Theorems 1 and 2 to the measure-ring defined by $(X, \mathbf{S}, \mu)$, we obtain the corollaries :

COROLLARY 1. *There exists a fundamental domain for* $(X, \mathbf{S}, \mu)$ *if and only if* (π_0), (π_0'), (ρ_0) *hold simultaneously.*

COROLLARY 2. $p = c$ *is equivalent to* (δ_0).

Let us consider a locally compact and σ-compact topological group H. We denote by μ the Haar measure on H and by $\mathbf{S}$ the ring of all μ-measurable sets in H. Let G be a countable subgroup of H. The left translations by elements of G form a group of measure preserving transformations of the measure-space $(H, \mathbf{S}, \mu)$. Evidently (π_0) holds. Hence (δ_0) is true and so $p = c$.

It follows from Corollary 1 that if G is discrete (in the topology induced by H), then a fundamental domain exists. This is however a known result [**1**]. It follows also from Corollary 1 that if G is not discrete, then no fundamental domain exists. But this can be proved also directly. In fact, a fundamental domain F is of positive measure and we have $F \cap gF = \phi$ for $g \in G - \{e\}$. Thus $G - \{e\}$ cannot intersect every neighbourhood of e (see [**4**]).

3. Two lemmas. Let us call coverings of $\mathbf{S}$ simply coverings and let us adopt the same convention for packings and fundamental domains.

LEMMA 1. *We assume that* (π) *holds. Then every covering* C *which is not a packing contains a covering* $C_0 \neq C$.

Proof. We have $C \cap g^{-1}C \neq 0$ for some $g \neq e$. Let $A = C \cap g^{-1}C$ and let $B \subset A$ satisfy (π). Thus $gB \subset gA \subset C$ and hence both B, gB are contained in C. Since they are disjoint it follows that $C_0 = C - B$ is also a covering.

LEMMA 2. *If* (π) *and* (ρ) *hold and there exists a covering* C *with* $0 < \mu C < \infty$, *then a fundamental domain exists.*

Proof. Let $\mathbf{C}$ be the family of all coverings of finite measure. We observe that a partial order is defined in $\mathbf{C}$ by the relation of inclusion. From Zorn's Lemma it follows that $\mathbf{C}$ contains a maximal decreasing chain $\mathbf{M}$, i.e. an ordered subfamily $\mathbf{M}$ of coverings such that no covering $C_0 \in \mathbf{C} - \mathbf{M}$ is contained in all $C \in \mathbf{M}$. Let $a = \inf_{C \in \mathbf{M}} \mu C$. There exist coverings $C_1, C_2, \ldots, C_n, \ldots \in \mathbf{M}$ such that $a = \lim_{n \to \infty} \mu C_n$. Put $F = \bigcap_{n=1}^{\infty} C_n$. Thus $\mu F = a$. Let $B = GF$. Since $1 = GC_n$ for each n it follows that $1 - B \subset G(C_n - F)$. We obtain from $\lim \mu(C_n - F) = 0$

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that $1-B$ has arbitrarily small coverings. Hence, by (ρ), we have $B = 1$; i.e. F is a covering.

Let us verify that $F \subset C$ holds for every $C \in \mathbf{M}$. Indeed, from $F \cap C \neq F$ for some $C \in \mathbf{M}$ follows $\mu(F \cap C) < \mu F = a$, and thus $\mu \bigcap_{n=0}^{m} C_n < a$ for $C_0 = C$ and sufficiently large m. This is a contradiction since $\bigcap_{n=0}^{m} C_n \in \mathbf{M}$. From $F \subset C$ for every $C \in \mathbf{M}$ we have that no covering $C_0 \neq F$ is contained in F. Since F is a covering, we obtain, by Lemma 1, that F is also a packing and thus F is a fundamental domain.

