CODIMENSION ONE FOLIATIONS IN POSITIVE CHARACTERISTIC

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Abstract We investigate the geometry of codimension one foliations on smooth projective varieties defined over fields of positive characteristic with an eye toward applications to the structure of codimension one holomorphic foliations on projective manifolds.

Contents

1. Introduction

The study of foliations (subsheaves of the tangent sheaf closed under Lie brackets) on algebraic varieties defined over fields of characteristic $p > 0$ traditionally focuses on foliations that are closed under p-th powers. For instance, Miyaoka studied these objects in order to establish the existence of rational curves on complex projective varieties in the presence of subsheaves of the tangent sheaf with positivity properties; see [35], [2, Chapter 9], and [36, Part I, Lecture III].

In the first part of this work, we focus instead on codimension one foliations which are not closed under p-th powers. We exploit the 20-year-old observation [38], [13, Theorem 6.2] that these foliations admit non-trivial infinitesimal transverse symmetries and, as such, can be defined by closed rational 1-forms, uniquely defined up to multiplication by constants of derivations (i.e., p-th powers of rational functions). The action of the Cartier operator on these 1-forms defines a companion distribution (not necessarily involutive subsheaf of the tangent sheaf), the Cartier transform of the foliation. The intersection of this distribution with the original foliation is a canonically defined subdistribution that we begin to study in this work.

Although we believe that the study of the geometry of foliations in positive characteristics is worth pursuing per se, our primary motivation is the potential for applications toward the study of holomorphic foliations in the spirit of earlier joint work of Loray, Touzet and the second author, [31, Section 7]. In the second part of this work, we use the results established in the first part to give new information about the space of codimension one foliations on projective spaces; see, for instance, Theorem 11.9, which exhibits previously unknown irreducible components, and Theorem 12.4, which characterizes the so-called logarithmic components.

2. Distributions and foliations

In this section, we present the definitions of distributions and foliations on arbitrary smooth varieties defined over an algebraically closed field k.

2.1. Distributions

A distribution $\mathcal{D}$ on a smooth algebraic variety consists of a pair $(T_{\mathcal{D}}, N_{\mathcal{D}}^*)$ of coherent subsheaves of T_X and Ω_X^1 such that $T_{\mathcal{D}}$ is the annihilator of $N_{\mathcal{D}}^*$ and $N_{\mathcal{D}}^*$ is the annihilator of $T_{\mathcal{D}}$. Explicitly,

$$T_{\mathcal{D}} = \{v \in T_X \,|\, \omega(v) = 0 \text{ for every } \omega \in N_{\mathcal{D}}^*\} \text{ and}$$
$$N_{\mathcal{D}}^* = \{\omega \in \Omega_X^1 \,|\, \omega(v) = 0 \text{ for every } v \in T_{\mathcal{D}}\}.$$

The sheaf $T_{\mathcal{D}}$ is the tangent sheaf of the distribution, and the sheaf $N_{\mathcal{D}}^*$ is the conormal sheaf of the distribution. It follows from the definition that both sheaves $T_{\mathcal{D}}$ and $N_{\mathcal{D}}^*$ are reflexive (i.e., canonically isomorphic to their bi-duals). The dimension of $\mathcal{D}$ is the generic rank of $T_{\mathcal{D}}$, and the codimension of $\mathcal{D}$ is the generic rank of $N_{\mathcal{D}}^*$.

The dual of $T_{\mathcal{D}}$ is the cotangent sheaf of $\mathcal{D}$ and will be denoted by $\Omega_{\mathcal{D}}^1$. Similarly, the dual of $N_{\mathcal{D}}^*$ is the normal sheaf of $\mathcal{D}$ and will be denoted by $N_{\mathcal{D}}$. These sheaves fit into the following exact sequences.

$$0 \longrightarrow T_{\mathcal{D}} \longrightarrow T_X \longrightarrow N_{\mathcal{D}}$$
$$0 \longrightarrow N_{\mathcal{D}}^* \longrightarrow \Omega_X^1 \longrightarrow \Omega_{\mathcal{D}}^1$$

For every i between 1 and $\dim \mathcal{D}$, we will define $\Omega_{\mathcal{D}}^i$ as the dual of $\wedge^i T_{\mathcal{D}}$; that is,

$$\Omega_{\mathcal{D}}^i = \text{Hom}\left(\bigwedge^i T_{\mathcal{D}}, \mathcal{O}_X\right).$$

Since the sheaves $\Omega_{\mathcal{D}}^i$ are defined as duals, they are reflexive sheaves. It is convenient to set $\Omega_{\mathcal{D}}^0 = \mathcal{O}_{\mathcal{D}}$ and $\Omega_{\mathcal{D}}^j = 0$ if $j \notin \{0, \dots \dim \mathcal{D}\}$.

We will denote the determinant of $\Omega_{\mathcal{D}}^1$ by $\omega_{\mathcal{D}}$; that is,

$$\omega_{\mathcal{D}} = \left(\bigwedge^{\dim \mathcal{D}} \Omega_{\mathcal{D}}^1\right)^{**}.$$

Since $T_{\mathcal{D}}$ is reflexive, [26, Proposition 1.10] implies that

$$\Omega_{\mathcal{D}}^1 \simeq \left(\bigwedge^{\dim \mathcal{D} - 1} T_{\mathcal{D}}\right)^{**} \otimes \omega_{\mathcal{D}}.$$

The singular set of $\mathcal{D}$ is the locus where $T_X/T_{\mathcal{D}}$ is not locally free and is denoted by $\text{sing}(\mathcal{D})$. Since $T_X/T_{\mathcal{D}}$ is torsion-free, the codimension of $\text{sing}(\mathcal{D})$ is at least two. Outside $\text{sing}(\mathcal{D})$, the rightmost arrows of both exact sequences above are surjective; see, for instance, [22, Section 2]. Taking determinants one gets an isomorphism of line-bundles

$$\omega_X \simeq \det(\Omega_{\mathcal{D}}^1) \otimes \det(N_{\mathcal{D}}^*), \tag{2.1}$$

where ω_X denotes the canonical sheaf of X.

Remark 2.1. Any coherent torsion-free sheaf injects into its double dual; see, for instance, [43, Tag 0AVT, Lemma 31.12.4]. Therefore, we can see $T_X/T_{\mathcal{D}}$ as a subsheaf of $N_{\mathcal{D}}$.

If $\mathcal{D}$ is a distribution of codimension q, then there exists $\omega \in H^0(X, \Omega_X^q \otimes \det(N_{\mathcal{D}}))$, without codimension one zeros, such that $T_{\mathcal{D}}$ is the kernel of the morphism

$$T_X \to \Omega_X^{q-1} \otimes \det(N_{\mathcal{D}})$$

defined by contraction with ω.

2.2. Foliations

A foliation $\mathcal{F}$ on a smooth algebraic variety X is a distribution with tangent sheaf $T_{\mathcal{F}}$ closed under Lie brackets. Explicitly, the $\mathcal{O}_X$-morphism (also called the O'Neill tensor)

$$\bigwedge^2 T_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$$

$$v \wedge w \mapsto [v,w] \mod T_{\mathcal{F}}$$

is identically zero. Here, we used Remark 2.1 to replace $T_X/T_{\mathcal{F}}$ by $N_{\mathcal{F}}$. Coherent subsheaves of T_X which are closed under Lie brackets are called involutive subsheaves.

Lemma 2.2. *Let $\mathcal{F}$ be a foliation of dimension r on a smooth algebraic variety X. If $x \notin \mathrm{sing}(\mathcal{F})$, then there exist $v_1, \ldots, v_r$ generators of $T_{\mathcal{F}} \otimes \mathcal{O}_{X,x}$ such that $[v_i, v_j] = 0$ for every $i, j \in \{1, \ldots, r\}$.*

Proof. See [13, Lemma 6.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

If X is a complex algebraic variety, then the vector fields given by Lemma 2.2 can be integrated, in an analytic or formal neighborhood of x, to a germ of action of an abelian Lie group of dimension r. The situation in positive characteristic is utterly different as will be discussed at length in Section 4.

A coherent subsheaf $\mathcal{L}$ of Ω_X^1 is called integrable if $d\mathcal{L}$ is contained in the saturation of $\mathcal{L} \wedge \Omega_X^1$ in Ω_X^2. Cartan's formula for the exterior derivative (see [10, Lemma 3] for its validity in positive characteristic) implies that the involutiveness of $T_{\mathcal{F}}$ is equivalent to the integrability of $N_{\mathcal{F}}^*$.

Example 2.3. The foliation $\mathcal{F}$ on $\mathbb{A}^2$ with conormal sheaf generated by $\omega = xdy - \lambda ydx$, λ different from 0 and -1, is such that $dN_{\mathcal{F}}^*$ is not contained in $N_{\mathcal{F}}^* \wedge \Omega_X^1$.

Proof. Notice that $d\omega = (1+\lambda)dx \wedge dy$. If $\lambda \neq 0,1$, then ω vanishes only at 0 ($\lambda \neq 0$) and $d\omega$ vanishes nowhere ($\lambda \neq -1$). Therefore, it does not exist in a regular 1-form η such that $d\omega = \eta \wedge \omega$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The integrability of $N_{\mathcal{F}}^*$ and the reflexiveness of $\Omega_{\mathcal{F}}^\bullet$ imply that the exterior derivative of differential forms on X descends to a k-linear morphism

$$d_{\mathcal{F}} : \Omega_{\mathcal{F}}^\bullet \to \Omega_{\mathcal{F}}^{\bullet+1}$$

which, like the exterior derivative, satisfies Leibniz's rule. Indeed, if $U \subset X$ is a sufficiently small open subset contained in $X - \mathrm{sing}(\mathcal{F})$ and ω belongs to $\Omega_{\mathcal{F}}^i(U)$, then we consider any lift $\hat{\omega}$ of ω to $\Omega_X^i(U)$ and define $d_{\mathcal{F}}\omega$ as the restriction of $d\hat{\omega}$ to $\wedge^{i+1} T_{\mathcal{F}}$. If $\hat{\omega}'$ is any other lift, then $\hat{\omega} - \hat{\omega}'$ belongs to $\Omega_X^{i-1}(U) \wedge N_{\mathcal{F}}^*(U)$ and the integrability condition implies that $d(\hat{\omega} - \hat{\omega}')$ vanishes on $\wedge^{i+1} T_{\mathcal{F}}$. If $j : X - \mathrm{sing}(\mathcal{F}) \to X$ denotes the inclusion, then we have just shown how to define a k-linear morphism $d_{\mathcal{F}}$ on $j^* \Omega_{\mathcal{F}}^\bullet$. The reflexiveness of $\Omega_{\mathcal{F}}^\bullet$

implies that $d_\mathcal{F}$ extends to a k-linear morphism defined on the whole X which inherits the Leibniz' property from d.

Every smooth variety X has two trivial foliations: one foliation of dimension zero given by the pair $(0, \Omega_X^1) \subset (T_X, \Omega_X^1)$, called the foliation by points, and one foliation of dimension $\dim X$ given by the pair $(T_X, 0)$, called the foliation with just one leaf.

Part 1. Geometry of foliations in positive characteristic

3. Differential geometry in positive characteristic

In this section, for later use, we briefly review the most fundamental properties of vector fields in positive characteristic (Subsection 3.2), and the Cartier correspondence (Subsection 3.3) for coherent sheaves endowed with flat connections with vanishing p-curvature.

3.1. Absolute Frobenius

Let X be a scheme over an algebraically closed field k of characteristic $p > 0$. The absolute Frobenius morphism of X is the endomorphism $\mathrm{Frob} = \mathrm{Frob}_X : X \to X$ of the scheme X, which acts as the identity on the topological space X and which acts on the structural sheaf of X by raising elements to their p-th powers.

Definition 3.1. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be coherent sheaves of $\mathcal{O}_X$-modules. A morphism $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ of sheaves of abelian groups is called

(1) p-linear if $\varphi(fe) = f^p e$; and

(2) p^{-1}-linear if $\varphi(f^p e) = fe$

for every $f \in \mathcal{O}_X(U)$, $e \in \mathcal{E}_1(U)$ and every open subset $U \subset X$.

A p^{-1}-linear morphism $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ is equivalent to an $\mathcal{O}_X$-linear morphism $\varphi : \mathrm{Frob}_* \mathcal{E}_1 \to \mathcal{E}_2$; see [5, Sections 2 and 3]. If $\mathcal{E}$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{E}$ and $\mathrm{Frob}_* \mathcal{E}$ are isomorphic, but the $\mathcal{O}_X$-module structure on $\mathrm{Frob}_* \mathcal{E}$ is different. If we denote the action of $\mathcal{O}_X$ on $\mathrm{Frob}_* \mathcal{E}$ by $\star$, then

$$f \star \sigma = f^p \sigma$$

for any $f \in \mathcal{O}_X$ and any $\sigma \in \mathrm{Frob}_* \mathcal{E}$.

If $\mathcal{E}$ still denotes a coherent sheaf of $\mathcal{O}_X$-modules, then

$$
\begin{aligned}
\mathrm{Frob}^* \mathcal{E} &= \mathcal{O}_X \otimes_{\mathrm{Frob}^{-1} \mathcal{O}_X} \mathrm{Frob}^{-1} \mathcal{E} && \text{(Definition of pull-back)} \\
&= \mathcal{O}_X \otimes_{\mathrm{Frob}^{-1} \mathcal{O}_X} \mathcal{E} && \text{(Frob is the identity on } |X|\text{)}.
\end{aligned}
$$

Consequently, in $\mathrm{Frob}^* \mathcal{E}$ we have that

$$1 \otimes f\sigma = f^p \otimes \sigma \tag{3.1}$$

for any $f \in \mathcal{O}_X$ and any $\sigma \in \mathcal{E}$. It follows that a p-linear morphism $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ is equivalent to an $\mathcal{O}_X$-linear morphism $\varphi : \mathrm{Frob}^* \mathcal{E}_1 \to \mathcal{E}_2$.

The $\mathcal{O}_X$-module $\mathrm{Frob}^* \mathcal{E}$ comes endowed with a canonical connection

$$\nabla^{\mathrm{can}} : \mathrm{Frob}^* \mathcal{E} \to \Omega_X^1 \otimes \mathrm{Frob}^* \mathcal{E}$$
$$f \otimes \sigma \mapsto df \otimes (1 \otimes \sigma).$$

It follows from Equation (3.1) that the kernel of ∇^{can} can be identified, as sheaves of abelian groups, with $1 \otimes \mathcal{E} \subset \mathrm{Frob}^* \mathcal{E}$.

3.2. Vector fields and the *p*-curvature of connections

Given a vector field/derivation $v \in T_X(U)$ on an open subset $U \subset X$ of a variety defined over a field k of characteristic $p > 0$, the p-th iteration of v is also a derivation since Leibniz's rule implies

$$v^p(f \cdot g) = \sum_{i=0}^p \binom{p}{i} v^i(f) \cdot v^{p-i}(g) = f v^p(g) + v^p(f) g$$

for every $f, g \in \mathcal{O}_X(U)$.

Thus, one can define a morphism of sheaves of sets $\times^p : T_X \to T_X$ which takes a vector field and raises it to its p-th power. This is not a morphism of sheaves of abelian groups, but a result of Jacobson ([28, Equation (5.2.4)]) says that

$$(v + w)^p = v^p + w^p + Q_p(v, w), \tag{3.2}$$

where Q_p is a Lie polynomial (i.e., a polynomial on (iterated Lie) brackets of v and w). Moreover, if $f \in \mathcal{O}_X$ and $v \in T_X$, then

$$(fv)^p = f^p v^p - f v^{p-1}(f^{p-1}) v, \tag{3.3}$$

according to [28, Equation (5.4.0)] modulo a sign misprint.

Recall from [28, Section 5] that the p-curvature of a connection $\nabla : \mathcal{E} \to \Omega_X^1 \otimes \mathcal{E}$ is the morphism of $\mathcal{O}_X$-modules

$$\psi : \mathrm{Frob}^* T_X \longrightarrow \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$$
$$v \mapsto (\nabla_v)^p - \nabla_{v^p},$$

where $\nabla_\bullet$ is the k-endomorphisms of $\mathcal{E}$ that send defined by

$$\mathcal{E} \longrightarrow \mathcal{E}$$
$$\sigma \mapsto (i_\bullet \otimes \mathrm{id}) \nabla \sigma,$$

where $i_\bullet$ denotes the contraction of 1-forms with the vector field $\bullet$.

Example 3.2. The canonical connection ∇^{can} defined on $\mathrm{Frob}^* \mathcal{E}$ is a flat connection with zero p-curvature.

