DENOMINATOR SEQUENCES OF CONTINUED FRACTIONS I

R. T. WORLEY

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Let α be an irrational number with simple continued fraction $\alpha = (a_0, a_1, a_2, \cdots)$. The problem studied is that of whether the sequence (q_n) of denominators of the convergents p_n/q_n to α has a subsequence $(B_n) = (q_{i_n}\}$ which is the sequence of denominators of convergents A_n/B_n to a different number α'. In other words, does there exist a subsequence $\{q_{i_n}\}$ which satisfies $q_{i_0} = 1$ and

(0) $$q_{i_{n+2}} \equiv q_{i_n} (\bmod q_{i_{n+1}});\ n \geqq 0?$$

For example, the sequence of denominators of convergents to $\frac{1}{2}(3-e)$ is a subsequence of the sequence of denominators of convergents to e.

In what follows we shall preserve the notation introduced above. We shall show that if the continued fraction for α satisfies a condition a little more general than periodicity then there usually exists at least one α' for which the denominators B_n of convergents form, apart from an initial few, a subsequence of the sequence q_n. Furthermore, either $B_{n+1} \geqq \frac{1}{2}B_n^2$ infinitely often or $\alpha' = a\alpha + b$ for rational numbers a and b.

We define, for odd integers $p \geqq 3$, the continued fraction $(a_0, a_1, a_2, \cdots)$ as nearly periodic with period (p, r) if $r \geqq 0$ is an integer such that for each integer $n \geqq 1$ at least one of the conditions

(1) (i) $$a_{np+r-i} = a_{(n+1)p+r-i} \qquad (0 \leqq i \leqq p-2)$$

(ii) $$a_{np+r-i} = a_{np+2+r+i} \qquad (0 \leqq i \leqq p-2)$$

holds. In other words the sequence $a_{(n-1)p+2+r}, \cdots a_{np+r}$ is repeated in the same or reverse order after a_{np+r+1}. For example the continued fraction for e, $(2, 1, 2, 1, 1, 4, \cdots)$, is nearly periodic with period $(3, 1)$. Note that r is not restricted to be less than p, so the nearly periodic property is not determined by the initial elements.

LEMMA 1. *Let k, l and m be positive integers with $m > l$. Define P_m, Q_m, R_m and S_m by*

$$P_m/Q_m = (0, a_{m+2}, \cdots, a_{m+k}),$$
$$R_m/S_m = (0, a_m, a_{m-1}, \cdots, a_{m-l+2})$$

where the right hand sides are to be interpreted as $0/1$ *if* $k = 1$ *or* $l = 1$. *Then*

$$q_{m+k} = P_m q_m + Q_m q_{m-1} + Q_m a_{m+1} q_m,$$
$$q_{m-l} = (-1)^{l-1}(-R_m q_m + S_m q_{m-1}),$$

and similarly with the q_i *replaced by* p_i.

PROOF. This is a direct consequence of the relations between the q_i and between the p_i.

COROLLARY. *If the continued fraction for* α *is nearly periodic with period* (p,r) *then for each integer* $n \geqq 1$,

(2) (i) $$q_{(n+1)p+r} = c_{n-1} q_{np+r} + q_{(n-1)p+r}, \quad \text{and}$$

(ii) $$p_{(n+1)p+r} = c_{n-1} p_{np+r} + p_{(n-1)p+r},$$

where the c_n *are positive integers.*

PROOF. Apply the lemma with $m = np + r$, $k = l = p$, taking

$$c_{n-1} = Q_m a_{m+1} + P_m + R_m,$$

observing that conditions (1) imply that $Q_m = S_m$ since $(0, a_m, \cdots, a_{m-p+2})$ and $(0, a_{m-p+2}, \cdots, a_m)$ have the same denominator.

LEMMA 2. *Let* $d_1, \cdots, d_{s-1}$ *be positive integers, let* X_0, X_1, Y_0, Y_1 *be non-negative integers and let* $X_2, \cdots, X_s$, $Y_2, \cdots, Y_s$ *be defined inductively by*

$$X_m = d_{m-1} X_{m-1} + X_{m-2} \qquad (2 \leqq m \leqq s),$$
$$Y_m = d_{m-1} Y_{m-1} + Y_{m-2} \qquad (2 \leqq m \leqq s).$$

Then

$$\frac{X_s}{Y_s} = \frac{X_1 \beta_s + X_0}{Y_1 \beta_s + Y_0}$$

where $\beta_s = (d_1, d_2, \cdots, d_{s-1})$.

PROOF. The result is trivial for $s = 2$. For $s > 2$ the result is proved by induction in a similar manner to the particular case in simple continued fraction theory.

