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ON HOMEOMORPHISMS OF THE UNIT CIRCLE PRESERVING ORIENTATION

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1. Introduction

Let Γ denote the unit circle in the complex plane $\boldsymbol{C}$, $C(\Gamma)$ the set of complex valued continuous functions on Γ which is a Banach space by the sup-norm $\|\cdot\|$, $A(z)$ the uniform closure of all polynomials in z on Γ, $H(\Gamma)$ the set of homeomorphisms of Γ, $H^+(\Gamma)$ the set of direction-preserving homeomorphisms and $H^-(\Gamma)$ the set of direction-reversing homeomorphisms. For $\psi \in H(\Gamma)$, let $A(\psi)$ denote the uniform closure of all polynomials in ψ on Γ.

Any map ψ belonging to $H^-(\Gamma)$ has the following property (A) (Browder and Wermer [2]):

(A) $\overline{A(z) + A(\psi)} = C(\Gamma)$.

The purpose of this paper is to investigate direction-preserving homeomorphisms which have the property (A). We observe the following Lemma 1 which is, in essence, contained in Browder and Wermer [2].

Lemma 1. *Any map $\psi \in H^+(\Gamma)$ has the property* (A) *provided it enjoys the following property* (B):

(B) $A(z) \cap A(\psi) = \boldsymbol{C}$.

In view of this it is important for our purpose to classify when $\psi \in H^+(\Gamma)$ has the property (B). We will give one sufficient condition for $\psi \in H^+(\Gamma)$ to the property (B) as follows:

Theorem 1. *If, for a given map $\psi \in H^+(\Gamma)$, there exists a Blaschke product B such that $\psi \neq B$ and the linear measure of $\{z \in \Gamma \mid \psi(z) = B(z)\}$ is positive, then ψ has the property* (B).

Corollary. *The set of maps in $H^+(\Gamma)$ possessing the property* (B) *is*

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dense in the space $H^+(\Gamma)$.

We mention here an example of maps in $H^+(\Gamma)$ which do not have the property (A). Consider the linear transformation

$$U(z; a) = \frac{z-a}{1-\bar{a}z} \qquad |a| < 1 .$$

Clearly U belongs to $H^+(\Gamma)$ and $A(U) = A(z)$ and a fortiori U does not have the property (A). We will later give another such examples in Theorem 2, among which a typical one is following: let $B(z) = \prod_{k=1}^{n} U(z; a_k)$ be with all different a_k. Then the map $(B(z))^{1/n}$ is an example.

In connection with the welding theory of Riemann surfaces, we will give an example which has the property (B). This example will be constructed from the Jordan curve γ in the following

THEOREM 3. *On any Jordan curve* γ *which contains a line segment, there dose not exist any nonconstant function which is bounded and continuous on the complex plane, analytic in the interior of and anti-analytic in the exterior of* γ.

Finally we will prove the following theorem in no. 5:

THEOREM 4. *If, for a given map* $\psi \in H(\Gamma)$, *there exists a nowhere constant function* $f \in C(\Gamma)$ *with the property* $f(\psi) = f$, *then there exists an integer* n *with the property* $\psi \circ \psi \circ \cdots \circ \psi$ *(*n *compositions)* $= z$, *where a function* $f \in (\Gamma)$ *is said to be nowhere constant if* f *is nonconstant on any open set of* Γ.

2. Proof of Theorem 1

For $\alpha, \beta \in \Gamma$ $\alpha \neq \beta$, there exist two arcs of Γ whose end points are α and β. Among them we denote by (α, β) the arc starting from α and ending at β in the positive direction and $[\alpha, \beta]$ the closed arc $\{\alpha\} \cup (\alpha, \beta) \cup \{\beta\}$. Although the proof of Lemma 1 is contained in the proof of Theorem of Browder and Wermer in [2] we include it here for the sake of convenience to the readers.