3.3. Relative Frobenius, *p*-closed flat connections and the Cartier correspondence

The absolute Frobenius is not a morphism of k-schemes. Nevertheless, if we set $S = \mathrm{Spec}\,\mathsf{k}$ and consider the fiber product $X^{(p)} = X \times_S S$ of the structure morphism of X with the

absolute Frobenius of S, then the absolute Frobenius Frob factors as in the following commutative diagram:

$$
\begin{array}{ccc}
& \text{Frob=Frob}_X & \\
X \xrightarrow{\text{Frob}_{X/S}} & X^{(p)} \xrightarrow{W_{X/S}} & X \\
\downarrow & & \downarrow \\
& S \xrightarrow{\text{Frob}_S} & S.
\end{array}
$$

The morphism $\text{Frob}_{X/S} : X \to X^{(p)}$ is a morphism of k-schemes and is called the relative Frobenius morphism of X over S; see [1, Éxpose XV]. Notice that $W_{X/S} : X^{(p)} \to X$ is an isomorphism of schemes (but not of k-schemes) defined through the twisting of the coefficients by means of the field automorphism $x \mapsto x^p$.

Given any coherent sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules on X, we set $\mathcal{E}^{(p)} = W_{X/S}^* \mathcal{E}$. In other words, $\mathcal{E}^{(p)}$ is the base change of $\mathcal{E}$ to $X^{(p)}$. As such, $\mathcal{E}^{(p)}$ is an $\mathcal{O}_{X^{(p)}}$-module such that $\text{Frob}^* \mathcal{E} = \text{Frob}_{X/S}^* \mathcal{E}^{(p)}$. If we consider $\mathcal{O}_{X^{(p)}}$ as a subsheaf of $\mathcal{O}_X$, then the kernel of the canonical connection ∇^{can} on $\text{Frob}_{X/S}^* \mathcal{E}^{(p)}$ is an $\mathcal{O}_{X^{(p)}}$-module that can be identified with $\mathcal{E}^{(p)}$. Reciprocally, if $\mathcal{E}$ is a coherent sheaf on X endowed with a flat connection ∇ of p-curvature zero, then the kernel of ∇ is, in a natural way, an $\mathcal{O}_{X^{(p)}}$-module such that

$$
\text{Frob}_{X/S}^*(\ker \nabla) \simeq \mathcal{E}.
$$

This is essentially the content of the Cartier correspondence: there exists an equivalence of categories between the category of quasi-coherent sheaves on $X^{(p)}$ and the category of quasi-coherent $\mathcal{O}_X$-modules with flat connections of p-curvature zero given by

$$
\mathcal{E} \mapsto (\text{Frob}_{X/S}^* \mathcal{E}, \nabla^{\text{can}}) \quad \text{and} \quad (\mathcal{E}, \nabla) \mapsto \ker \nabla;
$$

see, for instance, [28, Theorem 5.1].

Lemma 3.3. *Let $\mathcal{E}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{K} \subset \text{Frob}^* \mathcal{E}$ be a coherent subsheaf. If the morphism of $\mathcal{O}_X$-modules*

$$
\varphi : \mathcal{K} \longrightarrow \Omega_X^1 \otimes \frac{\text{Frob}^* \mathcal{E}}{\mathcal{K}}
$$

induced by ∇^{can} is identically zero, then there exists a coherent subsheaf $\mathcal{L} \subset \mathcal{E}$ such that $\mathcal{K} = \text{Frob}^* \mathcal{L}$

Proof. If φ vanishes, then the sheaf $\mathcal{K}$ is preserved by the canonical connection of $\text{Frob}^* \mathcal{E}$. By Cartier correspondence, there exists a subsheaf $\mathcal{L}^{(p)} \subset \mathcal{E}^{(p)}$ on $X^{(p)}$ such that $\mathcal{K} = \text{Frob}_{X/S}^* \mathcal{L}^{(p)}$. Since $W_{X/S}$ is an isomorphism, and $\mathcal{E}^{(p)} = W_{X/S}^* \mathcal{E}$, there exists a subsheaf $\mathcal{L} \subset \mathcal{E}$ on X such that $\mathcal{L}^{(p)} = W_{X/S}^* \mathcal{L}$. The sheaf $\mathcal{L}$ is such that $\mathcal{K} = \text{Frob}^* \mathcal{L}$. $\quad\square$

4. Particular features of foliations in positive characteristic

In this section, we introduce the key concepts for the first part of the paper: the p-curvature of the foliation (Subsection 4.1), p-dense foliations (Subsection 4.2), the degeneracy divisor of the p-curvature, the kernel of the p-curvature of a foliation (Subsection 4.3) and the Cartier transform of a foliation (Subsection 4.5).

4.1. The p-curvature of foliations

Let $\mathcal{F}$ be a foliation on X. It follows from Equations (3.2) and (3.3) that the map which sends $v \in T_{\mathcal{F}}$ to the class of v^p in $T_X/T_{\mathcal{F}} \subset N_{\mathcal{F}}$ is p-linear and additive. We can thus define the morphism of $\mathcal{O}_X$-modules

$$\psi_{\mathcal{F}} : \mathrm{Frob}^* T_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$$
$$\sum f_i \otimes v_i \mapsto \sum f_i v_i^p \mod T_{\mathcal{F}}.$$

The morphism $\psi_{\mathcal{F}}$ is called the p-curvature of $\mathcal{F}$. If $\psi_{\mathcal{F}} = 0$, then we say that the foliation is p-closed.

The foliation $\mathcal{F}$ determines a subsheaf $\mathcal{O}_{X/\mathcal{F}} \subset \mathcal{O}_X$ consisting of functions that are annihilated by any section of $T_{\mathcal{F}}$; that is,

$$\mathcal{O}_{X/\mathcal{F}} = \{a \in \mathcal{O}_X \,;\, v(a) = 0 \text{ for any } v \in T_{\mathcal{F}}\}.$$

Theorem 4.1. *If $\mathcal{F}$ is a p-closed foliation of dimension r on a smooth algebraic variety X, then $Y = \mathrm{Spec}_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_{X/\mathcal{F}}$ is a normal algebraic variety and the inclusion $\mathcal{O}_{X/\mathcal{F}} \subset \mathcal{O}_X$ determines a morphism $f : X \to Y$ which factors the relative Frobenius $\mathrm{Frob}_{X/\mathrm{Spec}\,\mathsf{k}} : X \to X^{(p)}$ and satisfies $[\mathsf{k}(X) : f^*\mathsf{k}(Y)] = p^r$. Reciprocally, if $\mathcal{O} \subset \mathcal{O}_X$ is a sheaf of $\mathcal{O}_{X^{(p)}}$-algebras integrally closed in $\mathcal{O}_X$, then there exists a p-closed foliation $\mathcal{F}$ such that $\mathcal{O} = \mathcal{O}_{X/\mathcal{F}}$.*

Proof. See [36, Proposition 1.9, Lecture II, Part I] or [40, Theorem 2]. $\qquad\qquad\square$

Corollary 4.2. *Let $\mathcal{F}$ be a foliation of codimension q on a smooth algebraic variety X. The foliation $\mathcal{F}$ is p-closed if and only if there exists q rational first integrals for $\mathcal{F}$, say $f_1, \ldots, f_q \in \mathsf{k}(X)$, such that $df_1 \wedge \cdots \wedge df_q \neq 0$.*

Proof. See [31, Section 7]. $\qquad\qquad\square$

4.2. p-dense foliations

Some authors use the term foliation only to refer to p-closed foliations. In this work, we will also consider foliations which are not p-closed.

Definition 4.3. Let $\mathcal{F}$ be a foliation on a smooth variety X. The p-closure of $\mathcal{F}$ is the (unique) p-closed foliation $\overline{\mathcal{F}}^p$ such that $\mathcal{O}_{X/\mathcal{F}} = \mathcal{O}_{X/\overline{\mathcal{F}}^p}$. When the p-closure of $\mathcal{F}$ coincides with the foliation with only one leaf (i.e., $\mathcal{O}_{X/\mathcal{F}} = \mathcal{O}_{X^{(p)}}$), we will say that $\mathcal{F}$ is p-dense.

Notice that a foliation is p-dense if and only if any rational first integral $f \in \mathsf{k}(X)$ of $\mathcal{F}$ is such that $df = 0$.

Proposition 4.4. *If $\mathcal{F}$ is a p-dense codimension one foliation on a smooth algebraic variety X defined by $\omega \in H^0(X, \Omega_X^1 \otimes N_\mathcal{F})$ and v is a rational vector field tangent to $\mathcal{F}$ such that $\omega(v^p) \neq 0$, then the rational 1-form*

$$\frac{\omega}{\omega(v^p)}$$

is closed.

The result above first appeared in [11, Theorem 6.2]. Its proof can be generalized to p-dense foliations of arbitrary codimension as shown by the next proposition.

Proposition 4.5. *If $\mathcal{F}$ is a p-dense foliation of codimension q on a smooth algebraic variety X, then there exists closed rational 1-forms $\omega_1, \ldots, \omega_q$ which vanish on $T_\mathcal{F}$ and satisfy $\omega_1 \wedge \cdots \wedge \omega_q \neq 0$.*

Proof. Let $r = \dim \mathcal{F}$. If $x \notin \mathrm{sing}(\mathcal{F})$, then there exist generators $v_1, \ldots, v_r$ of $T_\mathcal{F} \otimes \mathcal{O}_{X,x}$ such that $[v_i, v_j] = 0$ for every $i, j \in \{1, \ldots, r\}$; see Lemma 2.2. Notice that the iterated p-th powers of the vector fields v_i commute with $v_1, \ldots, v_r$ and also among themselves. Since $\mathcal{F}$ is p-dense, we can choose among the iterated p-powers of $v_1, \ldots, v_r$, vector fields $v_{r+1}, \ldots, v_{r+q}$, $r + q = \dim X$, such that

$$v_1 \wedge \cdots \wedge v_r \wedge v_{r+1} \wedge \cdots \wedge v_{r+q}$$

does not vanish identically. Therefore, we can interpret $v_1, \ldots, v_{r+q}$ as $\mathsf{k}(X)$-linearly independent commuting rational vector fields on X.

Let $\alpha_1, \ldots, \alpha_{r+q}$ be rational 1-forms on X dual to $v_1, \ldots, v_{r+q}$. In other words, the 1-forms α_i are characterized by $\alpha_i(v_j) = \delta_{ij}$. We claim that the 1-forms α_i are closed. As $v_1, \ldots, v_{r+q}$ are $\mathsf{k}(X)$-linearly independent rational vector fields, it suffices to check that $d\alpha_i(v_j, v_k) = 0$ for $i, j, k \in \{1, \ldots, r+q\}$. Since $v_1, \ldots, v_{r+q}$ commute, Cartan's formula for the exterior derivative implies that

$$d\alpha_i(v_j, v_k) = v_j(\alpha_i(v_k)) - v_k(\alpha_i(v_j)) - \alpha_i([v_j, v_k]) = 0,$$

establishing our claim. By construction, $\omega_1 = \alpha_{r+1}, \ldots, \omega_q = \alpha_{r+q}$ vanish on $T_\mathcal{F}$ and satisfy $\omega_1 \wedge \cdots \wedge \omega_q \neq 0$ as wanted. $\qquad \square$

The following example shows that without p-denseness of $\mathcal{F}$, Proposition 4.5 does not hold.

Example 4.6. If the k has characteristic two, $X = \mathbb{A}_k^3$ and $\mathcal{F}$ is the foliation on X generated by $v = x \frac{\partial}{\partial x} + zy \frac{\partial}{\partial y}$, then every closed rational 1-form containing $T_\mathcal{F}$ in its kernel is a rational multiple of dz.

A simple adaptation of the arguments used in the proof of Proposition 4.5 gives the following generalization.

Proposition 4.7. *If $\mathcal{F}$ is a not necessarily p-dense foliation on a smooth algebraic variety X, $\mathcal{G} = \overline{\mathcal{F}}^p$, and $q = \mathrm{codim}(\mathcal{F}) - \mathrm{codim}(\mathcal{G})$, then there exist rational sections $\omega_1, \ldots, \omega_q$ of $\Omega_\mathcal{G}^1$ which vanish on $T_\mathcal{F}$ such that $\omega_1 \wedge \cdots \wedge \omega_q \neq 0$ and $d_\mathcal{G} \omega_1 = \cdots = d_\mathcal{G} \omega_q = 0$.*

A $\mathcal{F}$-partial connection on a coherent sheaf $\mathcal{E}$ is a k-linear morphism $\nabla : \mathcal{E} \to \Omega^1_{\mathcal{F}} \otimes \mathcal{E}$ which, like usual connections, satisfies Leibniz's rule $\nabla(fs) = f\nabla(s) + d_{\mathcal{F}} f \otimes s$. Bott's partial connection is the $\mathcal{F}$-partial connection on the normal sheaf of $\mathcal{F}$ which sends the class of $v \mod T_{\mathcal{F}}$ to the 1-form (defined only along $T_{\mathcal{F}}$) that maps a local section w of $T_{\mathcal{F}}$ to $[v,w] \mod T_{\mathcal{F}}$.

Our next result provides an interpretation of Propositions 4.5 and 4.7 in terms of Bott's partial connection and the canonical connection on $\mathrm{Frob}^* T_{\mathcal{F}}$.

Proposition 4.8. *The morphism $\psi_{\mathcal{F}}$ is such that*

$$(r \otimes \psi_{\mathcal{F}}) \circ \nabla^{\mathrm{can}} = \nabla^{\mathcal{F}} \circ \psi_{\mathcal{F}},$$

where $r : \Omega^1_X \to \Omega^1_{\mathcal{F}}$ is the natural restriction morphism, ∇^{can} is the canonical connection on $\mathrm{Frob}^* T_{\mathcal{F}}$ and $\nabla^{\mathcal{F}}$ is Bott's connection on $N_{\mathcal{F}}$. In other words, $\psi_{\mathcal{F}}$ is a morphism of $\mathcal{F}$-partial connections between $(\mathrm{Frob}^* T_{\mathcal{F}}, \nabla^{\mathrm{can}}|_{T_{\mathcal{F}}})$ and $(N_{\mathcal{F}}, \nabla^{\mathcal{F}})$.

Proof. It suffices to show that for every local section $v \in T_{\mathcal{F}}$, the image of $1 \otimes v \in \mathrm{Frob}^* T_{\mathcal{F}}$ under $\psi_{\mathcal{F}}$ is a flat section of Bott's partial connection on $N_{\mathcal{F}}$. To do this, we proceed as in the proof of Proposition 4.5. We fix $x \in X$ sufficiently general and use Lemma 2.2 to guarantee the existence of commuting generators $v_1, \ldots, v_r$ of $T_{\mathcal{F}} \otimes \mathcal{O}_{X,x}$. Since $\psi_{\mathcal{F}}$ is $\mathcal{O}_X$-linear, it suffices to show that $v_1^p, \ldots, v_r^p$ are flat sections of Bott's partial connection. But this is clear since each of v_i^p commutes with generators of $T_{\mathcal{F}} \otimes \mathcal{O}_{X,x}$. $\square$

4.3. The degeneracy divisor and the kernel of the *p*-curvature

Proposition 4.8 implies that the kernel of $\psi_{\mathcal{F}}$ is invariant by the partial connection $\nabla^{\mathrm{can}}|_{T_{\mathcal{F}}}$. Therefore, the morphism

$$\ker \psi_{\mathcal{F}} \longrightarrow \Omega^1_{\mathcal{F}} \otimes \frac{\mathrm{Frob}^* T_{\mathcal{F}}}{\ker \psi_{\mathcal{F}}}$$

induced by $\nabla^{\mathrm{can}}|_{T_{\mathcal{F}}}$ is zero. This is not sufficient to apply Lemma 3.3 as it may happen that $\ker \psi_{\mathcal{F}}$ is not invariant by the connection ∇^{can}; that is, the morphism of $\mathcal{O}_X$-modules

$$\ker \psi_{\mathcal{F}} \longrightarrow \Omega^1_X \otimes \frac{\mathrm{Frob}^* T_{\mathcal{F}}}{\ker \psi_{\mathcal{F}}} \tag{4.1}$$

induced by ∇^{can} is not necessarily zero.

Lemma 4.9. *If $\mathcal{F}$ is p-dense, then the morphism (4.1) is zero. In particular, there exists a distribution $\mathcal{V}(\mathcal{F})$ contained in $\mathcal{F}$ such that*

$$\ker \psi_{\mathcal{F}} = \mathrm{Frob}^* T_{\mathcal{V}(\mathcal{F})}.$$

In this case, the distribution $\mathcal{V}(\mathcal{F})$ will be called the kernel of the p-curvature of $\mathcal{F}$.

Proof. The vanishing of the morphism (4.1) is assured by [39, Proposition 6.1]. The existence of the distribution $\mathcal{V}(\mathcal{F})$ follows from Lemma 3.3. $\square$

Definition 4.10. If $\mathcal{F}$ is a p-dense codimension one foliation, then the image of the p-curvature morphism $\psi_{\mathcal{F}}$ is of the form $N_{\mathcal{F}} \otimes \mathcal{I}_Z$ for some ideal sheaf $\mathcal{I}_Z$. The divisorial part of $\mathcal{I}_Z$ will be called the degeneracy divisor of the p-curvature and will be denoted by $\Delta_{\mathcal{F}}$.