4. Proof of Theorem 1. Suppose first that a fundamental domain F exists. We shall prove that both (π) and (ρ) hold. Assume that (π) is not true, i.e. that there exists an $A \in \mathbf{S}$ and $g \in G$ such that $A \neq 0$, $g \neq e$ and $B \cap gB \neq 0$ whenever $B \subset A$ and $B \neq 0$. From $A \subset GF$ we have that, for some $g_0 \in G$, the set $B = A \cap g_0 F$ is not empty. From $B \subset A$ and $B \neq 0$ it follows that $B \cap gB \neq 0$. This is a contradiction, since $B \subset g_0 F$, $gB \subset gg_0 F$ and $g \neq e$.

Now suppose that (ρ) is false. We assume that $A \neq 0$ has arbitrarily small coverings. It follows that the same is true for GA. Thus, by the theorem about packings and coverings, there exists no packing of GA except 0. Evidently $A \cap g_0 F \neq 0$ for some $g_0 \in G$. Thus $P = A \cap g_0 F$ is a packing of GA which is different from 0 and this is a contradiction.

Now let us suppose that (π) and (ρ) hold. We take a maximal set Φ of non-zero elements A of finite measure such that all elements GA ($A \in \Phi$) are disjoint. This set is countable since μ is σ-finite. Thus $\Phi = \{A_1, A_2, \ldots, A_n, \ldots\}$. Suppose that $\bigcup_{n=1}^{\infty} GA_n \neq 1$. By the σ-finiteness of μ the element $B = 1 - \bigcup_{n=1}^{\infty} GA_n$ contains an element $D \neq 0$ of finite measure. We have $GD \cap GA_n = 0$ for every n and this is a contradiction since Φ is maximal. Hence $1 = \bigcup_{n=1}^{\infty} GA_n$. Since A_n is a covering of GA_n, it follows from Lemma 2 that there exists a fundamental domain F_n for each GA_n. Thus $F = \bigcup_{n=1}^{\infty} F_n$ is a fundamental domain for $\mathbf{S}$.

5. Proof of Theorem 2. We assume first that (δ) does not hold and we shall prove that then $p \neq c$. Let $A \in \mathbf{S}$ and $g \in G$, $g \neq e$ be such that for $B \subset A$ and $B \neq 0$ we have $B \cap gB \neq 0$, but A does not have arbitrarily small coverings. It follows that the lower bound m of measures of coverings of GA is positive. Let us prove that every packing P of $\mathbf{S}$ is disjoint from GA. Assume the contrary. Then $g_1 A \cap g_2 P \neq 0$ for some $g_1, g_2 \in G$ and thus $B = A \cap g_1^{-1} g_2 P \neq 0$. We have $B \subset A$, $B \neq 0$ and thus it follows from $B \cap gB \neq 0$ that $g_1^{-1} g_2 P \cap g g_1^{-1} g_2 P \neq 0$. Therefore P cannot be a packing. We now define $Q = 1 - GA$. If C is an arbitrary covering of $\mathbf{S}$, then evidently $M = C \cap GA$ is a covering of GA and $N = C \cap Q$ is a covering of Q. Consequently $\mu M \geqslant m$. We have $M \cup N = C$, $M \cap N = 0$ and this implies $\mu N \leqslant \mu C - m$. Let P be a packing of $\mathbf{S}$. Since P is disjoint from GA, P is a packing of Q. Thus $\mu P \leqslant \mu N$, by the theorem about packings and coverings, and we obtain $\mu P \leqslant \mu C - m$. Therefore $p < c$ follows.

We assume now that (δ) holds and we shall prove that $p = c$. If (ρ) holds, then (π) follows by (δ), and then $p = c$ by Theorem 1. Suppose that (ρ) does not hold and take a maximal set Ω of non-zero elements such that each $A \in \Omega$ has arbitrarily small coverings and the elements GA, where $A \in \Omega$, are disjoint. Ω is countable by the σ-finiteness of μ and thus also $Q = \bigcup_{A \in \Omega} GA$ has arbitrarily small coverings. We shall now prove that there exists a fundamental domain for $1 - Q$. This follows from Theorem 1. Indeed, (ρ) holds for each $A \subset 1 - Q$ by the construction of Q and it remains to verify that (π) holds also. But if (π) is false for some $A \neq 0$ and $g \neq e$, then, by (δ), A has arbitrarily small coverings, contradicting (ρ). Let F be a fundamental domain for $1 - Q$. For each $\varepsilon > 0$ there exists a covering D of Q with $\mu D < \varepsilon$. It follows that $F \cup D$ is a covering and F a packing of S. Thus $c \leqslant p$. By the theorem about packings and coverings, we have $p \leqslant c$ and therefore $p = c$.