Proof of Lemma 1. Assuming the conclusion is false, by the Hahn-Banach theorem, there exists a non-zero measure on Γ with the following property:

$$\int_{|z|=1} z^n d\mu = 0\,, \quad \int_{|z|=1} \psi^n d\mu = 0 \qquad (n = 0, 1, 2, \cdots)\,.$$

By the former condition of the above, we have the Riesz's representation

$$d\mu = h(z)dz \qquad h \in L^1(\Gamma)\,.$$

If we set

$$H(z) = \int_{[1,z]} h(\zeta)d\zeta\,,$$

then $H \in C(\Gamma)$ and

$$\int_{|z|=1} z^n H(z)dz = \int_{|z|=1} z^n \int_{[1,z]} h(\zeta)d\zeta dz = \int_{|\zeta|=1} h(\zeta) \int_{[\zeta,1]} z^n dz d\zeta = 0\,.$$

Therefore $H \in A(z)$. Clearly H is non-constant and

$$\int_{|z|=1} H(\psi^{-1})z^n dz = \int_{|w|=1} H(w)\psi^n(w)d\psi(w)$$
$$= -\frac{1}{n+1}\int_{|w|=1} \psi^{n+1}(w)h(w)dw = 0\,.$$

Hence $H(\psi^{-1}) \in A(z)$. If we set $H(\psi^{-1}) = G$, then $G(\psi) = H$ and $A(\psi) \cap A(z) \neq C$. This is a contradiction.

The proof of Theorem 1. Suppose that $A(z) \cap A(\psi) \neq C$. There exist non-constant functions $f \in A(z) \cap A(\psi)$ and $g \in A(z)$ with the property $f = g(\psi)$. Since $g(B) \in A(z)$ and $B(z) = \psi(z)$ for $z \in E$, $f(z) = g(B(z))$ for $z \in E$. By the Fatou theorem, $f = g(B)$ and $g(\psi) = g(B)$. In view of $\psi(\alpha) = B(\alpha)$ and the following Lemma 2, $\psi = B$. This is a contradiction.

LEMMA 2. *Let the function $\tau(t)$ be monotone increasing in $\{t \geqq 0\}$ and satisfy $\tau(0) = 0$. If there exists a nowhere constant function f in $\{t \geqq 0\}$ with the property $f(\tau(t)) = f(t)$ for $t \geqq 0$, then $\tau(t) = t$.*

Proof. We set $F = \{t \geqq 0 | \tau(t) = t\}$. Since F is closed, $\{t \geqq 0\} - F$ consists of countably many disjoint open intervals. Among them, we choose an arbitrary interval (a, b). Assuming $\tau(t) > t$ on (a, b), there exists t_0 such that $a < t_0 < b$, $a < \tau(t_0) < b$ and $f(\tau(t_0)) \neq f(a)$. The sequence $\{\tau(t_0), t_0, \tau^{-1}(t_0), \tau^{-1} \circ \tau^{-1}(t_0), \cdots\}$ is contained in (a, b) and monotone decreasing. We denote by c the limit point of this sequence. Since $f(c) = f(\tau(t_0)) \neq f(a)$, we conclude that $c \neq a$. On the other hand, $\tau(c) = c$, a contradiction. Similarly it does not hold that $\tau(t) < t$.

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Proof of Corollary. Given $\psi \in H^+(\Gamma)$ and $\varepsilon > 0$, there exist two different points α and β belonging to Γ such that $|\psi(z_1) - \psi(z_2)| < \varepsilon/2$ for $z_1, z_2 \in (\alpha, \beta)$. Take a point c in (α, β). We denote by ϕ a linear transformation which is a map from Γ to Γ and maps α to $\psi(\alpha)$, β to $\psi(\beta)$ and c to any point of $(\psi(\alpha), \psi(\beta))$. The map ϕ is a finite Blaschke product. We denote by ψ^* a map which is equal to ϕ on $[\alpha, \beta]$ and ψ on $[\beta, \alpha]$. The map ψ^* belongs to $H^+(\Gamma)$ and, by choosing $\psi^*(c)$ suitably, satisfies the conditions in Theorem 1 and $|\psi^* - \psi| < \varepsilon$.