Example 4.11. Let $\mathcal{F}$ be a codimension one foliation on $X = \mathbb{A}^n_{\mathsf{k}}$ with $N^*_{\mathcal{F}}$ generated by 1-form

$$\omega = \left(\prod_{i=1}^r x_i\right)\left(\sum_{i=1}^r \lambda_i \frac{dx_i}{x_i}\right),$$

where $1 < r \leq n$ and $\lambda_1, \ldots, \lambda_r \in \mathsf{k}^*$. The tangent sheaf of $\mathcal{F}$ is locally free and generated by the vector fields $v_j = \lambda_j x_1 \frac{\partial}{\partial x_1} - \lambda_i x_j \frac{\partial}{\partial x_j}$ for $1 < j \leq r$ and $\frac{\partial}{\partial x_{r+1}}, \ldots, \frac{\partial}{\partial x_n}$. Observe that $\frac{\partial^p}{\partial x_{r+1}} = \ldots = \frac{\partial^p}{\partial x_n} = 0$ and that

$$\omega(v_j^p) = \left(\prod_{i=1}^r x_i\right)(\lambda_j^p \lambda_1 - \lambda_1^p \lambda_j)$$

for any $1 < j \leq r$. Hence, $\mathcal{F}$ is p-closed if and only if $(\lambda_1 : \ldots : \lambda_r) \in \mathbb{P}^{r-1}_{\mathsf{k}}(\mathbb{F}_p)$. Furthermore, if $\mathcal{F}$ is not p-closed, then $\Delta_{\mathcal{F}}$ is the simple normal crossing divisor given by $\{x_1 \cdots x_r = 0\}$,

Proposition 4.12. *Let $\mathcal{F}$ be a p-dense codimension one foliation on a smooth algebraic variety X of characteristic $p > 0$. If $\mathscr{V}(\mathcal{F})$ denotes the kernel of the p-curvature of $\mathcal{F}$, then the identity*

$$\mathcal{O}_X(\Delta_{\mathcal{F}}) = (\omega_{\mathcal{F}} \otimes \omega^*_{\mathscr{V}(\mathcal{F})})^{\otimes p} \otimes \det N_{\mathcal{F}}$$

holds in the Picard group of X.

Proof. The definition of $\Delta_{\mathcal{F}}$ implies that the sequence

$$0 \longrightarrow \mathrm{Frob}^* T_{\mathscr{V}(\mathcal{F})} \longrightarrow \mathrm{Frob}^* T_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}(-\Delta_{\mathcal{F}}) \longrightarrow 0 \tag{4.2}$$

is exact on the complement of a codimension two subset of X. The result follows by taking determinants. $\qquad\square$

Proposition 4.13. *Let X be a smooth algebraic variety of characteristic $p > 0$ and let $\mathcal{F}$ be a p-dense codimension one foliation on X. Then every $\mathcal{F}$-invariant hypersurface is contained in the support of $\Delta_{\mathcal{F}}$. Reciprocally, if p does not divide the coefficient of an irreducible component H in $\Delta_{\mathcal{F}}$, then H is $\mathcal{F}$-invariant.*

Proof. Suppose first that H is $\mathcal{F}$-invariant. Let $U \subset X$ be a sufficiently small open subset intersecting H and let $h \in \mathcal{O}_X(U)$ be a reduced equation for $H \cap U$. Since H is $\mathcal{F}$-invariant, for any local section $v \in T_{\mathcal{F}}(U)$, we have that $v(h)$ is a multiple of h. Therefore, the same holds true for $v^p(h)$. In other words, for any $v \in T_{\mathcal{F}}(U)$, the restriction of v and v^p to $H \cap U$ is contained in the kernel of $dh\big|_{H \cap U}$. Hence, the support of $\Delta_{\mathcal{F}}$ contains H.

Suppose now that $\mathrm{ord}_H(\Delta_{\mathcal{F}}) = \alpha \not\equiv 0 \mod p$. In the notation above, after restricting U if necessary, we can chose commuting generators $v_1, \ldots, v_r$ of $T_{\mathcal{F}}(U)$ such that

$$v_1 \wedge \cdots \wedge v_r \wedge v_1^p = h^\alpha \theta,$$

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where $r = \dim \mathcal{F}$, and $\theta \in \bigwedge^{r+1} T_X(U)$ does not vanish along H; see Lemma 2.2. Observe that

$$[v_1, h^\alpha \theta] = 0 \implies \alpha v_1(h) h^{\alpha-1} \theta = -h^\alpha [v_1, \theta].$$

Since $\alpha \not\equiv 0 \mod p$, it follows that h divides $v_1(h)$. This is sufficient to show that $H = \{h = 0\}$ is $\mathcal{F}$-invariant. $\square$

Proposition 4.14. *Let X be a smooth algebraic variety of characteristic $p > 0$ and let $\mathcal{F}$ be a p-dense codimension one foliation on X. If η is a closed rational 1-form defining $\mathcal{F}$, then the coefficients of the divisors $\Delta_{\mathcal{F}}$ and $((\eta)_\infty - (\eta)_0)$ coincide modulo p.*

Proof. Let η' be another closed rational 1-form defining $\mathcal{F}$. As both η and η' define the same codimension one foliation, there exists $h \in \mathsf{k}(X)$ such that $\eta = h\eta'$. The closedness of η and η' implies that $dh \wedge \eta' = 0$. Since $\mathcal{F}$ is p-dense, $dh = 0$ and we can write $\eta = g^p \eta'$ for some $g \in \mathsf{k}(X)$. In particular, the coefficients of the divisors $(\eta)_\infty - (\eta)_0$ and $(\eta')_\infty - (\eta')_0$ agree modulo p.

Let U be a sufficiently small open subset of $X - (\mathrm{sing}(\mathcal{F}) \cup \mathrm{sing}(\mathscr{V}(\mathcal{F})))$. Choose a nowhere vanishing section ω of $N_{\mathcal{F}}^*(U)$ and a vector field $v \in T_{\mathcal{F}}(U) \subset T_X(U)$ everywhere transverse to $T_{\mathscr{V}(F)}\big|_U$. Therefore, $f = \omega(v^p)$ generates $\mathcal{O}_U(-\Delta_{\mathcal{F}})$ and the rational 1-form ω/f is closed according to Proposition 4.4. Thus, the coefficients of $(\eta)_\infty - (\eta)_0$ and $\Delta_{\mathcal{F}}$ along hypersurfaces intersecting U agree modulo p. Since we can choose U intersecting any given hypersurface of X, the result follows. $\square$

4.4. The Cartier operator

Let X be a smooth algebraic variety defined over an algebraically closed field of characteristic $p > 0$.

The de Rham complex $(\Omega_X^\bullet, d)$ of X is a complex of sheaves of abelian groups but is not a complex of $\mathcal{O}_X$-modules. Nevertheless, for any $f \in \mathcal{O}_X$ and any $\omega \in \Omega_X^i$, we have that

$$d(f^p \cdot \omega) = f^p \cdot \omega.$$

It follows that $(\mathrm{Frob}_* \Omega_X^\bullet, d)$ is a complex of $\mathcal{O}_X$-modules. If we denote by $Z\Omega_X^\bullet$ the sheaf of closed differential, and by $B\Omega_X^\bullet$ the sheaf of locally exact differentials, then both $\mathrm{Frob}_* Z\Omega_X^\bullet$ and $\mathrm{Frob}_* B\Omega_X^\bullet$ are coherent $\mathcal{O}_X$-modules. The sheaf cohomology groups of the complex $(\mathrm{Frob}_* \Omega_X^\bullet, d)$ are the $\mathcal{O}_X$-modules

$$\mathscr{H}^i \mathrm{Frob}_* \Omega_X^\bullet = \frac{\mathrm{Frob}_* Z\Omega_X^i}{\mathrm{Frob}_* B\Omega_X^i}.$$

Cartier proved in [9] the existence of a unique morphism $\mathbf{C} : \mathrm{Frob}_* Z\Omega_X^\bullet \to \Omega_X^\bullet$ of sheaves of graded-commutative $\mathcal{O}_X$-algebras such that

(1) $\ker \mathbf{C} = \mathrm{Frob}_* B\Omega_X^\bullet$; and

(2) $\mathbf{C}(f^{p-1} df) = df$ for every $f \in \mathcal{O}_X$; and

(3) $\mathbf{C}$ is surjective and, consequently, induces an isomorphism of graded-commutative $\mathcal{O}_X$-algebras between $\mathscr{H}^\bullet \mathrm{Frob}_* \Omega_X^\bullet$ and $\Omega_X^\bullet$.

The morphism $\mathbf{C}$ is the (absolute) Cartier operator on X. If ω is a closed q-form, we will say that $\mathbf{C}(\omega)$ is the Cartier transform of ω

Remark 4.15. In the literature, one also finds the relative Cartier operator $\mathrm{Frob}_* Z\Omega_X^\bullet \to \Omega_{X^{(p)}}^\bullet$. As we are working over $S = \mathrm{Spec}(\mathsf{k})$ where k is an algebraically closed field, the two operators are essentially equivalent, cf. [27, pages 122-123]. For details on the construction of the Cartier operator, see, for instance, [28, Section 7], [7, Chapter 1, Section 1.3], or [27, Section 3].

Lemma 4.16. *Let X be a smooth variety defined over a field k of characteristic $p > 0$. If $\omega \in \Omega_X^1$ and $v \in T_X$, then*

$$(i_v \mathbf{C}(\omega))^p = i_{v^p}\omega - v^{p-1}(i_v\omega).$$

More generally, if $\omega \in \Omega_X^j$ and $v \in T_X$, then

$$i_v \mathbf{C}(\omega) = \mathbf{C}\left(i_{v^p}\omega - (L_v)^{p-1}(i_v\omega)\right).$$

Proof. The first formula is due to Cartier and appears (without proof) in [9]. A proof can be found in [40, Proposition 3]. The second formula is the content of [4, Proposition 2.6]. $\square$

4.5. The Cartier transform of a *p*-dense foliation

Let $\mathcal{F}$ be a p-dense foliation of codimension q on a smooth algebraic variety X. Let $\omega_1, \ldots, \omega_q$ be closed rational 1-forms defining $\mathcal{F}$. The existence of such 1-forms is guaranteed by Proposition 4.5. If ω is any closed 1-form vanishing on $T_\mathcal{F}$, we can write

$$\omega = \sum_{i=1}^{q} f_i \omega_i$$

for some rational functions $f_i \in \mathsf{k}(X)$. If we differentiate the expression above and take the wedge product of the result with $\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_q$, we see that $df_i \wedge \omega_1 \wedge \cdots \wedge \omega_q = 0$ for any i. Since $\mathcal{F}$ is p-dense, it follows that $df_i = 0$ for every i (i.e., there exists $g_i \in \mathsf{k}(X)$ such that $f_i = g_i^p$). Therefore, the identity

$$\mathbf{C}(\omega) = \sum_{i=1}^{q} g_i \mathbf{C}(\omega_i)$$

holds. It follows that the distribution defined by the Cartier transform of all closed rational 1-forms vanishing on $T_\mathcal{F}$ can be defined by the Cartier transform of only $q = \mathrm{codim}\,\mathcal{F}$ closed rational 1-forms.

Definition 4.17. Let $\mathcal{F}$ be a p-dense foliation of codimension q on a smooth algebraic variety X. The Cartier transform of $\mathcal{F}$ is the distribution $\mathscr{C}(\mathcal{F})$ on X defined by the intersection of the kernels of the Cartier transform of all closed rational 1-forms vanishing on $T_\mathcal{F}$.

Example 4.18. The Cartier transform of a p-dense foliation is not necessarily a foliation. For instance, the codimension one foliation on $\mathbb{A}_\mathsf{k}^3$ defined by the closed 1-form

$\omega = y^{p-1}dy + z^p x^{p-1}dx$ has Cartier transform defined by the non-integrable 1-form $\mathbf{C}(\omega) = dy + zdx$.

4.6. The Cartier transform and the kernel of the p-curvature

The proposition below gives an alternative description of the kernel of the p-curvature of a p-dense foliation $\mathcal{F}$.

Proposition 4.19. *Let $\mathcal{F}$ be a p-dense foliation on a smooth algebraic variety X. The kernel of the p-curvature of $\mathcal{F}$ coincides with the saturation of $T_{\mathcal{F}} \cap T_{\mathscr{C}(\mathcal{F})}$ inside T_X.*

Proof. Lemma 4.9 implies the existence of a distribution $\mathscr{V}(\mathcal{F})$ on X such that $\ker \psi_{\mathcal{F}} = \mathrm{Frob}^* T_{\mathscr{V}(\mathcal{F})}$.

Let $v \in T_{\mathcal{F}}$. For any closed rational 1-form vanishing on $T_{\mathcal{F}}$, Lemma 4.16 implies that

$$(i_v \mathbf{C}(\omega))^p = i_{v^p}\omega - v^{p-1}(i_v \omega) = i_{v^p}\omega.$$

If $v \in T_{\mathscr{V}(\mathcal{F})}$, then, by the definition of $\mathscr{V}(\mathcal{F})$, $i_{v^p}\omega = 0$ for any $\omega \in N_{\mathcal{F}}^*$. Therefore, when ω is closed, $i_v \mathbf{C}(\omega) = 0$. This shows that $T_{\mathscr{V}(\mathcal{F})}$ is contained in the saturation of $T_{\mathcal{F}} \cap T_{\mathscr{C}(\mathcal{F})}$.

Reciprocally, if $v \in T_{\mathcal{F}} \cap T_{\mathscr{C}(\mathcal{F})}$, then, by definition, $i_v \mathbf{C}(\omega) = 0$ for any closed rational rational 1-form ω vanishing on $T_{\mathcal{F}}$. Consequently, $i_{v^p}\omega = 0$ for any closed rational section of $N_{\mathcal{F}}^*$. Since $N_{\mathcal{F}}^*$ is generically generated by closed rational 1-forms, as proved in Proposition 4.5, this is sufficient to show that $T_{\mathcal{F}} \cap T_{\mathscr{C}(\mathcal{F})}$ is contained in $T_{\mathscr{V}(\mathcal{F})}$. $\qquad\square$

4.7. Global expression for the kernel of the p-curvature

Assume that $\mathcal{F}$ is a p-dense foliation of codimension one on a smooth algebraic variety X. For later use, we will present a twisted section of $\Omega_{\mathcal{F}}^1$ defining the distribution $\mathscr{V}(\mathcal{F})$.

Let U be the open set $X - (\mathrm{sing}(\mathcal{F}) \cup \mathrm{sing}(\mathscr{V}(\mathcal{F})))$ and consider a sufficiently fine open covering U_i of U. Choose for each U_i a nowhere vanishing section ω_i of $N_{\mathcal{F}}^*(U_i)$ and a vector field $v_i \in T_{\mathcal{F}}(U_i) \subset T_X(U_i)$ everywhere transverse to $T_{\mathscr{V}(F)}|_{U_i}$. Therefore, $f_i = \omega_i(v_i^p)$ generates $\mathcal{O}_{U_i}(-\Delta_{\mathcal{F}})$, and the rational 1-form ω_i/f_i is closed.

Let η_i be the restriction of the rational 1-form $\mathbf{C}(\omega_i/f_i)$ to $T_{\mathcal{F}}|_{U_i}$. We claim that η_i is a regular section of $\Omega_{\mathcal{F}}^1|_{U_i}$ without codimension one zeros. Indeed, if $v \in T_{\mathcal{F}}(U_i)$, then, using Lemma 4.16, we can write

$$\omega_i(v^p) = f_i\left(\frac{\omega_i}{f_i}\right)(v^p) = f_i\left(\mathbf{C}\left(\frac{\omega_i}{f_i}\right)(v)\right)^p.$$

Since $\omega_i(v^p) \in \mathcal{O}_X(-\Delta_{\mathcal{F}})(U_i) = f_i\mathcal{O}_X(U_i)$, we deduce that $\eta_i(v) = \mathbf{C}\left(\frac{\omega_i}{f_i}\right)(v) \in \mathcal{O}_X(U_i)$ for no matter which $v \in T_{\mathcal{F}}(U_i)$. Moreover, η_i has no zeros since $\eta_i(v_i) = 1$.

Taking into account Proposition 4.12, we deduce that the collection $\{\eta_i\}$ defines a regular section η of the sheaf $(\Omega_{\mathcal{F}}^1 \otimes \omega_{\mathscr{V}(\mathcal{F})} \otimes \omega_{\mathcal{F}}^*)|_U = (\Omega_{\mathcal{F}}^1 \otimes \det N_{\mathscr{V}(\mathcal{F})} \otimes N_{\mathcal{F}}^*)|_U$. Since $\Omega_{\mathcal{F}}^1 \otimes \omega_{\mathscr{V}(\mathcal{F})} \otimes \omega_{\mathcal{F}}^*$ is reflexive, η extends to an element of

$$H^0(X, \Omega_{\mathcal{F}}^1 \otimes \omega_{\mathscr{V}(\mathcal{F})} \otimes \omega_{\mathcal{F}}^*) = H^0(X, \Omega_{\mathcal{F}}^1 \otimes \det N_{\mathscr{V}(\mathcal{F})} \otimes N_{\mathcal{F}}^*),$$

without codimension one zeros, defining the distribution $\mathscr{V}(\mathcal{F}) \subset \mathcal{F}$.