THEOREM. *Let* α *be such that for positive integers* $c_0, c_1, c_2, \cdots$

(3) (i) $$q_{i_{n+1}} = c_{n-1} q_{i_n} + q_{i_{n-1}} \qquad (n \geqq 1), \quad \textit{and}$$

(ii) $$p_{i_{n+1}} = c_{n-1} p_{i_n} + p_{i_{n-1}} \qquad (n \geqq 1),$$

for a subsequence (q_{i_n}) of (q_i). If q_{i_0} and q_{i_1} are relatively prime then there exists α' of the form $a\alpha + b$ with a and b rational such that $B_{n+u} = q_{i_n}$ for $n \geqq 0$, where $u \geqq 0$ is integral and $u = 0$ if $q_{i_0} = 1$.

PROOF. If $q_{i_0} = 1$ then plainly the q_{i_n} are the denominators to $\alpha' = (0, q_{i_1}, c_0, c_1, c_2, \cdots)$. However if $q_{i_0} > 1$, define $q_{i_{-1}}, q_{i_{-2}}, \cdots$, and $c_{-1}, c_{-2}, \cdots$, inductively by

$$q_{i_{j+2}} = c_j q_{i_{j+1}} + q_{i_j} \qquad (0 \leqq q_{i_j} \leqq q_{i_{j+1}} - 1)$$

where the process is terminated when $q_{i_t} = 0$ is reached. Then it is easy to verify that the denominators of the convergents to $\alpha' = (0, c_t, c_{t+1}, \cdots)$ are precisely $q_{i_{t+1}}, q_{i_{t+2}}, \cdots$.

To find the relation between α and α' we observe that, by lemma 2 with $\beta = (c_0, c_1, \cdots)$

$$\alpha = \lim_{n \to \infty} p_{i_n}/q_{i_n} = (p_{i_1}\beta + p_{i_0})/(q_{i_1}\beta + q_{i_0}) \tag{4}$$

and

$$\alpha' = (y\beta + x)/(q_{i_1}\beta + q_{i_0})$$

where $y/q_{i_1} = (0, c_t, \cdots, c_{-1})$ and $x/q_{i_0} = (0, c_t, \cdots, c_{-2})$, these expressions being interpreted as $1/q_{i_1}$ and $0/1$ if $q_{i_0} = 1$. Then comparing the above formulae gives

$$\alpha' = \{p_{i_0}q_{i_1} - p_{i_1}q_{i_0}\}^{-1}\{\alpha\{xq_{i_1} - yq_{i_0}\} + yp_{i_0} - xp_{i_1}\}$$

which shows α' to be of the desired form. It remains to set $u = -(t+1)$.

It should be remarked that if $(q_{i_0}, q_{i_1}) = d > 1$ then $d \mid q_{i_n}$ for $n \geqq 2$ as well. Replacing the q_{i_j} by $q_{i_j}^* = q_{i_j}d^{-1}$ we have 3(i) holding for the $q_{i_j}^*$. Since $(q_{i_j})^2 \left|\alpha - p_{i_j}/q_{i_j}\right| < 1$ implies $(q_{i_j}^*)^2 \left|d\alpha - p_{i_j}/q_{i_j}^*\right| < d^{-1} \leqq \frac{1}{2}$, then $p_{i_j}/q_{i_j}^*$ is a convergent to $\alpha^* = d\alpha$. It follows that the $q_{i_j}^*$ are denominators of convergents to some number α^*, i.e. that the q_{i_j} are multiples by $d = (q_{i_1}, q_{i_0})$ of a subsequence of denominators of convergents to α^*.

COROLLARY. *If the continued fraction for α is nearly periodic with period (p, r), where $p \geqq 3$ is odd, then*

(a) *If $r = 0$ or $r = a_1 = 1$ then $q_r = 1$ and the theorem, with $i_n = pn + r$, yields the existence of α' the denominators of convergents to which form a subsequence $(B_n) = (q_{i_n})_{n \geqq 0}$ of the denominator sequence of convergents to α,*

(b) *If $(q_r, q_{p+r}) = 1$ then the theorem, with $i_n = pn + r$, yields the existence of α' the denominators of convergents to which form, apart from an initial few, a subsequence $(B_{n+u}) = (q_{i_n})_{n \geqq 0}$ of the denominator sequence of convergents to α, and*

(c) *If $(q_r, q_{p+r}) = d > 1$ then by the remarks above there exists α' the denominators of convergents to which, apart from an initial few, when multi-*

plied by d form a subsequence of the denominator sequence of convergents to α.

It will be observed that by lemma 1 $(q_{p+r}, q_r) = (Q, q_r)$ where Q is the denominator of $(0, a_{r+2}, \cdots, a_{r+p})$, and so Q and q_r are completely independent of one another.

REMARKS. (i) If we take $\frac{1}{2}(\sqrt{5}-1)$ as our α, so that the q_n are the Fibonacci numbers, then the continued fraction for α is nearly periodic with periods $(p, 0)$ and $(p, 1)$ for any odd number p. Thus there are infinitely many α' for this choice of α.