6. Rings with a non σ-finite measure. We give first an example of a measure-ring $(\mathbf{S}, \mu)$ and a group G such that (π) and (ρ) hold but no fundamental domain exists. Let $\mathbf{S}$ be the ring of all these sets of real numbers which either are countable or have a countable complement. Let G be the group of translations by integers. The measure μ of $A \in \mathbf{S}$ is defined to be the number of elements in A (μA is infinite if A is infinite). Then (π) and (ρ) hold. We observe that every packing of $\mathbf{S}$ is a countable set and every covering is not countable. Therefore no fundamental domain exists.

Now let us give an example where (δ) holds and $p \neq c$. Let L be the ring of all Lebesgue-measurable sets of real numbers and let $N \subset L$ be the ideal of all sets of measure 0. We denote by L^* the quotient ring L/N. Let T be an infinite non-countable set and let to each $\tau \in T$ correspond a replica L_τ^* of L^*. We consider the product $\mathbf{S} = \prod_{\tau \in T} L_\tau^*$. For $A \in \mathbf{S}$ we denote by A_τ the τ-coordinate of A $(A_\tau \in L_\tau^*)$. Let m denote the Lebesgue measure in L^*. We define μ on $\mathbf{S}$ by

$$\mu A = \sum_{\tau \in T} m A_\tau,$$

where the sum of a non-countable collection of positive numbers is defined to be infinite. For $A, B \in \mathbf{S}$ let $C = A \cup B$ if $C_\tau = A_\tau \cup B_\tau$ for every τ. Similarly we define in $\mathbf{S}$ the operations $-$ and $\cap$. Let G be the group of translations of elements of L by rational numbers. Thus for every $A \in \mathbf{S}$ and $g \in G$ we can define $g(A_\tau)$ for each τ. Let us define gA by $(gA)_\tau = g(A_\tau)$. Consequently (π) is true and (δ) follows. Let us observe that if P is a packing of $\mathbf{S}$, then each P_τ is a packing of L_τ^* and thus $P_\tau = 0$ by the theorem about packings and coverings. Hence 0 is the only packing of $\mathbf{S}$ and we have $p = 0$. We easily observe that if C is a covering of $\mathbf{S}$, then $\mu C = \infty$. Therefore $p \neq c$.

7. Proofs of the corollaries. Let N be the ideal of all subsets of X which are of measure 0 ; these sets form an ideal since the measure μ is complete. We consider the measure-ring $(\mathbf{S}^*, \mu)$, where $\mathbf{S}^*$ is the quotient ring S/N. Let us denote by $A^* \in \mathbf{S}^*$ the image of $A \in \mathbf{S}$ by the natural mapping of $\mathbf{S}$ onto $\mathbf{S}^*$.

We first prove Corollary 1. Suppose that (π_0), (π_0') and (ρ_0) hold. Then (π) and (ρ) hold for $\mathbf{S}^*$. Thus, by Theorem 1, there exists an $F \in \mathbf{S}$ such that F^* is a fundamental domain for $\mathbf{S}^*$. It follows that

$$P = F - (G - \{e\})F$$

is a packing of $\mathbf{S}$ such that $Q = X - GP \in N$. Evidently Q is a union of sets Gx where $x \in Q$. Let $D \subset Q$ be any set which contains exactly one element from each of these sets Gx. We have $D \in N$, and thus D is measurable. It follows from (π_0') that D is a fundamental domain for Q. Consequently $P \cup D$ is a fundamental domain for $\mathbf{S}$.

Conversely, if F is a fundamental domain for **S**, then evidently F^* is a fundamental domain for **S*** and the necessity of (π_0) and (ρ_0) follows. The necessity of (π_0') is obvious.

To prove Corollary 2 it suffices to observe that to every packing of **S** corresponds a packing of **S*** of the same measure and conversely, and that the same is true for coverings.

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