5. Degeneracy divisor of the *p*-curvature

This section investigates the degeneracy divisor of the *p*-curvature of codimension one foliations. It starts by describing how the degeneracy divisor behaves under two natural operations: pull-back by dominant rational maps (Subsection 5.3) and restriction to non-invariant subvarieties (Subsection 5.6). It then moves, in Subsection 5.7, to review work by the first author on the reducibility of the degeneracy divisor of the *p*-curvature of sufficiently general foliations on the projective plane and Hirzebruch surfaces. The results on the reducibility and on the behavior under pull-backs will be essential for the applications developed in the second part of the paper.

5.1. Equidimensional and dominant rational maps

Let X and Y be smooth varieties of dimensions n and m, respectively, defined over an algebraically closed field k and let $\varphi : X \dashrightarrow Y$ be a dominant rational map. Throughout this section, we will make use of the following assumption on φ.

Assumption 5.1. There exist open subsets $X^\circ \subset X$ and $Y^\circ \subset Y$ such that φ restricts to a morphism $\varphi^\circ : X^\circ \to Y^\circ$ satisfying

(1) the set $X - X^\circ$ has codimension at least two;

(2) φ° is equidimensional and, consequently, the Zariski closure in Y° of the image of any irreducible hypersurface of X° is either a hypersurface of Y° or the whole Y°;

(3) the differential of φ° has generic rank equal to $m = \dim Y$, and the differential of the restriction of φ° to any irreducible hypersurface $H \subset X^\circ$ has generic rank equal to $m = \dim Y^\circ$ when H dominates Y° or equal to $m - 1$ when the image of H is contained in a hypersurface.

When k has characterisitic zero, Item (3) follows from Item (2), but this is no longer true when k has positive characteristic.

Define the ramification divisor $\mathrm{Ram}(\varphi^\circ) \in \mathrm{Div}(X^\circ)$ of φ° as

$$\mathrm{Ram}(\varphi^\circ) = \sum_H (\varphi^\circ)^* H - ((\varphi^\circ)^* H)_{\mathrm{red}}, \tag{5.1}$$

where the sum runs over all irreducible hypersurfaces of Y°. Assumption 5.1, Item (3), guarantees that this is a finite sum. Moreover, each irreducible component of the support of $\mathrm{Ram}(\varphi^\circ)$ is mapped (dominantly) to an irreducible hypersurface of Y°. Define the branch divisor of φ°, $\mathrm{Branch}(\varphi^\circ)$, as the reduced divisor with support equal to the image of $\mathrm{Ram}(\varphi^\circ)$. Set $\mathrm{Ram}(\varphi)$ as the divisor on X given by the closure of $\mathrm{Ram}(\varphi^\circ)$.

If $\mathcal{L}$ is a line-bundle on Y, then we will abuse the notation and write $\varphi^* \mathcal{L}$ for the unique line-bundle on X such that $\varphi^* \mathcal{L}\big|_{X^\circ} = (\varphi^\circ)^* \big(\mathcal{L}\big|_{Y^\circ}\big)$. Since X is smooth and $\mathrm{codim}\, X - X^\circ \geq 2$, such a line-bundle always exists.

The next proposition, as well as Proposition 5.2 below, are well-known for equidimensional morphisms between normal complex varieties; see, for instance, [20, Subsection 2.9].

Proposition 5.1. *Let $\varphi : X \dashrightarrow Y$ be a rational map satisfying Assumption 5.1. If $\mathcal{V}(\varphi)$ is the foliation on X defined by φ, then*

$$\omega_{\mathcal{V}(\varphi)} = \omega_X \otimes \varphi^* \omega_Y^* \otimes \mathcal{O}_X(-\operatorname{Ram}(\varphi)).$$

Proof. Further restricting X° and Y° (keeping the assumption $\operatorname{codim} X - X^\circ \geq 2$), we can, and will, assume that all irreducible components of $\operatorname{Ram}(\varphi^\circ)$ and $\operatorname{Branch}(\varphi^\circ)$ are smooth, pairwise disjoint and that the rank of the differential of φ° is equal to $m - 1 = \dim Y - 1$ on the support of $\operatorname{Ram}(\varphi^\circ)$ and equal to m everywhere else.

Set $B = \operatorname{Branch}(\varphi^\circ)$ and $R = ((\varphi^\circ)^*(B))_{red}$. Notice that $\operatorname{Ram}(\varphi^\circ) = (\varphi^\circ)^* B - R$. As usual, we will write $T_{X^\circ}(-\log R)$ for the subsheaf of T_{X° formed by vector fields tangent to R and similarly for $T_{Y^\circ}(-\log B)$.

Let $\mathcal{V}(\varphi)$ be the foliation defined by φ. Assumption 5.1, Item (3), (or rather the stronger version of it made on the first paragraph of this proof) implies that the tangent sheaf of $\mathcal{V}(\varphi)\big|_{X^\circ}$ fits into the exact sequence

$$0 \longrightarrow T_{\mathcal{V}(\varphi)}\big|_{X^0} \longrightarrow T_{X^\circ}(-\log R) \longrightarrow (\varphi^\circ)^*(T_{Y^\circ}(-\log B)) \to 0,$$

where the morphism $T_{X^\circ}(-\log R) \longrightarrow (\varphi^\circ)^*(T_{Y^\circ}(-\log B))$ is induced the differential $d\varphi^\circ$ of φ°. It suffices to take the determinant of this exact sequence to conclude.   □

5.2. Pull-backs of foliations under dominant rational maps

Before continuing the discussion, we introduce two notations:

(1) If D is an arbitrary divisor on smooth variety carrying a foliation $\mathcal{F}$, we will write $D = D_{\mathcal{F}} + D_{\mathcal{F}^\perp}$ for the unique decomposition of D as the sum of divisors where all the irreducible components of the support of $D_{\mathcal{F}}$ are $\mathcal{F}$-invariant and all irreducible components of the support of $D_{\mathcal{F}^\perp}$ are not $\mathcal{F}$-invariant.

(2) We will write $T_{\mathcal{F}}(-\log D)$ for the subsheaf of $T_{\mathcal{F}}$ formed by vector fields tangent to $\mathcal{F}$ and D. This definition implies that $T_{\mathcal{F}}(-\log D) = T_{\mathcal{F}}(-\log D_{\mathcal{F}^\perp})$ and $T_{\mathcal{F}}(-\log D_{\mathcal{F}}) = T_{\mathcal{F}}$. Note that $\det T_{\mathcal{F}}(-\log D) = \det T_{\mathcal{F}} \otimes \mathcal{O}_X(-D_{\mathcal{F}^\perp})$.

Proposition 5.2. *Let $\varphi : X \dashrightarrow Y$ be a rational map satisfying Assumption 5.1 and let $\mathcal{G}$ be a foliation on Y. If $\mathcal{F}$ is the pull-back of $\mathcal{G}$ under φ, then*

$$\det(N_{\mathcal{F}}^*) = \varphi^* \det(N_{\mathcal{G}}^*) \otimes \mathcal{O}_X(\operatorname{Ram}(\varphi)_{\mathcal{F}}), \text{ and}$$
$$\omega_{\mathcal{F}} = \varphi^* \omega_{\mathcal{G}} \otimes \omega_{\mathcal{V}(\varphi)} \otimes \mathcal{O}_X(\operatorname{Ram}(\varphi)_{\mathcal{F}^\perp}).$$

Proof. The restriction of the tangent sheaf of $\mathcal{F}$ to X° is defined by the exact sequence

$$0 \longrightarrow T_{\mathcal{F}}\big|_{X^\circ} \longrightarrow T_{X^\circ} \longrightarrow \varphi^*(N_{\mathcal{G}}\big|_{Y^\circ}),$$

where the rightmost arrow is given by the composition of the differential of φ° with the natural projection $T_{Y^0} \to N_{\mathcal{G}}\big|_{Y^\circ}$.

As in the proof of Proposition 5.1, set $B = \operatorname{Branch}(\varphi^\circ)$ and $R = ((\varphi^\circ)^*(B))_{red}$. Assumption 5.1, Item (3), gives an exact sequence

$$0 \longrightarrow T_{\mathcal{V}(\varphi)}\big|_{X^0} \longrightarrow T_{\mathcal{F}}\big|_{X^\circ}(-\log R_{\mathcal{F}^\perp}) \longrightarrow (\varphi^\circ)^*\left(T_{\mathcal{G}}\big|_{Y^\circ}(-\log B_{\mathcal{G}^\perp})\right)$$

with the rightmost morphism surjective in codimension one. Taking determinants, we obtain that

$$\omega_{\mathcal{F}} = \varphi^* \omega_{\mathcal{G}} \otimes \omega_{\mathcal{V}(\varphi)} \otimes \mathcal{O}_X \left(\mathrm{Ram}(\varphi)_{\mathcal{F}^{\perp}} \right),$$

as claimed. The formula for the conormal bundle of $\mathcal{F}$ follows from Equation (2.1) (adjunction for foliations) combined with Proposition 5.1. $\square$

5.3. Degeneracy divisor of the pull-back under a dominant rational map

Throughout this subsection, k is an algebraically closed field of characteristic $p > 0$. We will determine a formula comparing the degeneracy divisor of the p-curvature of a p-dense codimension one foliation $\mathcal{G}$ on a smooth projective variety with the degeneracy divisor of its pull-back under a dominant rational map $\varphi : X \dashrightarrow Y$ satisfying Assumption 5.1.

Proposition 5.3. *Let $\varphi : X \dashrightarrow Y$ be a rational map and let $\mathcal{G}$ be a p-dense codimension one foliation on Y. If φ satisfies Assumption 5.1 and $\mathcal{F} = \varphi^* \mathcal{G}$, then $\Delta_{\mathcal{F}}$ is linearly equivalent to*

$$\varphi^* \Delta_{\mathcal{G}} - \mathrm{Ram}(\varphi)_{\mathcal{F}} + p \left(\mathrm{Ram}(\varphi)_{\mathcal{F}^{\perp}} - \mathrm{Ram}(\varphi)_{\mathcal{V}(\mathcal{F})^{\perp}} \right).$$

Proof. The proof builds on a combination of Proposition 4.12 with Proposition 5.2.

First, observe that $\mathcal{V}(\mathcal{F})$, the kernel of p-curvature of $\mathcal{F} = \varphi^* \mathcal{G}$, coincides with $\varphi^* \mathcal{V}(\mathcal{G})$. Therefore, Proposition 5.2 implies that

$$N_{\mathcal{F}} = \varphi^* N_{\mathcal{G}} \otimes \mathcal{O}_X \left(-\mathrm{Ram}(\varphi)_{\mathcal{F}} \right)$$
$$\omega_{\mathcal{F}} = \varphi^* \omega_{\mathcal{G}} \otimes \omega_{\mathcal{V}(\varphi)} \otimes \mathcal{O}_X \left(\mathrm{Ram}(\varphi)_{\mathcal{F}^{\perp}} \right)$$
$$\omega_{\mathcal{V}(\mathcal{F})} = \varphi^* \omega_{\mathcal{V}(\mathcal{G})} \otimes \omega_{\mathcal{V}(\varphi)} \otimes \mathcal{O}_X \left(\mathrm{Ram}(\varphi)_{\mathcal{V}(\mathcal{F})^{\perp}} \right).$$

Hence, we deduce from Proposition 4.12 that $\mathcal{O}_X(\Delta_{\mathcal{F}})$ is isomorphic to

$$\varphi^* \mathcal{O}_Y(\Delta_{\mathcal{G}}) \otimes \mathcal{O}_X \left(-\mathrm{Ram}(\varphi)_{\mathcal{F}} + p \left(\mathrm{Ram}(\varphi)_{\mathcal{V}(\mathcal{F})^{\perp}} - \mathrm{Ram}(\varphi)_{\mathcal{F}^{\perp}} \right) \right)$$

in the Picard group of X. $\square$

Corollary 5.4. *Assumptions as in Proposition 5.3. Assume also that X has no non-zero effective divisor linearly equivalent to zero. If $\mathrm{Ram}(\varphi)_{\mathcal{V}(\mathcal{F})^{\perp}} = \mathrm{Ram}(\varphi)_{\mathcal{F}^{\perp}} = 0$ and all the coefficients of $\Delta_{\mathcal{F}}$ are strictly smaller than the characteristic of k, then*

$$\Delta_{\mathcal{F}} = \varphi^* \Delta_{\mathcal{G}} - \mathrm{Ram}(\varphi)_{\mathcal{F}}.$$

Proof. According to Proposition 4.4, $\mathcal{G}$ is defined by a closed rational 1-form η. Since $\varphi^* \eta$ is a closed 1-form defining $\mathcal{F}$, Proposition 4.14 implies that

$$\Delta_{\mathcal{G}} = (\eta)_{\infty} - (\eta)_0 \mod p,$$
$$\Delta_{\mathcal{F}} = (\varphi^* \eta)_{\infty} - (\varphi^* \eta)_0 \mod p.$$

A local computation gives that $(\varphi^*\eta)_\infty - (\varphi^*\eta)_0 = \varphi^*((\eta)_\infty - (\eta)_0) - \mathrm{Ram}(\varphi)_\mathcal{F}$ in accordance with Proposition 5.2. Consequently, we can write

$$\Delta_\mathcal{F} = \varphi^*\Delta_\mathcal{G} - \mathrm{Ram}(\varphi)_\mathcal{F} \mod p.$$

Our assumption $\mathrm{Ram}(\varphi)_{\mathscr{V}(\mathcal{F})^\perp} = \mathrm{Ram}(\varphi)_{\mathcal{F}^\perp} = 0$ combined with Proposition 5.3 implies that $\Delta_\mathcal{F}$ and $\varphi^*\Delta_\mathcal{G} - \mathrm{Ram}(\varphi)_\mathcal{F}$ are linearly equivalent. Since $\mathcal{G}$-invariant hypersurfaces are contained $\Delta_\mathcal{G}$ according to Proposition 4.13, the divisor $\varphi^*\Delta_\mathcal{G} - \mathrm{Ram}(\varphi)_\mathcal{F}$ is effective. Therefore, $\Delta_\mathcal{F}$ and $\varphi^*\Delta_\mathcal{G} - \mathrm{Ram}(\varphi)_\mathcal{F}$ are effective divisors, with the same reduction modulo p, and the coefficients of $\Delta_\mathcal{F}$ are smaller than p; it follows from our assumptions on X that they must be same divisor. $\qquad\square$

Example 5.5. Let $n \geq 2$ and $\ell > 0$ be integers such that p does not divide ℓ. The table below, where $H = \{x_n = 0\}$, describes the behaviour of the degeneracy divisor of the p-curvature of the pull-back $\mathcal{F} = \varphi^*\mathcal{G}$ of a foliation $\mathcal{G}$ on $\mathbb{A}_\mathsf{k}^n$ under the tame morphism

$$\varphi : \mathbb{A}_\mathsf{k}^n \to \mathbb{A}_\mathsf{k}^n$$
$$(x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n^\ell).$$

H is $\mathcal{F}$-invariant	H is $\mathscr{V}(\mathcal{F})$-invariant	Degeneracy divisor
No	No	$\varphi^*\Delta_\mathcal{G}$
No	Yes	$\varphi^*\Delta_\mathcal{G} + p(\ell-1)H$
Yes	Yes	$\varphi^*\Delta_\mathcal{G} - (\ell-1)H$

5.4. Non-invariant hypersurfaces and differents

Let $\mathcal{F}$ be a distribution on a smooth algebraic variety X and let $H \subset X$ be a smooth hypersurface. Let δ be the natural composition

$$T_\mathcal{F}\big|_H \to T_X\big|_H \to N_H.$$

If H is not $\mathcal{F}$-invariant, then the image of δ is equal to $N_H \otimes \mathcal{I}$ for some ideal sheaf $\mathcal{I} \subset \mathcal{O}_H$. The effective divisor defined by the codimension one part of $\mathcal{I}$ will be called the different of $\mathcal{F}$ and H and will be denoted by $\mathrm{diff}_H(\mathcal{F})$. Notice that the kernel of δ is the tangent sheaf of the distribution $\mathcal{H} = \mathcal{F}\big|_H$.

Note that when X is a surface and H is a curve, $\mathrm{diff}_H(\mathcal{F})$ is nothing but the tangency divisor of $\mathcal{F}$ and H as defined by Brunella [8, Chapter 3]. The terminology adopted here is in accordance with [42, Section 3] and is justified by the analogy with the homonymous concept from birational geometry (see, for instance, [29, Section 4.1]) suggested by the following observation.