(ii) If α has an ultimately periodic continued fraction then by the result of Schmidt [1] it follows that any α' must be of the form $a\alpha + b$ for rationals a and b unless it is transcendental.

(iii) If we take $e - 1 = (1, 1, 2, 1, 1, 4, 1, 1, 6, \cdots)$ as α then the continued fraction is nearly periodic with period $(3, 1)$, and we find $x = 0$, $y = 1$, $c_j = 10 + 4j$ and $\beta = (10, 14, 18, 22, \cdots)$. Hence using (4) we obtain a proof that $e - 1 = (1, 1, 2, 1, \cdots)$ given that $(0, 2, 6, 10, \cdots) = (e-1)/(e+1)$.

(iv) If some terms are dropped from the beginning of the sequence (q_{i_n}), α' as defined in the proof of the theorem is unaffected. Hence if an α' exists which is not of the form $a\alpha + b$ for rational a and b then the sequence i_n defined by $B_{n+u} = q_{i_n}$ must violate 3(ii) infinitely often where c_{n-1} is defined by 3(i). If Q_{i_n} and S_{i_n} are obtained by setting $m = i_n$, $k = i_{n+1} - i_n$, $l = i_n - i_{n-1}$ in the formulae of lemma 1, then plainly $Q_{i_n} - (-1)^{l-1} S_{i_n} = Q_{i_n} \pm S_i$ must be non-zero infinitely often, for when it is zero 3(i) and 3(ii) hold. But q_{i_n} divides $q_{i_{n+1}} - q_{i_{n-1}}$ and is relatively prime to $q_{i_{n-1}}$. Hence, infinitely often, $|Q_{i_n} \pm S_{i_n}| \geqq q_{i_n}$. But $Q_{i_n} \geqq \frac{1}{2} q_{i_n}$ implies that $q_{i_{n+1}} \geqq \frac{1}{2} q_{i_n}^2$, and $S_{i_n} \geqq \frac{1}{2} q_{i_1}$ implies $Q_{i_{n-1}} \geqq \frac{1}{2} q_{i_{n-1}}$ $(S_{i_n} = Q_{i_{n-1}})$ which in turn yields $q_{i_n} \geqq \frac{1}{2} q_{i_{n-1}}^2$. Thus we may conclude that if α' is not of the form $a\alpha + b$ for rational a and b then $B_{n+1} \geqq \frac{1}{2} B_n^2$ infinitely often.

(v) If α is a quadratic irrational it is easy to construct an α' not of the form $a\alpha + b$ for rational a and b since we only need to ensure $q_{i_{n+1}} \equiv q_{i_{n-1}} \bmod q_{i_n}$ and that $q_{i_{n+1}} \geqq \frac{1}{2} q_{i_n}^2$ (or, indeed, any sufficiently rapidly increasing function of q_{i_n}). The α' defined by such a sequence cannot be a quadratic irrational and so certainly cannot be of the form $a\alpha + b$. Such a sequence (q_{i_n}) can be chosen since the congruence classes $\bmod\, q_{i_n}$ of the q_j recur in cyclic pattern.

(vi) It is possible to construct an α, not a quadratic irrational, having an α' for which $B_n \geqq \frac{1}{2} B_{n+1}^2$ infinitely often, but in this case the transcendence of α' does not guarantee the non-existence of a relation of the type $\alpha' = a\alpha + b$ for a and b rational.

(vii) It can be deduced that if α and α_1 are such that they have a common subsequence (B_n) of the sequence of denominators of convergents satisfying

$B_n < \frac{1}{2}B_{n+1}^2$ for all but finitely many n, then $\alpha_1 = a\alpha + b$ for rational a and b. This raises the problem of how dense a common subsequence of denominators, with or without extra conditions, must be in order to guarantee $\alpha_1 = a\alpha + b$ for rationals a and b. Of course it may be assumed that one of α and α_1 is transcendental, as the result of Schmidt shows that if α, α_1 are both algebraic with an infinite number of common denominators then $\alpha_1 = a\alpha + b$ with a and b rational.

(viii) It should be noted that the sequence of fractional parts $\{B_n\alpha\}$ converges to 0 mod 1. Thus if α and α' are as in (v) then $1, \alpha, \alpha'$ are independent over the rationals and the sequence $\{B_n\alpha\}$ is not dense in $[0, 1]$. This result sheds little light on the unsolved problem of whether or not the sequence $\{F_n\sqrt{3}\}$ is dense in $[0, 1]$, where the Fibonacci numbers F_n are the denominators of the convergents to $\frac{1}{2}(\sqrt{5}-1)$, since though 1, $\frac{1}{2}(\sqrt{5}-1), \sqrt{3}$ are independent over the rationals, the second of these is a quadratic irrational whereas α' from (v) is transcendental.

Reference

[1] W. M. Schmidt, 'On simultaneous approximations of two algebraic numbers by rationals' *Acta Math.* 119 (1967), 27–50.

Monash University
Clayton, Vic. 3168
Australia

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