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RESEARCH ARTICLE

A hyperbolic free-by-cyclic group determined by its finite quotients

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Abstract
We show that the group $\langle a, b, c, t : a^t = b, b^t = c, c^t = ca^{-1}\rangle$ is profinitely rigid amongst free-by-cyclic groups, providing the first example of a hyperbolic free-by-cyclic group with this property.

1. Introduction

All groups considered will be finitely generated and residually finite. Two groups are said to be *profinitely isomorphic* if they share the same set of finite quotients, and a group G is *profinitely rigid* (within a class $\mathcal{C}$) if any group (within $\mathcal{C}$), which is profinitely isomorphic to G is isomorphic to G. One can ask which groups within which classes are profinitely rigid, though this question can be subtle even when the class is very restricted: finitely generated abelian groups are profinitely rigid, free groups are profinitely rigid among themselves but it is an open question due to Remeslennikov [11], Question 5.48] whether this holds among finitely generated groups, and there are pairs of non-isomorphic virtually free (even virtually $\mathbb{Z}$!) groups which are profinitely isomorphic [1].

Grothendieck asked if there are groups that are profinitely isomorphic to one of their proper, non-isomorphic subgroups: Platonov and Tavgen [12] provide examples of this phenomenon for finitely generated groups, and Bridson and Grunewald [2], Theorem 1.1] for finitely presented groups. For a discussion of these facts, as well as a broader introduction to profinite rigidity of groups, see for instance [13].

Herein, we consider profinite rigidity within the class of free-by-cyclic groups. A free-by-cyclic group is a semidirect product $G := F_r \rtimes_\varphi \mathbb{Z}$, where F_r denotes the free group of finite rank r. If two automorphisms φ, ψ of F_r represent conjugate or conjugate inverse elements of $\mathrm{Out}(F_r)$ then they define isomorphic free-by-cyclic groups [4], Lemma 2.1]. The converse holds when the abelianisation has rank 1, $b_1(G) = 1$ [4], Theorem 2.4], though not in general [4], p1678].

We provide the first known example of a hyperbolic free-by-cyclic group that is profinitely rigid amongst free-by-cyclic groups.

Theorem 1. *The group $G \cong \langle a, b, c, t : a^t = b, b^t = c, c^t = ca^{-1}\rangle$ is profinitely rigid amongst free-by-cyclic groups.*

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Hughes and Kudlinska [10] have shown that if G has $b_1(G)=1$ then in many cases G is *almost* profinitely rigid: it shares a profinite isomorphism class with at most finitely many non-isomorphic free-by-cyclic groups. These cases include when the defining automorphism is irreducible (it has no non-trivial preserved free factor system) and when the rank of the kernel is 3. Building off work of Bridson, Reid and Wilton [8] they also show that if the kernel has rank 2 then G is profinitely rigid among free-by-cyclic groups. Note that in the rank 2 case G is never hyperbolic.

In a different direction Bridson and Piwek have very recently shown that free-by-cyclic groups with centre [5], Theorem 1.1], or where the cyclic group is instead required to be *finite* cyclic [5], Theorem 1.2], are profinitely rigid among respectively all free-by-(infinite cyclic) groups and free-by-(finite cyclic) groups.

Following the analogy between free-by-cyclic groups and three manifolds, we recall two cognate results on the profinite rigidity of three manifolds. First, the fundamental group of the figure-eight knot complement is profinitely rigid amongst all fundamental groups of three manifolds [6], Theorem A]: the proof goes via the identification of this manifold with a once punctured torus bundle over the circle, which algebraically corresponds to some $F_2 \rtimes \mathbb{Z}$. Second, and stronger, the fundamental group of the Weeks manifold (the unique closed orientable hyperbolic three manifold of minimal volume) is profinitely rigid, with no need to restrict the class concerned [3], Theorem 9.1].

Proof of the theorem. The proof will be an application of Hughes and Kudlinska's characterisation of properties detected by the profinite isomorphism class of a free-by-cyclic group (Theorem 2), together with Hillen's work (Corollary 3) controlling stretch factors of elements of $\mathrm{Out}(F_r)$:

Theorem 2 (Hughes–Kudlinska). *Suppose $G \cong F_r \rtimes \mathbb{Z}$ is a free-by-cyclic group that is hyperbolic and has $b_1(G)=1$. If $H \cong F_s \rtimes \mathbb{Z}$ is profinitely isomorphic to G, then*

- *H is hyperbolic (Theorem C of [10]).*
- *$r=s$ (Theorem B(1) of [10] and Lemma 3.1 of [6]).*
- *The set of stretch factors $\{\lambda_G^+, \lambda_G^-\}$ associated to the defining outer automorphism of G (and its inverse) agrees with $\{\lambda_H^+, \lambda_H^-\}$ (End of Theorem B of [10]).*

We briefly recap the concepts introduced within the theorem. Brinkmann showed that G is hyperbolic if and only if its defining automorphism is *atoroidal:* it has no periodic conjugacy classes [7], Theorem 1.2]. The stretch factor λ of an automorphism is defined to be

$$\sup_{w\in F_r} \limsup \sqrt[n]{\|\varphi^n(w)\|},$$

where $\|w\|$ is the cyclically reduced word length of the element w; it records "how fast" elements grow under repeated application of the automorphism, and is an $\mathrm{Out}(F_r)$-conjugacy class invariant. Elements of $\mathrm{Out}(F_r)$ can and often do have different stretch factors from their inverses, so we must record both. Note that the set $\{\lambda_H^+, \lambda_H^-\}$ is well defined since the profinite isomorphism class of a group determines its abelianisation (see for instance [13], Remark 3.2]), and in particular H must also have $b_1(H)=1$ and so there is a unique (up to inverting and $\mathrm{Out}(F_r)$-conjugacy) defining automorphism for H.

Corollary 3. *[9], Corollary 8.1] The element ψ of $\mathrm{Out}(F_3)$ defined by sending $a \mapsto b, b \mapsto c, c \mapsto ca^{-1}$ defines the unique $\mathrm{Out}(F_3)$-conjugacy class of infinite order irreducible elements realising the minimal stretch factor $\lambda \approx 1.167$.*

Proof of Theorem 1. First we check that G satisfies the hypotheses of Theorem 2. A quick computation with the abelianisation verifies that $b_1(G)=1$, while hyperbolicity follows from Brinkmann's theorem [7], Theorem 1.2] since the single fold representative has no periodic Nielsen paths so there cannot be a periodic conjugacy class.

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Now, suppose H is another free-by-cyclic group that is profinitely isomorphic to G. From Theorem 2 we know that H is some $F_3 \rtimes \mathbb{Z}$, that it is hyperbolic, and that the defining automorphism and its inverse will have the same stretch factor(s) as ψ and its inverse; in particular one of them must be λ. Let φ be the choice with smaller stretch factor.

It follows from Brinkmann's theorem that φ is atoroidal. We also observe, as for instance in the proof of [10], Corollary F], that if an element of $\mathrm{Out}(F_3)$ is atoroidal then it must be irreducible (the point is that a preserved free factor would have rank 1 or 2, and in either case there is a periodic conjugacy class). But then, by Corollary 3, we have that φ and ψ are conjugate as elements of $\mathrm{Out}(F_3)$, and so they define isomorphic free-by-cyclic groups. □

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