Lemma 5.6. *Let $\mathcal{F}$ be a distribution of arbitrary dimension on a smooth algebraic variety X and let $H \subset X$ be a smooth hypersurface. If H is not $\mathcal{F}$-invariant and $\mathcal{H} = \mathcal{F}\big|_H$ is the restriction of $\mathcal{F}$ to H, then*

$$(\omega_{\mathcal{F}} \otimes \mathcal{O}_X(H))\big|_H \simeq \omega_{\mathcal{H}} \otimes \mathcal{O}_H(\operatorname{diff}_H(\mathcal{F})),$$
$$(\det N_{\mathcal{F}})\big|_H \simeq \det N_{\mathcal{H}} \otimes \mathcal{O}_H(\operatorname{diff}_H(\mathcal{F})),$$

where we adopt the convention that when $\mathcal{H}$ is the foliation/distribution by points, then $\omega_{\mathcal{H}} = \mathcal{O}_H$ and $N_{\mathcal{H}} = T_H$.

Proof. The first formula follows from the fact that the kernel of δ equals $T_{\mathcal{H}}$. The second formula is deduced from the first by adjunction. $\qquad\square$

Lemma 5.7. *Let $\mathcal{F}$ be a codimension q distribution on a smooth algebraic variety X defined by $\omega \in H^0(X, \Omega_X^q \otimes \det N_{\mathcal{F}})$. Let $\mathcal{G} \subset \mathcal{F}$ be a subdistribution of codimension r in $\mathcal{F}$ defined by $\eta \in H^0(X, \Omega_{\mathcal{F}}^r \otimes \det N_{\mathcal{G}} \otimes \det N_{\mathcal{F}}^*)$. If $H \subset X$ is a smooth hypersurface not invariant by $\mathcal{F}$ and $\mathcal{G}$, $i: H \to X$ is the natural inclusion, and $\mathcal{H}$ is the restriction of $\mathcal{F}$ to H, then*

(1) *the zero divisor of $i^*\omega \in H^0(H, \Omega_H^q \otimes \det N_{\mathcal{F}}\big|_H)$ is equal to $\operatorname{diff}_H(\mathcal{F})$; and*

(2) *the zero divisor of $i^*\eta \in H^0(H, \Omega_{\mathcal{H}}^r \otimes (\det N_{\mathcal{G}} \otimes \det N_{\mathcal{F}}^*)\big|_H)$ is equal to the difference of differents $\operatorname{diff}_H(\mathcal{G}) - \operatorname{diff}_H(\mathcal{F})$.*

Proof. Item (1) is a reinterpretation of the proof of Lemma 5.6. To verify Item (2), first observe that we can define unambiguously the product $\beta = \omega \wedge \eta \in H^0(X, \Omega_X^{q+r} \otimes \det N_{\mathcal{G}})$ by considering an arbitrary rational lift $\hat{\eta}$ of η to $\Omega_X^r \otimes \det N_{\mathcal{G}} \otimes \det N_{\mathcal{F}}^*)$ and wedging it with ω. The resulting twisted $(q+r)$-form does not depend on the lift and is a regular twisted $(q+r)$-form without codimension one zeros defining the distribution $\mathcal{G}$. According to Item (1), the zero divisor of $i^*\beta$ is equal to $\operatorname{diff}_H(\mathcal{G})$. It follows that the zero divisor of $i^*\eta$ is equal to the difference $\operatorname{diff}_H(\mathcal{G}) - \operatorname{diff}_H(\mathcal{F})$ as claimed. $\qquad\square$

5.5. Differents and non-invariant subvarieties of higher codimension

A variant of the discussion about differents carried out in Subsection 5.4 can be made for the restriction of a distribution $\mathcal{F}$ to a smooth subvariety Y of codimension greater than one. One extra difficulty comes from the fact that the natural morphism

$$\delta: T_{\mathcal{F}}\big|_Y \to N_Y$$

can have any rank between 0 and $\operatorname{codim} Y$. When it has rank zero, the subvariety Y is $\mathcal{F}$-invariant. If the rank of δ is equal to $q = \operatorname{codim} Y$, then we will say that Y is strongly transverse to $\mathcal{F}$. In this case, the image of $\wedge^q \delta$ is of the form $\det N_Y \otimes \mathcal{I}$ for some ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$, and we define the different $\operatorname{diff}_Y(\mathcal{F})$ as the effective divisor on Y defined by the codimension one part of $\mathcal{I}$.

The definition implies that

$$(\omega_{\mathcal{F}} \otimes \det N_Y)\big|_Y \simeq \omega_{\mathcal{Y}} \otimes \mathcal{O}_Y(\operatorname{diff}_Y(\mathcal{F}))$$
$$(\det N_{\mathcal{F}})\big|_Y \simeq \det N_{\mathcal{Y}} \otimes \mathcal{O}_Y(\operatorname{diff}_Y(\mathcal{F}))$$

for any smooth subvariety $Y \subset X$ strongly transverse to $\mathcal{F}$. Likewise, we also have a natural analogue of Lemma 5.7.

Lemma 5.8. *Let $\mathcal{F}$ be a codimension q distribution on a smooth algebraic variety X defined by $\omega \in H^0(X, \Omega_X^q \otimes \det N_{\mathcal{F}})$. Let $\mathcal{G} \subset \mathcal{F}$ be a subdistribution of codimension r in $\mathcal{F}$ defined by $\eta \in H^0(X, \Omega_{\mathcal{F}}^r \otimes \det N_{\mathcal{G}} \otimes \det N_{\mathcal{F}}^*)$. If $Y \subset X$ is a smooth subvariety strongly transverse to $\mathcal{F}$ and $\mathcal{G}$, $i : Y \to X$ is the natural inclusion, and $\mathcal{Y}$ is the restriction of $\mathcal{F}$ to Y, then*

(1) *the zero divisor of $i^*\omega \in H^0(Y, \Omega_Y^q \otimes \det N_{\mathcal{F}}|_Y)$ is equal to $\mathrm{diff}_Y(\mathcal{F})$; and*

(2) *the zero divisor of $i^*\eta \in H^0(Y, \Omega_{\mathcal{Y}}^r \otimes (\det N_{\mathcal{G}} \otimes \det N_{\mathcal{F}}^*)|_Y)$ is equal to the difference of differents $\mathrm{diff}_Y(\mathcal{G}) - \mathrm{diff}_Y(\mathcal{F})$.*

Proof. Analogous to the proof of Lemma 5.7. $\qquad\square$

5.6. The degeneracy divisor of the p-curvature for the restriction to strongly transverse subvarieties

We are now ready to state and prove a result that offers a comparison between the degeneracy divisor of the p-curvature of a foliation and its restriction to a smooth subvariety.

Proposition 5.9. *Let $\mathcal{F}$ be a codimension one foliation on a smooth algebraic variety X defined over a field k of characteristic $p > 0$. If $Y \subset X$ is a smooth subvariety strongly transverse to by $\mathscr{V}(\mathcal{F})$ (consequently, strongly transverse to $\mathcal{F}$) and $\mathcal{Y}$ is the restriction of $\mathcal{F}$ to Y, then $\Delta_{\mathcal{Y}}$, the degeneracy divisor of the p-curvature of $\mathcal{Y}$, is equal to*

$$\left.(\Delta_{\mathcal{F}})\right|_Y + p \cdot \left(\mathrm{diff}_Y(\mathscr{V}(\mathcal{F})) - \mathrm{diff}_Y(\mathcal{F})\right) - \mathrm{diff}_Y(\mathcal{F}).$$

Proof. Let $\omega \in H^0(X, \Omega_X^1 \otimes N_{\mathcal{F}})$ be a twisted 1-form defining $\mathcal{F}$. According to Lemma 5.7 the zero divisor $i^*\omega$ is $\mathrm{diff}_Y(\mathcal{F})$. We can thus write $i^*\omega = \delta \cdot \alpha$, where δ belongs to $H^0(Y, \mathcal{O}_Y(\mathrm{diff}_Y(\mathcal{F})))$ and $\alpha \in H^0(Y, \Omega_Y^1 \otimes N_{\mathcal{Y}})$ is a twisted 1-form defining $\mathcal{Y} = \mathcal{F}|_Y$.

If $f \in H^0(X, \mathcal{O}_X(\Delta_{\mathcal{F}}))$ defines the degeneracy divisor of the p-curvature of $\mathcal{F}$ and $\eta \in H^0(X, \Omega_{\mathcal{F}}^1 \otimes \det N_{\mathscr{V}(\mathcal{F})} \otimes N_{\mathcal{F}}^*)$ is the element exhibited in Subsection 4.7 that defines $\mathscr{V}(\mathcal{F})$, then, using Lemma 4.16, we can write

$$\alpha(v^p) = \frac{i^*\omega(v^p)}{\delta} = \frac{i^*f}{\delta}\left(i^*\eta(v)\right)^p$$

for every $v \in T_{\mathcal{Y}}$. Since the degeneracy divisor of the p-curvature of $\mathcal{Y}$ is the largest effective divisor $\Delta_{\mathcal{Y}}$ such that $\Delta_{\mathcal{Y}} \le (\alpha(v^p))_0$ for every $v \in T_{\mathcal{Y}}$, we deduce that

$$\Delta_{\mathcal{Y}} = i^*(f)_0 - (\delta)_0 + p \cdot (i^*\eta)_0 = \left.\Delta_{\mathcal{F}}\right|_Y - \mathrm{diff}_Y(\mathcal{F}) + p \cdot (i^*\eta)_0.$$

The result follows from Item (2) of Lemma 5.8 . $\qquad\square$

5.7. The degeneracy divisor of the p-curvature on $\mathbb{P}_{\mathsf{k}}^2$ and $\mathbb{P}_{\mathsf{k}}^1 \times \mathbb{P}_{\mathsf{k}}^1$

In this subsection, we review a result on the structure of the degeneracy divisor of the p-curvature of sufficiently general foliations on $\mathbb{P}_{\mathsf{k}}^2$ and on Hirzebruch surfaces, and sketch its proof, obtained by the first author in [34]; see also [33]. We refer the reader to these works for details.

Theorem 5.10. *Let* k *be an algebraically closed field of characteristic $p > 0$. Let X be equal to $\mathbb{P}^2_k$ or a Hirzebruch surface. If $\mathcal{F}$ is a general foliation on X with canonical bundle $\omega_{\mathcal{F}}$, then at least one of the following assertions holds true.*

(1) *The line-bundle $\omega_{\mathcal{F}}$ is not pseudo-effective and $\mathcal{F}$ is a p-closed foliation.*

(2) *The surface is a Hirzebruch surface different from $\mathbb{P}^1_k \times \mathbb{P}^1_k$, the line-bundle $\omega_{\mathcal{F}}$ is pseudo-effective but not nef, the foliation $\mathcal{F}$ is p-dense but the unique curve of negative self-intersection appears as with multiplicity p in the degeneracy divisor of the p-curvature of $\mathcal{F}$.*

(3) *The image of $\omega_{\mathcal{F}}$ under the group morphism $\mathrm{Pic}(X) \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is zero.*

(4) *The degeneracy divisor of the p-curvature of $\mathcal{F}$ is reduced.*

Sketch of the proof. Let us first consider the case of $X = \mathbb{P}^2_k$. If $\mathcal{F}$ is a foliation of $\mathbb{P}^2_k$, then the degree of $\mathcal{F}$, $\deg(\mathcal{F})$, is defined by the number of tangencies between of $\mathcal{F}$ a non-invariant line ℓ, counted with multiplicities. By definition, $\deg(\mathcal{F}) \geq 0$ and a simple computation shows that $N_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^2}(\deg(\mathcal{F})+2)$ and $\omega_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^2_k}(\deg(\mathcal{F})-1))$.

If $\deg(\mathcal{F}) = 0$, then $\mathcal{F}$ is always p-closed. Indeed, if $\mathcal{F}$ is p-dense, then the degeneracy divisor of the p-curvature would be a section of $N_{\mathcal{F}} \otimes \omega_{\mathcal{F}}^{\otimes p} = \mathcal{O}_{\mathbb{P}^2_k}(2-p)$ containing the singular set of $\mathcal{F}$. A clear absurdity for $p > 2$. In the case $p = 2$, we also reach a contradiction since $\mathrm{sing}(\mathcal{F})$ is non-empty because $c_2(\Omega^1_{\mathbb{P}^2_k}(2)) = 1$.

If $\deg(\mathcal{F}) = 1$, then $\mathcal{F}$ is defined by a regular vector field v (since $\omega_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^2_k}$). In suitable affine coordinates, where 0 is a singularity of v and the line at infinity is invariant, a general v takes the form

$$v = \alpha x \frac{\partial}{\partial x} + \beta y \frac{\partial}{\partial y}.$$

If $\alpha/\beta \notin \mathbb{F}_p$, then $\mathcal{F}$ is p-dense with $\Delta_{\mathcal{F}}$ equal to the sum of the lines $\{x = 0\}$, $\{y = 0\}$ and the line at infinity.

If $\deg(\mathcal{F}) = 2$, then we consider the foliation on $\mathbb{P}^2_k$ defined, in affine coordinates, by

$$\omega = (xdy - ydx) + \omega_2,$$

where ω_2 is a general homogeneous 1-form with polynomial coefficients of degree two. The foliation defined by ω leaves the line at infinity invariant and three lines through $0 \in \mathbb{A}^2_k$ defined by the vanishing of $\omega(R) = \omega_2(R)$, where $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is the radial (or Euler) vector field. It follows that the degeneracy divisor is supported on the union of four lines above and a curve C of degree at most p, since the degeneracy divisor of the p-curvature of $\mathcal{F}$ has degree $p+4$. Blowing up the origin, one sees that the strict transform of C must intersect the strict transform of a line through 0 in exactly one point, and this implies, after some work, that for ω_2 generic, C is an irreducible and reduced curve of degree p invariant by $\mathcal{F}$.

For $\deg(\mathcal{F}) > 2$, we start with a foliation of $\mathcal{G}$ on $\mathbb{P}^2_k$ of degree two, with reduced $\Delta_{\mathcal{G}}$ and leaving invariant three lines in general position, say $\{xyz = 0\}$. We consider the endomorphism for every positive integer $\ell \geq 2$ not divisible by p; one considers the endomorphism of $\mathbb{P}^2_k$ defined by $\varphi_\ell(x:y:z) = (x^\ell:y^\ell:z^\ell)$. The pull-back foliation $\mathcal{F}_\ell = \varphi_\ell^* \mathcal{G}$

is a foliation with $\omega_{\mathcal{F}_\ell} = \mathcal{O}_{\mathbb{P}^2_k}(\ell)$, hence of degree $\ell + 1$. Moreover, since φ_ℓ is étale on the complement of the invariant divisor $\{xyz = 0\}$, Proposition 5.3 implies that $\Delta_{\mathcal{F}_\ell}$ is a reduced divisor. This shows that when Assertions (1), (2) and (3) are not valid, then Assertion (4) holds true.

To prove the result on Hirzebruch surfaces, one constructs examples of p-dense foliations with p-curvature having reduced degeneracy divisor by applying a series of birational transformations and ramifications as above starting from examples on $\mathbb{P}^2_k$. We refer to [34] for details. □

Remark 5.11. We point out, for later use, that the proofs presented in [34] show that whenever there exists a foliation on $\mathbb{P}^2_k$ or on a Hirzebruch surface with reduced degeneracy divisor for its p-curvature, there exists a foliation with the same property and same normal bundle that leaves invariant the boundary of X, seen as a toric surface.

6. Families of foliations

This section studies the behavior of the kernel of the p-curvature and the degeneracy divisor of the p-curvature under deformations.

6.1. Families of foliations

Let T be an irreducible algebraic variety. For us, a family of codimension q foliations parametrized by T consists of an algebraic variety $\mathcal{X}$, a smooth and projective morphism $\pi : \mathcal{X} \to T$, a line bundle $N_{\mathcal{F}}$ on $\mathcal{X}$, and a foliation $\mathcal{F}$ on $\mathcal{X}$ of codimension $q + \dim T$ defined by a relative q-form $\omega \in H^0(\mathcal{X}, \Omega^q_{\mathcal{X}/T} \otimes N_{\mathcal{F}})$ with codimension two singular set over any point $t \in T$ such that the kernel of

$$T_{\mathcal{X}/T} \longrightarrow \Omega^{q-1}_{\mathcal{X}/T} \otimes N_{\mathcal{F}}$$
$$v \mapsto i_v \omega$$

is an involutive subsheaf of $T_{\mathcal{X}}$ of rank $\dim \mathcal{X} - \dim T - q$.

When $\mathcal{X} = X \times T$, we will say that $\mathcal{F}$ is a family of foliations on X.

Remark 6.1. There are other possibilities for defining a family of foliations parameterized by T. One can, for instance, impose that a family of foliations is defined by an involutive subsheaf of $T_{\mathcal{X}/T}$ which is flat over T. This would lead to a more stringent concept for a family of foliations, which many authors have already studied. Our definition above aims to study the irreducible components of the space of foliations on projective spaces as considered by Cerveau, Lins Neto and others; see [11, 17] and references therein.

6.2. Behavior of the kernel of the p-curvature under deformations

If X is a polarized smooth and connected projective variety of dimension n, with polarization given by an ample divisor H, and $\mathcal{E}$ is a reflexive sheaf on X, we set the degree of $\mathcal{E}$ with respect to the polarization H equal to $\det(\mathcal{E}) \cdot H^{n-1}$.