3. Examples which do not satisfy the property (A)

For a finite Blaschke product $B(z) = \prod_{k=1}^{n} U(z; a_k)$ $|a_k| < 1$, we set $\psi(z) = B(z)^{1/n}$. The map $\psi(z)$ belongs to $H^+(\Gamma)$ and does not satisfy the property (A) because of the following Theorem 2. We denote by $L^1(\Gamma)$ the set of integrable functions on Γ, $C^1(\Gamma)$ the set of continuously differentiable functions on Γ and $H^1(|z| < 1)$ the subset of $L^1(\Gamma)$ with the following property:

$$\int_{|z|=1} f(z) z^n dz = 0 \,, \qquad n = 0, 1, 2, \cdots .$$

For a function $f(z)$ defined on Γ we denote by $\dfrac{\delta f}{\delta z}$ the limit

$$\lim_{y \to z} \frac{f(y) - f(z)}{y - z} \,,$$

if it exists. The differential operator $\dfrac{\delta}{\delta z}$ has the following properties:

1° For $z = e^{i\theta}$, $\dfrac{\delta f}{\delta z}(z) = -ie^{-i\theta} \dfrac{d}{d\theta}(f(e^{i\theta}))$.

2° For f $C^1(\Gamma)$, $\displaystyle\int_{[1,z]} \frac{\delta f}{\delta z}(z) dz = f(z) - f(1)$.

3° If f is analytic at $z \in \Gamma$, then $\dfrac{\delta f}{\delta z}(z) = \dfrac{df}{dz}(z)$.

4° For $f \in C^1(\Gamma)$ and $\phi \in H(\Gamma) \cap C^1(\Gamma)$, if we set $z = \phi(\zeta)$,

$$\frac{\delta}{\delta \zeta}\{f(\phi(\zeta))\} = \frac{\delta f}{\delta z}(\zeta) \,.$$

THEOREM 2. *If a map $\psi \in H^+(\Gamma)$ is conformal on some neighborhood of Γ and there exists a non-constant function $f(z)$ on Γ with the following*

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property:

$$\frac{\delta f}{\delta z}(z) \in H^1(|z| < 1) \quad and \quad f(\psi) \in A(z)\ ,$$

then the map ψ *does not have the property* (A).

Proof. We set

$$d\mu = \frac{\delta}{\delta z}\{f(\psi(z))\}dz\ .$$

We will show that the measure μ satisfies

$$\int_{|z|=1} z^n d\mu = 0\ , \qquad \int_{|z|=1} \psi^n d\mu = 0 \qquad (n = 0, 1, 2, \cdots)\ .$$

Since $f(\psi) \in A$, we have

$$\int_{|z|=1} z^n d\mu = \int_{|z|=1} z^n \frac{\delta}{\delta z}\{f(\psi(z))\}dz$$
$$= -n\int_{|z|=1} f(\psi(z))z^{n-1}dz = 0\ .$$

We set

$$g(z) = \int_{[1,z]} \zeta^n \frac{\delta}{\delta \zeta}(\zeta)d\zeta\ .$$

By the former property of f, the function $g(\dot{z})$ belongs $A(z)$. Therefore

$$\int_{|z|=1} z^m g(z)dz = 0 \quad \text{and} \quad \frac{\delta g}{\delta z} = z^n\frac{\delta f}{\delta z}\ .$$

Since the map ψ is conformal on Γ,

$$\frac{\delta}{\delta \zeta}\{f(\psi(\zeta))\} = \frac{\delta f}{\delta z}(\psi(\zeta))\,\frac{d\psi}{dz}(z)$$

and

$$\frac{\delta}{\delta \zeta}g(\psi(\zeta)) = \frac{\delta g}{\delta z}(\psi(\zeta))\frac{d\psi}{dz}(\zeta)$$
$$= \psi^n(\zeta)\frac{\delta f}{\delta z}(\psi(\zeta))\frac{d\psi}{dz}(\zeta) = \psi^n(\zeta)\frac{\delta}{\delta \zeta}\{f(\psi(\zeta))\}\ .$$

Therefore

$$\int_{|\zeta|=1} \psi^n(\zeta)d\mu(\zeta) = \int_{|\zeta|=1} \psi^n(\zeta)\frac{\delta}{\delta\zeta}\{f(\psi(\zeta))\}d\zeta$$
$$= \int_{|\zeta|=1} \frac{\delta}{\delta\zeta}\{g(\psi(\zeta))\}d\zeta = 0 .$$