Recall from the statement of Lemma 4.9 that $\mathscr{V}(\mathcal{F})$ is the distribution with tangent sheaf determined by the kernel of the p-curvature of a p-dense foliation $\mathcal{F}$; that is,

$$\mathrm{Frob}^* T_{\mathscr{V}(\mathcal{F})} = \ker \psi_{\mathcal{F}}.$$

Naturally, $\omega_{\mathscr{V}(\mathcal{F})} = \det(T_{\mathscr{V}(\mathcal{F})})^*$ is the canonical bundle of $\mathscr{V}(\mathcal{F})$.

Lemma 6.2. *Let $\mathscr{F}$ be a family of codimension one foliations on a smooth projective variety X parameterized by an algebraic variety T. If $0 \in T$ is a closed point such that $\mathscr{F}_0$ is p-dense, then there exists an open neighborhood $U \subset T$ of 0 such that for every closed point $t \in U$, the foliation $\mathscr{F}_t$ is p-dense and the inequality*

$$\deg_H \omega_{\mathscr{V}(\mathscr{F}_t)} \leq \deg_H \omega_{\mathscr{V}(\mathscr{F}_0)}$$

holds true for any polarization H.

Proof. Consider the relative p-curvature morphism

$$\psi_{\mathscr{F}/T} : \mathrm{Frob}^* T_{\mathscr{F}/T} \longrightarrow N_{\mathscr{F}}$$
$$v \mapsto \omega(v^p),$$

where $\mathrm{Frob} : \mathscr{X} \to \mathscr{X}$ is the absolute Frobenius.

Let $\mathcal{C}$ be the support of $\mathrm{coker}\,\psi_{\mathscr{F}/T}$ and fix an embedding of X in some $\mathbb{P}^N$. According to [25, Chapter III, Theorem 9.9], for every $m \gg 1$, the Hilbert function $h_{\mathcal{C}}(m,t) = \chi(X, \mathcal{O}_{\mathcal{C}_t}(m))$ is lower semi-continuous, where $\mathcal{O}_{\mathcal{C}_t}(m) = \mathcal{O}_{\mathcal{C}_t} \otimes \mathcal{O}_{\mathbb{P}^N}(m)$ as usual. Let Z be the subset of T such that for every $z \in Z$, corresponding to the p-closed foliations in the family. For every t in the complement $U = T - Z$, the subscheme $\mathcal{C}_t$ of X is different from X and has divisorial part equal to $\Delta_{\mathscr{F}_t}$. There exists a positive constant c such that $c \cdot \deg(\Delta_{\mathscr{F}_t})$ is the leading coefficient of the Hilbert function $h_{\mathcal{C}}(m,t)$.

Since $\det(\ker \psi_{\mathscr{F}_t})^*$, the Frobenius pull-back of the canonical bundle of the kernel of the p-curvature of $\mathscr{F}_t$, is isomorphic to $\det(\mathrm{Frob}^* T_{\mathscr{X}_t})^* \otimes \det(N_{\mathscr{F}_t} \otimes \mathcal{I}_{\mathcal{C}_t})$, we deduce from the lower semi-continuity of $h_{\mathscr{C}}$, the upper-semicontinuity of the function $t \mapsto \deg(\det(\ker \psi_{\mathscr{F}_t}))$ on U as wanted. $\square$

If one further assumes that the degeneracy divisor of a p-dense codimension one foliation is free from p-th powers, then one gets the reverse inequality for the degree of the kernel of the p-curvature.

Lemma 6.3. *Let $\mathscr{F}$ be a family of codimension one foliations on a smooth projective variety X parameterized by an algebraic variety T. If $0 \in T$ is a closed point such that $\mathscr{F}_0$ is p-dense and the degeneracy divisor of its p-curvature is free from p-th powers, then there exists a non-empty open subset $V \subset T$ such that for every closed point $t \in U$, the foliation $\mathscr{F}_t$ is p-dense and the equality*

$$\deg_H \omega_{\mathscr{V}(\mathscr{F}_t)} = \deg_H \omega_{\mathscr{V}(\mathscr{F}_0)}$$

holds true for any polarization H.

Proof. Thanks to Lemma 6.2, it suffices to show the existence of an open neighborhood U of $0 \in T$ such that

$$\deg_H \omega_{\mathcal{V}(\mathscr{F}_t)} \geq \deg_H \omega_{\mathcal{V}(\mathscr{F}_0)}$$

for any $t \in U$. For that, let $\mathcal{L}$ be a very-ample line bundle on $\mathscr{X} = X \times T$ such that $T_{\mathscr{F}} \otimes \mathcal{L}$ is generated by global sections. Choose $v \in H^0(\mathscr{X}, T_{\mathscr{F}} \otimes \mathcal{L})$ such that v_0, the restriction of v to $X \times \{0\}$, is generically transverse to $\mathcal{V}(\mathscr{F}_0)$. If $\omega \in H^0(\mathscr{X}, \Omega^1_{\mathscr{X}/T} \otimes N_{\mathscr{F}})$ is a twisted 1-form defining the family of foliations $\mathscr{F}$, then the section $\sigma = \omega(v^p) \in H^0(\mathscr{X}, N_{\mathscr{F}} \otimes \mathcal{L}^{\otimes p})$ defines a divisor $\mathscr{D}$ on $\mathscr{X}$. The choice of v guarantees that the support of $\mathscr{D}$ does not contain X_0. Consequently, there exists an open neighborhood of $0 \in T$ such that the support of $\mathscr{D}$ does not contain $\mathscr{X}_t$. For any $t \in U$, $\mathscr{D}_t := \mathscr{D}|_{\mathscr{X}_t}$ is, by construction, equal to $\Delta_{\mathscr{F}_t} + pR_t$ for some effective divisor R_t. Let $\mathscr{R} \in \mathrm{Div}(\mathscr{X})$ be the maximal effective divisor such that $p\mathscr{R} \leq \mathscr{D}$. Semi-continuity implies that $\mathscr{R}_t := \mathscr{R}|_{\mathscr{X}_t}$ is equal to R_t for every t in an non-empty open subset V of T (not necessarily containing 0).

Since we are assuming that $\Delta_{\mathscr{F}_0} = \mathscr{D}_0 - pR_0$ is free from p-th powers, we have that $\mathscr{R}_0 \leq R_0$. Using that the degrees $\deg_H \mathscr{D}_t$ and $\deg_H \mathscr{R}_t$ do not depend on t, we obtain that

$$\begin{aligned}
\deg_H \Delta_{\mathscr{F}_0} &\leq \deg_H (\Delta_{\mathscr{F}_0} + p(R_0 - \mathscr{R}_0)) \\
&= \deg_H (\mathscr{D}_0) - \deg_H (p\mathscr{R}_0) \\
&= \deg_H (\mathscr{D}_t) - \deg_H (p\mathscr{R}_t) = \deg_H \Delta_{\mathscr{F}_t}
\end{aligned}$$

for every $t \in V$. Proposition 4.12 implies the result. $\square$

7. Integrability of the Cartier transform

So far, we have studied foliations in positive characteristic without reference to foliations in characteristic zero. In this section, we show that foliations in positive characteristic obtained through reduction of foliations in characteristic zero are far from being arbitrary as the Cartier transform of them are integrable/involutive, a property that does not hold for arbitrary foliations.

7.1. Lifts of foliations modulo p^2

Let k be an algebraically closed field of characteristic $p > 0$, let $W_2(\mathsf{k})$ be the ring of Witt vectors of length 2 with values in k, and set $S = \mathrm{Spec}(W_2(\mathsf{k}))$. A lift of a smooth algebraic variety X module p^2 is a flat smooth S-scheme X' such that $X = X' \otimes \mathsf{k}$. Likewise, a lift modulo p^2 of a foliation on X consists of a lift X' of X modulo p^2 and subsheaf $T_{\mathcal{F}'} \subset T_{X'/S}$, flat over S, such that

(1) the sheaf $T_{\mathcal{F}'}$ is a lift of $T_{\mathcal{F}}$, *i.e.*, $T_{\mathcal{F}} = T_{\mathcal{F}'} \otimes \mathsf{k}$; and
(2) the sheaf $T_{\mathcal{F}'}$ is involutive; that is, the morphism

$$\bigwedge^2 T_{\mathcal{F}'} \to \frac{T_{X'/S}}{T_{\mathcal{F}'}}$$

induced by Lie brackets is identically zero.

As already noted by Miyaoka in [35, Example 4.2], not every foliation admits a lift modulo p^2. As it will be essential in what follows, we present below a variant of Miyaoka's example.

Example 7.1. Let p be an odd prime, let k be the algebraic closure of $\mathbb{F}_p$ and let ω be the 1-form $x^{p-1}dx + z^p y^{p-1}dy$ on $\mathbb{A}^3_{\mathbb{Z}}$. The reduction modulo p of ω defines a foliation $\mathcal{F}$ on $\mathbb{A}^3_{\mathbb{Z}}$ that does not admit a lift modulo p^2.

Proof. First, observe that

$$\omega \wedge d\omega = p(xyz)^{p-1} dx \wedge dy \wedge dz,$$

and, therefore, the reduction modulo p of ω defines an involutive subsheaf of $T_{\mathbb{A}^3_k}$. Let $\mathcal{F}$ be the corresponding foliation on $\mathbb{A}^3_k$. If $\mathcal{F}$ admits a lift modulo p^2 and we denote by $\overline{\omega}$ the reduction of ω modulo p^2, then such lift is defined by the 1-form $\overline{\omega} + p\eta$ for a suitable 1-form $\eta \in \Omega^1_{\mathbb{A}^3_k}$ which satisfies

$$(\overline{\omega} + p\eta) \wedge d(\overline{\omega} + p\eta) = 0 \quad \text{in} \quad \Omega^1_{\mathbb{A}^3_{W_2(k)}}. \tag{7.1}$$

Expanding the left-hand side of Equation (7.1), we get

$$p\left(z^p y^{p-1} dy \wedge d\eta + (xyz)^{p-1} dx \wedge dy \wedge dz + x^{p-1} dx \wedge d\eta \right) = 0.$$

Since $d\eta$ has no terms that are scalar multiples of $(xy)^{p-1}$, this expression is always non-zero. It follows that $\mathcal{F}$ does not admit a lift modulo p^2. $\square$

Remark 7.2. Observe that the arguments presented in Example 7.1 not only show that $\mathcal{F}$ does not admit a lift modulo p^2 but also show that the restriction of $\mathcal{F}$ to any open subset $U \subset \mathbb{A}^3_k$ containing the origin does not admit a lift modulo p^2.

7.2. Integrability of the Cartier transform of liftable foliations

Our next result is an amplification of Miyaoka's Example 7.1.

Theorem 7.3. *Let $\mathcal{F}$ be a codimension one foliation on a smooth projective variety X defined over a field k of characteristic $p > 2$. If $\mathcal{F}$ is p-dense and lifts to a foliation over the ring of Witt vectors $W_2(k)$, then the Cartier transform of $\mathcal{F}$ is a foliation.*

Proof. We will prove the contrapositive, (i.e., if the Cartier transform of a p-dense codimension one foliation $\mathcal{F}$ is not a foliation, then $\mathcal{F}$ does not lift modulo p^2).

Proposition 4.5 guarantees the existence of a closed rational 1-form ω defining $\mathcal{F}$. The Cartier transform of $\mathcal{F}$ is defined by the rational 1-form $\theta = \mathbf{C}(\omega)$. By assumption, the 3-form $\theta \wedge d\theta$ does not vanish identically. In particular, $\dim X \geq 3$.

Assume first that $\dim X = 3$. Let $a \in X$ be a closed point such that θ is regular (no poles) at x and $\theta \wedge d\theta(x) \neq 0$. Denote by $\mathfrak{m} \subset \mathcal{O}_{X,a}$ the corresponding maximal ideal. Let $x, y \in \mathfrak{m}$ be such that $d\theta = dx \wedge dy \mod \mathfrak{m}\Omega^2_{X,a}$. We can *integrate* the 2-form $d\theta$ modulo $\mathfrak{m}^2$ and write

$$\theta = xdy + dz + w^{p-1} dw \mod \mathfrak{m}^2 \Omega^1_{X,a}$$

for some $z,w \in \mathfrak{m}$. If $p > 2$, then we can assume that w is equal to zero as $w^{p-1} \in \mathfrak{m}^2$. The non-vanishing of $\theta \wedge d\theta$ at a implies that x,y,z generate $\mathfrak{m}$.

Since $\theta = \mathbf{C}(\omega)$, we can write

$$\omega = x^p y^{p-1} dy + z^{p-1} dz + df \mod \mathfrak{m}^2 \Omega^1_{X,a}$$

for some $f \in \mathfrak{m}$.

Let X' be a lift of X, let a' be a lift of a and let ω' be a 1-form in $\Omega^1_{X',a'}$ that defines a lift $\mathcal{F}'$ of $\mathcal{F}$ to X' and with reduction modulo p equal to ω. We can write

$$\omega' = (x^p y^{p-1} dy + z^{p-1} dz + df') + p\eta \mod \langle p^2, \mathfrak{m}^{2p} \Omega^1_{X',a'} \rangle$$

for any lift f' of f and a certain $\eta \in \Omega^1_{X,a}$. Expanding $\omega' \wedge d\omega'$, we get

$$p \left((x^p y^{p-1} dy + z^{p-1} dz + df') \wedge d\eta + (xyz)^{p-1} dx \wedge dy \wedge dz \right) +$$
$$+ p \left((xy)^{p-1} df' \wedge dx \wedge dy \right)$$

modulo $\langle p^2, \mathfrak{m}^{3p-2} \Omega^1_{X',a'} \rangle$. As in Example 7.1, for no matter which choice of f' and η, the term $(xyz)^{p-1} dx \wedge dy \wedge dz$ will not be cancelled. This shows that no lift of ω defines a foliation when $\dim X = 3$.

When $\dim X > 3$, we reduce to the previous case by restricting to the intersection of $\dim X - 3$ sufficiently general hyperplane sections. □

As pointed out by the referee, thanks to Chow's Lemma [43, Tag 0200], the result above also holds for proper smooth varieties.

8. Lifting the kernel of the *p*-curvature

This section defines integral models for holomorphic foliations and discusses the lifting, back to characteristic zero, of properties of the reduction to positive characteristic of a holomorphic foliation.

8.1. Reduction modulo *p* of holomorphic foliations on projective manifolds

Let $\mathcal{F}$ be a singular holomorphic foliation defined on a complex projective manifold X. The variety X and the subsheaf $T_{\mathcal{F}} \subset T_X$ can be viewed as objects defined over a finitely generated $\mathbb{Z}$-algebra $R \subset \mathbb{C}$. More precisely, there exists a projective scheme $\mathscr{X}$ flat over $\mathrm{Spec}(R)$ and subsheaf $T_{\mathscr{F}} \subset T_{\mathscr{X}/R}$ flat over $\mathrm{Spec}(R)$ such that X is isomorphic to the base change $\mathscr{X} \otimes_R \mathbb{C}$ and $\mathcal{F}$ is isomorphic to $\mathscr{F} \otimes_R \mathbb{C}$.

The pair $(\mathscr{X}, \mathscr{F})$ is called an integral model for $(X, \mathcal{F})$ and is not uniquely determined by $(X, \mathcal{F})$. Roughly speaking, it depends on the choice of generators for the ideal of X, on a finite presentation of $T_{\mathcal{F}}$, as well as on the choice of an open subset of the affine scheme determined by the coefficients used to define the ideal and the presentation.

If $\mathfrak{p} \subset R$ is a maximal ideal, then the residue field $\mathsf{k}(\mathfrak{p}) = R/\mathfrak{p}$ is a finite field of characteristic $p > 0$ with algebraic closure $\mathsf{k} = \overline{\mathsf{k}(\mathfrak{p})} \simeq \overline{\mathbb{F}}_p$. The reduction of $\mathcal{F}$ modulo $\mathfrak{p}$ is the foliation $\mathcal{F}_{\mathfrak{p}}$ of $X_{\mathfrak{p}} = \mathscr{X} \otimes_R \mathsf{k}$ determined by $T_{\mathcal{F}_{\mathfrak{p}}} = T_{\mathscr{F}} \otimes_R \mathsf{k}$.

Example 8.1. Let $\mathcal{F}$ be the codimension one foliation on $\mathbb{A}^3_\mathbb{C}$ defined by the 1-form

$$\omega = \sqrt{-1} \cdot \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}.$$

We can take the $\mathbb{Z}$-algebra $R = \mathbb{Z}[\sqrt{-1}] \simeq \mathbb{Z}[\alpha]/(\alpha^2 + 1)$ as the ring of definition of $\mathcal{F}$. The ring R has the following (maximal) prime ideals:

(1) only one prime ideal over 2, namely $(1 + \sqrt{-1})R = (1 - \sqrt{-1})R$;

(2) for each rational prime $p = a^2 + b^2$ congruous to 1 modulo 4, two distinct prime ideals, namely $(a \pm \sqrt{-1}b)R$; and

(3) for each rational prime p congruous to 3 modulo 4, the prime ideal pR.

If we reduce $\mathcal{F}$ modulo $(1 + \sqrt{-1})R$, we get a p-closed foliation over $\overline{\mathbb{F}}_2$ defined by $d\log(xyz)$. Likewise, if we reduce $\mathcal{F}$ modulo $(a \pm b\sqrt{-1})R$, $p = a^2 + b^2$ being a prime congruous to 1 modulo 4, we get a p-closed foliation over $\overline{\mathbb{F}}_p$ defined by $d\log(xyx)$. In contrast, if we reduce $\mathcal{F}$ modulo $\mathfrak{p} = pR$, where p is a rational prime congruous to 3 modulo 4, we get a p-dense foliation with $\mathscr{V}(\mathscr{F}_\mathfrak{p})$ defined by the kernel of the 2-form

$$\omega_\mathfrak{p} \wedge \mathbf{C}(\omega_\mathfrak{p}) = (\alpha_\mathfrak{p} \cdot \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}) \wedge (\alpha_\mathfrak{p}^{1/p} \cdot \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}),$$

where $\alpha_\mathfrak{p}$ stands for the class of α in the algebraic closure of $R/\mathfrak{p} \simeq \mathbb{F}_{p^2}$.

Example 8.2. Now let $\mathcal{F}$ be the codimension one foliation on $\mathbb{A}^3_\mathbb{C}$ defined by the 1-form

$$\omega = \alpha \cdot \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z},$$

with α equal to an arbitrary transcendental number. We can take the $\mathbb{Z}$-algebra $R = \mathbb{Z}[\alpha]$ as the ring of definition of $\mathcal{F}$. The maximal ideals of the ring R are of the form $(p, f(\alpha))R$, where p is a rational prime and $f \in \mathbb{Z}[\alpha]$ is a monic polynomial with irreducible reduction modulo p. The reduction of $\mathcal{F}$ modulo $\mathfrak{p} = (p, f(\alpha))R$ is p-closed if the degree of f is one; otherwise, $\mathscr{F}_\mathfrak{p}$ is p-dense.

8.2. A conjecture by Ekedahl, Shepherd-Barron and Taylor

In general, it is not easy to draw conclusions about the geometry of the holomorphic foliation $\mathcal{F}$ from the behavior of the reduction modulo $\mathfrak{p}$ for different primes $\mathfrak{p}$.

Conjecture 8.3 (Ekedahl, Shepherd-Barron and Taylor). *Let $\mathcal{F}$ be a holomorphic foliation on a complex projective manifold X with an integral model $(\mathscr{X}, \mathscr{F})$ defined over a finitely generated $\mathbb{Z}$-algebra R. The foliation $\mathcal{F}$ is algebraically integrable (i.e., all its leaves are algebraic) if and only if $\mathcal{F}_\mathfrak{p}$ is p-closed for every maximal prime (i.e., closed point) $\mathfrak{p}$ in a non-empty Zariski open subset of $\mathrm{Spec}(R)$.*

Despite the important particular cases settled by Bost in [6], Conjecture 8.3 is wide open. It must be mentioned that Conjecture 8.3 generalizes a conjecture by Grothendieck and Katz, which is also wide open despite recent advances [41, 37].

Under the assumptions of Conjecture 8.3, Theorem 4.1 implies the existence of algebraic subvarieties invariant by $\mathscr{F}_\mathfrak{p}$ that cover $X_\mathfrak{p}$ for a Zariski open and dense set of maximal primes $\mathfrak{p} \subset \mathrm{Spec}(R)$. This is not enough to guarantee the existence of algebraic subvarieties invariant by $\mathcal{F}$ covering X as the degrees of the subvarieties of $X_\mathfrak{p}$ invariant by $\mathscr{F}_\mathfrak{p}$ may be unbounded as a function of $\mathfrak{p}$.

8.3. Lifting subvarieties/subdistributions of bounded degree

If one can guarantee the existence of subvarieties/subdistributions for the reduction of a foliation modulo a Zariski dense set of maximal primes with degree uniformly bounded, then the lemma below implies the existence of the same type of objects in characteristic zero respecting the same bounds.

Lemma 8.4. *Let $\mathcal{F}$ be a holomorphic foliation on a polarized projective manifold (X, H) and let $(\mathscr{X}, \mathscr{H}, \mathscr{F})$ be an integral model for $(X, H, \mathcal{F})$ with ring of definition R. If there are integers M, m, and a Zariski dense set of maximal primes $\mathscr{P} \subset \mathrm{Spec}(R)$, such $\mathscr{F}_\mathfrak{p}$ has, for every $\mathfrak{p} \in \mathscr{P}$,*

(1) *an invariant subvariety of dimension m and degree at most M; or*

(2) *a subdistribution $\mathscr{G}_\mathfrak{p}$ of dimension m with $\deg_{\mathscr{H}_\mathfrak{p}} \omega_{\mathscr{G}_\mathfrak{p}} \leq M$.*

Then $\mathcal{F}$ has, respectively,

(i) *an invariant subvariety of dimension m and degree at most M; or*

(ii) *a subdistribution $\mathcal{G}$ of dimension m with $\deg_H \omega_\mathcal{G} \leq M$.*

Proof. The implication $(1) \implies (i)$ is the content of [31, Proposition 7.1]. The proof of the implication $(2) \implies (ii)$ follows the same principles and is very similar to the proof of [39, Proposition 6.2]. We proceed to give details of this later implication.

Observe that subdistributions of $\mathcal{F}$, resp. $\mathscr{F}_\mathfrak{p}$, are in one-to-one correspondence with torsion-free quotients of $T_\mathcal{F}$, resp. $T_{\mathscr{F}_\mathfrak{p}}$. Once the degree of the quotient is bounded from above, [24, Corollary 2.3] guarantees that the possible Hilbert polynomials for the quotients belong to a finite set. Moreover, bounds for the degree of the quotient are equivalent, through adjunction, to bounds for the degree of the canonical bundle of the subdistribution.

The formation of Quot schemes commute with base change, [24]. Therefore, for a fixed Hilbert polynomial χ, the scheme $\mathrm{Quot}_\chi(X, T_\mathcal{F})$ is non-empty if and only if its reduction modulo $\mathfrak{p}$, $\mathrm{Quot}_\chi(X_\mathfrak{p}, T_{\mathscr{F}_\mathfrak{p}})$, is non-empty for a Zariski dense set of primes $\mathfrak{p}$. This is sufficient to prove that $(2) \implies (ii)$. $\qquad\square$

To obtain a priori upper bounds for the degree of invariant subvarieties of the reduction to positive characteristic does not seem more manageable than obtaining the same upper bounds in characteristic zero. In contrast, if $\mathcal{F}$ is a holomorphic foliation such that the reduction modulo $\mathfrak{p}$ is not p-closed for a Zariski dense set of primes $\mathfrak{p} \subset \mathrm{Spec}(R)$, then the degree of the canonical bundle of $\mathscr{V}(\mathscr{F}_\mathfrak{p})$ is uniformly bounded as a function of $\mathfrak{p}$. Therefore, one gets from the lemma above the existence of certain subdistributions in characteristic zero. In Section 9, we will explore this fact to obtain new results about

the irreducible components of the space of holomorphic codimension one foliations on projective spaces.

8.4. A cautionary remark

For foliations of codimension one, Conjecture 8.3 can be rephrased as follows:

the general leaf of a codimension one holomorphic foliation $\mathcal{F}$ is Zariski dense if and only if $\mathcal{F}_{\mathfrak{p}}$ is p-dense for a non-empty Zariski open set of maximal primes $\mathfrak{p} \in \operatorname{Spec}(R)$.

By dropping the codimension one assumption, one obtains a strengthening of Conjecture 8.3, which turns out to be false as we now proceed to show.

Proposition 8.5. *Let X be a complex projective manifold of dimension $2n+1$. Assume that $H^0(X, \Omega_X^2)$ is generated by a 2-form ω satisfying $\omega^n \neq 0$ and let $\mathcal{F}$ be the foliation on X of dimension one defined by the kernel of ω. If $(\mathscr{X}, \mathscr{F})$ is an integral model for $(X, \mathcal{F})$ defined over a finitely generated $\mathbb{Z}$-algebra R, then $\mathcal{F}_{\mathfrak{p}}$ is not p-dense for any maximal prime $\mathfrak{p}$ in an open and dense subset of $\operatorname{Spec}(R)$.*

Proof. By semi-continuity, there exists an open and dense subset $U \subset \operatorname{Spec}(R)$ such that $h^0(\mathscr{X}_{\mathfrak{p}}, \Omega_{\mathscr{X}_{\mathfrak{p}}}^2) = 1$ for every maximal prime $\mathfrak{p} \in U$. For $\mathfrak{p} \in U$, let $\omega_{\mathfrak{p}}$ be the generator of $H^0(\mathscr{X}_{\mathfrak{p}}, \Omega_{\mathscr{X}_{\mathfrak{p}}}^2)$ or, equivalently, the reduction modulo $\mathfrak{p}$ of a generator of $H^0(\mathscr{X}, \Omega_{\mathscr{X}}^2)$. Replacing U by an open subset, we can assume that $\omega_{\mathfrak{p}}^n \neq 0$ for every $\mathfrak{p} \in U$.

Fix an arbitray maximal prime $\mathfrak{p} \in U$. Since $\omega_{\mathfrak{p}}^n \neq 0$, the kernel of the morphism from $T_{X_{\mathfrak{p}}}$ to $\Omega_{X_{\mathfrak{p}}}^1$ defined by contraction with $\omega_{\mathfrak{p}}$ is the tangent sheaf of a foliation $\mathcal{F}_{\mathfrak{p}}$ of dimension one. Let v be any rational section of $T_{\mathcal{F}_{\mathfrak{p}}}$. Since $h^0(X, \Omega_{X_{\mathfrak{p}}}^2) = 1$, $\mathbf{C}(\omega_{\mathfrak{p}})$ is a multiple of $\omega_{\mathfrak{p}}$. Consequently, $i_v \mathbf{C}(\omega_{\mathfrak{p}}) = 0$. According to Lemma 4.16,

$$0 = i_v \mathbf{C}(\omega_{\mathfrak{p}}) = \mathbf{C}(i_{v^p} \omega_{\mathfrak{p}} - (L_v)^{p-1}(i_v \omega_{\mathfrak{p}})) = \mathbf{C}(i_{v^p} \omega_{\mathfrak{p}}).$$

The vanishing of $\mathbf{C}(i_{v^p} \omega)$ implies the existence of a rational function f on $X_{\mathfrak{p}}$ such that $i_{v^p} \omega_{\mathfrak{p}} = df$ (i.e., $i_{v^p} \omega_{\mathfrak{p}}$ is exact). If $df = 0$, then $\mathcal{F}_{\mathfrak{p}}$ is p-closed; if not, then f is a non-trivial rational first integral for $\mathcal{F}_{\mathfrak{p}}$. In both cases, $\mathcal{F}_{\mathfrak{p}}$ is not p-dense as claimed. $\square$

If X is a complex projective symplectic manifold of dimension $2n$ with symplectic form ω and $Y \subset X$ is a smooth hypersurface with big normal bundle, then the characteristic foliation of Y (i.e., the one-dimensional foliation on Y induced by the pull-back of ω^{n-1} to Y) has Zariski dense general leaf [3, Theorem 1.7]. In contrast, Proposition 8.5 above implies that any integral model for $\mathcal{F}$ is not p-dense for almost every prime $\mathfrak{p}$.

Part 2. The space of holomorphic foliations on projective spaces

9. Foliations on complex projective spaces

In the second part of the paper, we will focus on codimension one foliations on complex projective spaces. We will use the results presented in the first part to obtain new information about the irreducible components of the scheme $\mathsf{Fol}_d(\mathbb{P}_{\mathbb{C}}^n)$ parameterizing degree d codimension one foliations on $\mathbb{P}_{\mathbb{C}}^n$.

In this section, we settle the notation, recall the definition of the space of foliations on projective spaces and establish the conventions concerning the reduction to positive characteristic used throughout the second part of the paper.

9.1. Foliations on projective spaces

Let k be an arbitrary algebraically closed field and let $\mathcal{F}$ be a codimension $q \geq 1$ foliation on the projective space $\mathbb{P}^n_{\mathsf{k}}$, $n \geq q+1$. Let $\omega \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^q_{\mathbb{P}^n_{\mathsf{k}}} \otimes \det N_{\mathcal{F}})$ be a twisted q-form with coefficients in the line-bundle $\det N_{\mathcal{F}}$ such that $T_{\mathcal{F}} \subset T_{\mathbb{P}^n_{\mathsf{k}}}$ equals to the kernel of morphism

$$T_{\mathbb{P}^n_{\mathsf{k}}} \longrightarrow \Omega^{q-1}_{\mathbb{P}^n_{\mathsf{k}}} \otimes \det N_{\mathcal{F}}$$

defined by contraction with ω. The degree of $\mathcal{F}$ is, by definition, the degree of the zero locus of $i^*\omega \in H^0(\mathbb{P}^q_{\mathsf{k}}, \Omega^q_{\mathbb{P}^q_{\mathsf{k}}} \otimes i^* N_{\mathcal{F}})$, where $i : \mathbb{P}^q_{\mathsf{k}} \to \mathbb{P}^n_{\mathsf{k}}$ is a general linear embedding of $\mathbb{P}^q_{\mathsf{k}}$ into $\mathbb{P}^n_{\mathsf{k}}$. In other words, the degree of $\mathcal{F}$ is the degree of the tangency divisor of $\mathcal{F}$ with a general $\mathbb{P}^q_{\mathsf{k}}$ linearly embedded in $\mathbb{P}^n_{\mathsf{k}}$. Since $\Omega^q_{\mathbb{P}^q_{\mathsf{k}}} = \mathcal{O}_{\mathbb{P}^q_{\mathsf{k}}}(-(q+1))$, it follows that

$$\det N_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}(\deg(\mathcal{F})+q+1)$$

for any codimension q foliation $\mathcal{F}$ of degree $\deg(\mathcal{F})$. Adjunction formula implies that any such foliation $\mathcal{F}$ has canonical sheaf satisfying

$$\omega_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}(\deg(\mathcal{F})-\dim(\mathcal{F})).$$

It follows from Euler's sequence that we can identify $H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^q_{\mathbb{P}^n}(d+q+1))$ with the vector space of homogenous polynomial q-forms

$$\sum_{0 \leq i_1 \leq \ldots \leq i_q \leq n}^{n} a_{i_1,\ldots,i_q}(x_0,\ldots,x_n) dx_{i_1} \wedge \cdots \wedge dx_{i_q}$$

with coefficients $a_{i_1,\ldots,i_q} \in \mathsf{k}[x_0,\ldots,x_n]$ of degree $d+1$, which are annihilated by the radial/Euler vector field $R = \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$. We will say that a homogeneous polynomial q-form on $\mathbb{A}^{n+1}_{\mathsf{k}}$ representing an element of $H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^q_{\mathbb{P}^n}(d+q+1))$ is a projective q-form of degree $d+q+1$.

9.2. The space of codimension one foliations on projective spaces

The space/scheme of codimension one foliations of degree d on the projective space $\mathbb{P}^n_{\mathsf{k}}$ is, by definition, the locally closed subscheme $\mathsf{Fol}_d(\mathbb{P}^n_{\mathsf{k}})$ of $\mathbb{P}H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(d+2))$ defined by the conditions

$$[\omega] \in \mathsf{Fol}_d(\mathbb{P}^n_{\mathsf{k}}) \text{ if, and only if, } \mathrm{codim}\,\mathrm{sing}(\omega) \geq 2 \text{ and } \omega \wedge d\omega = 0.$$

The space/scheme of codimension q foliations of degree d foliations of $\mathbb{P}^n_{\mathsf{k}}$ can be defined similarly; we refer to [15] for details. Here we will only mention that besides the integrability/Frobenius condition, one also has to add the decomposability/Plücker conditions to consider only q-forms with kernels having corank q.

9.3. Conventions about the reduction to positive characteristic

Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}_{\mathbb{C}}^n$ (i.e., $\mathcal{F}$ is a codimension one holomorphic foliation of a complex projective space of dimension n). Throughout this part of the paper, we will need to consider an integral model $(\mathscr{X}, \mathscr{F})$ for $\mathcal{F}$ as defined in Section 8. Concretely, if ω is a projective 1-form defining $\mathcal{F}$ and $R_0 \subset \mathbb{C}$ is any finitely generated $\mathbb{Z}$-algebra containing the coefficients of ω, then we take R equal to a finitely generated extension of R_0 obtained by inverting finitely many elements of R_0, $\mathscr{X} = \mathbb{P}_R^n = \operatorname{Proj} R[x_0, \ldots, x_n]$, and $\mathscr{F}$ as the foliation defined by ω. Notice that one needs to pass to the open subset $\operatorname{Spec}(R) \subset \operatorname{Spec}(R_0)$ to obtain the flatness properties required by the definition.

We will say that a property P for a foliation $\mathcal{F}$ holds for almost every prime if and only if there exists an integral model of $\mathcal{F}$ defined over a finitely generated $\mathbb{Z}$-algebra R and there exists a non-empty open subset $U \subset \operatorname{Spec}(R)$ such that for every maximal prime $\mathfrak{p} \in U$, the property P holds for the reduction of $\mathcal{F}$ modulo $\mathfrak{p}$.