4. Welding

Given a Jordan curve γ on the complex plane, we denote by Ω (Ω^*, resp.) the interior of γ (the exterior of γ, resp.), by D (D^*, resp.) the interior of Γ (the exterior of Γ^*, resp.) and by χ (χ^*, resp.) a Riemann's conformal map from D (D^*, resp.) to Ω (Ω^*, resp.) which is also a homeomorphism on the closure of the given region. From now on we assume that $\chi(1) = \chi^*(1)$. If we set

$$\psi(e^{i\theta}) = \chi^{*^{-1}} \circ \chi(e^{i\theta}) ,$$

then $\psi \in H^+(\Gamma)$. We denote by $H^+_w(\Gamma)$ all of $\psi \in H^*(\Gamma)$ with $\psi = \chi^{*^{-1}} \circ \chi$ for some Jordan curve γ. By the theorem of Oikawa [5], if we define a map ψ by z^3 on $[1, e^{2\pi i/5}]$ and $\sqrt{z}$ on $[e^{2\pi i/5}, 1]$ whose branches are chosen in such a way that the map ψ is continuous, then $\psi \in H^+(\Gamma)$ and $\psi \notin H^+_w(\Gamma)$. For $\psi \in H^+_w(\Gamma)$, there exist infinitely many Jordan curves corresponding to ψ, and among them we choose a certain γ and γ'. Then there is a homeomorphism Φ on C which maps γ onto γ' and the interior of γ (the exterior of γ, resp.) onto the interior of γ' (the exterior of γ', resp.) and is analytic off γ. The map Φ is not necessarily conformal on C. For example, if the area of γ is positive, the map Φ is not conformal for some γ ([5]). By the welding theory ([5]), it is sufficient for $\psi \in H^+(\Gamma)$ to belong to $H^+_w(\Gamma)$ that the map ψ has the following condition: for any $z \in \Gamma$, there exist $\varepsilon > 0$ and $\rho > 0$ (dependant on z) such that for any ζ and t with $(\zeta e^{-it}, \zeta e^{it}) \subsetneqq (ze^{-i\varepsilon}, ze^{i\varepsilon})$

$$\frac{1}{\rho} \leqq \left| \frac{\psi(\zeta e^{it}) - \psi(\zeta)}{\psi(\zeta) - \psi(\zeta e^{-it})} \right| \leqq \rho .$$

By virtue of this theorem, if ψ belongs to $C^1(\Gamma)$ and satisfies $\frac{\delta\psi}{\delta z}(z) \neq 0$, or if Γ is divided into finite intervals and on each interval ψ is equal to a linear transformation, then ψ belongs to $H^+_w(\Gamma)$.

DEFINITION. If there exist no non-constant functions that are bounded and continuous on C, analytic in the interior of γ and anti-analytic in the exterior of γ, then we denote the fact by $\gamma \in 0$.

LEMMA 3. *Given $\psi \in H_w^+(\Gamma)$ and choose a Jordan curve γ correspondent to ψ. Then ψ has the property* (B) *if and only if $\gamma \in 0$.*

Proof. We suppose that $f = g(\psi)$, where f and $g \in A$. Since $f = g(\chi^{*-1} \circ \chi)$ on Γ, we have $f(\chi^{-1}) = g(\chi^{*-1}) = g(1/(\bar{\chi}^*)^{-1})$ on γ. Therefore, by observing that the function is equal to $f(\chi^{-1})$ in the interior of γ and $g(1/(\bar{\chi}^*)^{-1})$ in the exterior of γ, we see the validity of our lemma.

THEOREM 3. *Any Jordan curve containing a line segment satisfies* $\gamma \in 0$.

Proof. We suppose that the line segment is on the real axis. There exists an open disk D such that the center of D is on the real axis and D does not have common points with γ except for the line segment. We denote by Ω the interior of γ, Ω^* the exterior of γ and Ω_1^* the domain obtained from Ω^* by reflecting it with respect to the real axis. We take a function f which is bounded and continuous on $\boldsymbol{C}$, analytic in the interior of γ and anti-analytic in the exterior of γ. The function $f(\bar{z})$ is analytic in Ω_1^* and continuous on $\bar{\Omega}_1^*$. By considering $f(z)$ and $f(\bar{z})$ in $\Omega \cap D$, we see that $f(z)$ and $f(\bar{z})$ satisfy $f(z) = f(\bar{z})$ on $\partial\Omega \cap D$ and are analytic in $\Omega \cap D$. By Fatou's theorem, $f(z) = f(\bar{z})$ in $\Omega \cap D$ and $f(z)$ is analytic on $\Omega_1 \cap \Omega^*$. We denote by Ω_1 the domain obtained from Ω by reflection and by Ω' the component of $\overline{\Omega \cup \Omega_1}^c$ containing the point at infinite. The domain Ω' is a Jordan region, whose boundary will be denoted by γ'. The Jordan curve γ' is symmetric with respect to real axis. If two points z_1 and z_2 on γ' satisfy $z_2 = \bar{z}_1$, then one of them is on γ. We may suppose that $z_1 \in \gamma$. There exists a curve ℓ which connects the center of D with z_1 and is contained in Ω^* except for its end points. If we denote by ℓ_1 the curve obtained from ℓ by reflection with respect to the real axis, ℓ_1 connects the center of D with z_2 and is contained in Ω_1^* except for its end points. Since $f(z) = f(\bar{z})$ in $\Omega_1^*, f(z_1) = f(z_2)$. Hence f is analytic Ω', bounded and continuous on $\bar{\Omega}'$ and satisfies $f(z) = f(\bar{z})$ on γ'. Since Ω' is symmetric, the analytic function $f(z)$ and the anti-analytic function $f(\bar{z})$ has the same boundary values, the function $f(z)$ is constant.

5. Proof of Theorem 4

Since $\psi \circ \psi \in H^+(\Gamma)$ for $\psi \in H^-(\Gamma)$, we will show the theorem in the case of $\psi \in H^+(\Gamma)$. For simplicity we denote by ψ^n an n-iterated map $\psi \circ \psi \circ$

$\cdots \circ \psi$.

LEMMA 4. *Under the assumption of Theorem* 4, *if* $\psi \neq z$, *then* $f([x, \psi(x)]) = f(\Gamma)$.

Proof. We will show that $f([x, \psi(x)]) \subset f([\psi(x), x])$ for any x and any $\psi \in H^+(\Gamma)$ with $f(\psi) = f$. Then from $\psi^{-1} \in H^+(\Gamma)$ and $f = f(\psi^{-1})$, it follows that the reverse inclusion holds if we take $\psi(x)$ in place of x. We suppose that $\alpha \in f([x, \psi(x)])$ and $\alpha \notin f([\psi(x), x])$. When three points α, β and x satisfy $[x, \alpha] \subset [x, \beta]$, we say that α is closer to x than β. We denote by t_0 the closest point to x among $\{t \in [x, \psi(x)] \mid f(t) = \alpha\}$. The map ψ sends (x, t_0) to $(\psi(x), (t_0))$ preserving direction. Since $f(\psi(t_0)) = f(t_0) = \alpha$ and $\alpha \notin f([\psi(x), x]$, $\psi(t_0) \notin [\psi(x), x]$ and $\psi(t_0) \in [x, \psi(x)]$. If $t_0 = \psi(t_0)$, then from Lemma 2 it follows that $\psi = z$. We may assume that $\psi(t_0) \in (t_0, \psi(x))$. It follows that $t_0 \in (\psi(x), \psi(t_0))$ and $\psi^{-1}(t_0) \in (x, t_0)$. Since $f(\psi^{-1}(t_0)) = \alpha$, this contradicts that t_0 is the closest to x.

Proof of Theorem 4. If there exist a point $x \in \Gamma$ and an integer n with $\psi^n(x) = x$, then $\psi^n = z$ because of $\psi^n \in H^+(\Gamma)$ and Lemma 2. We now discuss the rest. Given $x_0 \in \Gamma$, we may assume that $f(x_0) = 0$. From $f \in c(\Gamma)$, it follows that there exists a positive number δ such that $|f(x) - f(y)| < 1/2\|f\|$ for $|x - y| < \delta$. The sequence $\{\psi^n(x_0)\}$ has the following property: (1) if $m \neq n$, $\psi^m(x_0) \neq \psi^n(x_0)$, (2) $f(\psi^n(x_0)) = f(x_0) = 0$. If we take m and n with $|\psi^m(x_0) - \psi^n(x_0)| < \delta$, then any $y \in (\psi^m(x_0), \psi^n(x_0))$ satisfies $|f(y)| < 1/2\|f\|$. But from $\psi^{n-m} \in H^+(\Gamma)$ and Lemma 4 it follows that $f([\psi^m(x_0), \psi^n(x_0)]) = f(\Gamma)$, this is a contradiction.

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