Likewise, we will say a property P holds for a Zariski dense set of primes if and only if there exists an integral model of $\mathcal{F}$ defined over a finitely generated $\mathbb{Z}$-algebra R and there exists a Zariski dense subset $Z \subset \operatorname{Spec}(R)$ formed only by maximal primes such that for every $\mathfrak{p} \in Z$, the property P holds for the reduction modulo $\mathfrak{p}$ of $\mathscr{F}$.

These two notational shortcuts will allow us to shorten the statements quite a bit.

10. Subdistributions of minimal degree

If $\mathcal{F}$ is a codimension one foliation on $\mathbb{P}_{\mathsf{k}}^n$ and i is an integer between 1 and $n-1$, then we define

$$\delta_i(\mathcal{F}) = \min_{\mathcal{D} \subset \mathcal{F}} \deg(\mathcal{D}),$$

where $\mathcal{D}$ ranges over all codimension i distributions everywhere tangent to in $\mathcal{F}$. By definition, $\delta_1(\mathcal{F}) = \deg(\mathcal{F})$.

If $\Sigma \subset \mathsf{Fol}_d(\mathbb{P}_{\mathsf{k}}^n)$ is an irreducible component, then we will set $\delta_i(\Sigma)$ equal to $\delta_i(\mathcal{F})$, where $\mathcal{F}$ is a generic foliation $\mathcal{F} \in \Sigma$. Semi-continuity implies that

$$\delta_i(\Sigma) \geq \delta_i(\mathcal{F})$$

for every $\mathcal{F} \in \Sigma$.

10.1. Intersections and spans of pairs of distributions

Let X be a projective manifold defined over an algebraically closed field k. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two distributions on X. We define $\mathcal{D}_1 \cap \mathcal{D}_2$ as the distribution on X with tangent sheaf equal to the intersection $T_{\mathcal{D}_1} \cap T_{\mathcal{D}_2} \subset T_X$. If we consider the natural morphism

$$\varphi : T_{\mathcal{D}_1} \oplus T_{\mathcal{D}_2} \longrightarrow T_X$$
$$v_1 \oplus v_2 \mapsto v_1 + v_2,$$

then $T_{\mathcal{D}_1 \cap D_2}$ is isomorphic to the kernel of φ.

Likewise, we define the span of $\mathcal{D}_1$ and $\mathcal{D}_2$, denoted by $\mathcal{D}_1 + \mathcal{D}_2$, as the distribution on X with tangent sheaf equal to the saturation of the image of φ in T_X.

Our definitions imply that $T_{\mathcal{D}_1 \cap \mathcal{D}_2}$ and $T_{\mathcal{D}_1 + \mathcal{D}_2}$ fit into the exact sequence

$$0 \to T_{\mathcal{D}_1 \cap \mathcal{D}_2} \to T_{\mathcal{D}_1} \oplus T_{\mathcal{D}_2} \xrightarrow{\varphi} T_{\mathcal{D}_1 + \mathcal{D}_2}.$$

Observe that the rightmost arrow has torsion cokernel. Taking determinant, we get the existence of an effective divisor $\mathrm{defect}(\mathcal{D}_1, \mathcal{D}_2) \in \mathrm{Div}(X)$ with support contained in the support of the cokernel of the rightmost arrow such that

$$\omega_{\mathcal{D}_1} \otimes \omega_{\mathcal{D}_2} = \omega_{\mathcal{D}_1 \cap \mathcal{D}_2} \otimes \omega_{\mathcal{D}_1 + \mathcal{D}_2} \otimes \mathcal{O}_X(\mathrm{defect}(\mathcal{D}_1, \mathcal{D}_2)).$$

When $\mathrm{codim}\,\mathcal{D}_1 \cap \mathcal{D}_2 = \mathrm{codim}\,\mathcal{D}_1 + \mathrm{codim}\,\mathcal{D}_2$, then the divisor $\mathrm{defect}(\mathcal{D}_1, \mathcal{D}_2)$ is equal to the zero divisor of $\omega_1 \wedge \omega_2$, where ω_1 and ω_2 are, respectively, differential forms with coefficients in $\det N_{\mathcal{D}_1}$ and $\det N_{\mathcal{D}_2}$ defining $\mathcal{D}_1$ and $\mathcal{D}_2$.

Specializing to distributions on projective spaces (i.e., $X = \mathbb{P}^n_{\mathsf{k}}$), we obtain the identity

$$\deg(\mathcal{D}_1 \cap \mathcal{D}_2) = \deg(\mathcal{D}_1) + \deg(\mathcal{D}_2) - \deg(\mathcal{D}_1 + \mathcal{D}_2) - \deg(\mathrm{defect}(\mathcal{D}_1, \mathcal{D}_2)), \qquad (10.1)$$

where we adopt the conventions that the foliation with only one leaf has degree -1 and that the foliation by points has degree zero. These conventions are made in order to ensure that $\omega_{\mathcal{F}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n_{\mathsf{k}}}(\deg(\mathcal{F}) - \dim(\mathcal{F}))$.

10.2. Bounds for the minimal degrees of subdistributions

In this subsection, we establish bounds for the integers $\delta_i(\mathcal{F})$. We will make repeated use of the following observation.

Lemma 10.1. *Let* k *be an algebraically closed field and let* $\mathcal{F}$ *be a codimension one foliation on* $\mathbb{P}^n_{\mathsf{k}}$. *Then the vector space*

$$H^0\big(\mathbb{P}^n_{\mathsf{k}}, \Omega^{i-1}_{\mathcal{F}}(\delta_i(\mathcal{F}) - \deg(\mathcal{F}) + (i-1))\big)$$

is non-zero for any i *between* 1 *and* $n-1$.

Proof. Let $\omega \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(\deg(\mathcal{F}) + 2))$ be a twisted 1-form defining $\mathcal{F}$. Let $\mathcal{D}$ be a distribution on $\mathbb{P}^n_{\mathsf{k}}$ of degree δ and codimension q. As such, $\mathcal{D}$ is defined by a twisted q-form $\eta \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^q_{\mathbb{P}^n_{\mathsf{k}}}(\delta + q + 1))$. If $\mathcal{D}$ is everywhere tangent to $\mathcal{F}$, then locally, away from $\mathrm{sing}(\mathcal{F})$, we can decompose η as the wedge product of ω with a $(q-1)$-form α. The $(q-1)$-form is not uniquely defined since $\omega \wedge \alpha = \omega \wedge (\alpha + \omega \wedge \beta)$ for any $(q-2)$-form β, but this is the only ambiguity in the definition of α. Hence, $\alpha\big|_{T_{\mathcal{F}}}$ is unique and we have a global non-zero section of $\Omega^{q-1}_{\mathcal{F}}(\delta + q + 1 - (\deg(\mathcal{F}) + 2))$ on $\mathbb{P}^n_{\mathsf{k}} - \mathrm{sing}(\mathcal{F})$. Since $\mathrm{codim}\,\mathrm{sing}(\mathcal{F}) \geq 2$, we obtain a non-zero element of $H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^{q-1}_{\mathcal{F}}(\delta - \deg(\mathcal{F}) + q - 1))$ as claimed. $\qquad \square$

Lemma 10.2. *Let* k *be an algebraically closed field and let* $\mathcal{F}$ *be a codimension one foliation on* $\mathbb{P}^n_{\mathsf{k}}$. *For any* i *between* 1 *and* $n-1$, *we have an injective morphism*

$$\Omega^i_{\mathcal{F}} \hookrightarrow \Omega^{i+1}_{\mathbb{P}^n_{\mathsf{k}}}(\deg(\mathcal{F})+2) \tag{10.2}$$

with image in $N^*_{\mathcal{F}} \otimes \Omega^i_{\mathbb{P}^n_{\mathsf{k}}}(\deg(\mathcal{F})+2)$.

Proof. Let $\omega \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}}(\deg(\mathcal{F})+2))$ be a twisted 1-form defining $\mathcal{F}$. Since both $\Omega^i_{\mathcal{F}}$ and $\Omega^{i+1}_{\mathbb{P}^n_{\mathsf{k}}}(\deg(\mathcal{F})+2)$ are reflexive sheaves, it suffices to define the morphism on the complement of $\mathrm{sing}(\mathcal{F})$. If η is a germ of section of $\Omega^i_{\mathcal{F}}|_{X-\mathrm{sing}(\mathcal{F})}$, then we can consider an arbitrary lift of η to Ω^i_X and take the wedge product with ω. Since different lifts differ by 1-forms proportional to ω, this defines unambiguously the sought morphism. $\square$

Proposition 10.3. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_{\mathsf{k}}$, $n \geq 3$. If the characteristic of k does not divide $\deg(\mathcal{F})+2 = \deg(N_{\mathcal{F}})$, then $\delta_{i+1}(\mathcal{F}) \leq \delta_i(\mathcal{F})$. Otherwise, $\delta_{i+1}(\mathcal{F}) \leq \delta_i(\mathcal{F})+1$.*

Proof. Let us first verify that $\delta_{i+1}(\mathcal{F}) \leq \delta_i(\mathcal{F})+1$. Let $\mathcal{D}$ be a codimension i distribution of degree $\delta_i(\mathcal{F})$. To prove the claim, it suffices to construct a distribution of codimension $i+1$ and degree $\delta_i(\mathcal{F})+1$. For that, Equation (10.1) implies that it suffices to consider the intersection of $\mathcal{D}$ with general codimension one foliation $\mathcal{G}$ of degree zero.

Assume from now on that the characteristic of k does not divide $\deg(N_{\mathcal{F}}) = \deg(\mathcal{F})+2$. Let H be a hyperplane of $\mathbb{P}^n$ which is not invariant by $\mathcal{F}$. If h is a linear form on $\mathbb{A}^{n+1}_{\mathsf{k}}$ cutting out H, then the logarithmic differential $-\deg(N_{\mathcal{F}})\frac{dh}{h}$ defines a flat logarithmic connection ∇ on $N_{\mathcal{F}}$. Therefore, there exists $\eta \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathcal{F}}(\log H))$ such that

$$\nabla^{\mathrm{B}} - \nabla = \eta,$$

where $\nabla^{\mathrm{B}} : N_{\mathcal{F}} \to N_{\mathcal{F}} \otimes \Omega^1_{\mathcal{F}}$ is Bott's partial connection. Since both ∇^{B} and ∇ are flat $\mathcal{F}$-connections, $d_{\mathcal{F}}\eta = 0$. The non-invariance of H and the hypothesis on the characteristic of the base field imply that $\eta \neq 0$.

Let $\mathcal{D}$ be a foliation of codimension i and degree $\delta_i(\mathcal{F})$. As such, $\mathcal{D}$ is defined by $\alpha \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^i_{\mathbb{P}^n_{\mathsf{k}}}(\delta_i(\mathcal{F})+i+1)$. If $\mathcal{D}$ is everywhere tangent to $\mathcal{F}$, then we can, unambiguously, define the wedge product

$$\alpha \wedge \eta \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^{i+1}_{\mathbb{P}^n_{\mathsf{k}}}(\log H)(\delta_i(\mathcal{F})+i+1)));$$

see Lemma 10.2. Clearing out the denominator (i.e., multiplying $\omega \wedge \eta$ by a defining equation of H), we can interpret $\alpha \wedge \eta$ as a regular section of $\Omega^{i+1}_{\mathbb{P}^n_{\mathsf{k}}}(\delta_i(\mathcal{F})+i+2)$. Moreover, if H is sufficiently general, then $\alpha \wedge \eta$ is non-zero and defines a codimension $i+1$ distribution contained in $\mathcal{F}$ and of degree at most $\delta_i(\mathcal{F})$, as wanted. $\square$

Remark 10.4. The proof of the Proposition 10.3 can be phrased in more elementary terms. Choose an affine subspace $\mathbb{A}^n_{\mathsf{k}} \subset \mathbb{P}^n_{\mathsf{k}}$ such that H becomes the hyperplane at infinity. Since H is not invariant, the restriction of $\mathcal{F}$ to $\mathbb{A}^n_{\mathsf{k}}$ is defined by a 1-form ω with coefficients of degree at most $d+1 = \deg(\mathcal{F})+1$. If we expand ω as the sum of its homogeneous components $\omega_0 + \ldots + \omega_{d+1}$, then $\omega_{d+1} \neq 0$ and its contraction with the Euler vector field R is identically zero. Euler's formula implies that

$$i_R d\omega_{d+1} = (d+2)\omega_{d+1} = (\deg(N_{\mathcal{F}}))\,\omega_{d+1}.$$

Thus, if the characteristic of k does not divide $\deg(N_\mathcal{F})$, then $d\omega$ is a non-zero two-form defining a codimension two foliation $\mathcal{G}$ tangent to $\mathcal{F}$. The foliation $\mathcal{G}$ coincides with the foliation defined by the section $\eta \in H^0(\mathbb{P}_k^n, \Omega^1_\mathcal{F}(\log H))$ figuring in the proof presented above. To conclude, Equation 10.1 implies that it suffices to take a codimension i distribution realizing $\delta_i(\mathcal{F})$ and consider the intersection of it with the foliation $\mathcal{G}$.

Example 10.5. The assumption on the characteristic of the base field in Proposition 10.3 is necessary. If $\mathcal{F}$ is a general codimension one foliation of degree zero on $\mathbb{P}_k^n$, $n \geq 3$, and the characteristic of k is two, then $\delta_2(\mathcal{F}) = 1 = \deg(\mathcal{F}) + 1$.

10.3. Subdistributions of foliations defined by logarithmic 1-forms

In this subsection, we will compute the integers $\delta_2(\mathcal{F})$ for a codimension one holomorphic foliation defined by a logarithmic 1-form with simple normal crossing divisors. In particular, we will show that the bound provided by Proposition 10.3 is sharp.

Proposition 10.6. *Let D be a simple normal crossing divisor on $\mathbb{P}_\mathbb{C}^n$. Let $\omega \in H^0(\mathbb{P}_\mathbb{C}^n, \Omega^1_{\mathbb{P}_\mathbb{C}^n}(\log D))$ be a logarithmic 1-form with polar divisor equal to D. If $\mathcal{F}$ is the foliation defined by ω and $\eta \in H^0(\mathbb{P}_\mathbb{C}^n, \Omega^i_\mathcal{F})$, then $d_\mathcal{F}\eta = 0$. Moreover, if $j > 0$, then $H^0(\mathbb{P}_\mathbb{C}^n, \Omega^i_\mathcal{F}(-j)) = 0$ for every i between 1 and $\dim \mathcal{F} - 1 = n - 2$.*

Proof. If $\eta \in H^0(\mathbb{P}_\mathbb{C}^n, \Omega^i_\mathcal{F})$ is non-zero, then we can define the wedge product of η with ω, as in Lemma 10.2, and obtain a non-zero section of $H^0(\mathbb{P}_k^n, \Omega^{i+1}_{\mathbb{P}_\mathbb{C}^n}(\log D))$. Since logarithmic forms with poles on simple normal crossing divisors are closed according to Deligne's result [18, Corollary 3.2.14], we deduce that $d(\omega \wedge \eta) = 0$. It follows that $d_\mathcal{F}\eta = 0$, showing the first claim of the statement.

Let now j be a strictly positive integer, let i be a positive integer between 1 and $n - 2$, and let η be a non-zero element of $H^0(\mathbb{P}_\mathbb{C}^n, \Omega^i_\mathcal{F}(-j))$. If f_1, f_2 are two arbitrary sections of $\mathcal{O}_{\mathbb{P}_\mathbb{C}^n}(j)$, then $f_1\eta$ and $f_2\eta$ belong to $H^0(\mathbb{P}_\mathbb{C}^n, \Omega^i_\mathcal{F})$. Therefore, $d_\mathcal{F}(f_1\eta) = d_\mathcal{F}(f_2\eta) = 0$ as shown above. It follows that

$$d_\mathcal{F}\left(\frac{f_1}{f_2}\right) \wedge \eta$$

is identically zero. If $i < \dim \mathcal{F}$, then f_1/f_2 is a first integral for the subdistribution of $\mathcal{F}$ defined by η. Since f_1 and f_2 are arbitrary sections of $\mathcal{O}_{\mathbb{P}_\mathbb{C}^n}(j)$, we obtain a contradiction proving the second claim. □

Corollary 10.7. *Let $n \geq 3$ be a positive integer and let D be a simple normal crossing divisor on $\mathbb{P}_\mathbb{C}^n$ with $\ell \geq 2$ distinct irreducible components. Let $\omega \in H^0(\mathbb{P}_\mathbb{C}^n, \Omega^1_{\mathbb{P}_\mathbb{C}^n}(\log D))$ be a logarithmic 1-form with polar divisor equal to D, and let $\mathcal{F}$ be the codimension one foliation defined by ω. Then $\delta_2(\mathcal{F}) = \deg(\mathcal{F})$ when $\ell = 2$, and $\delta_2(\mathcal{F}) = \deg(\mathcal{F}) - 1$ otherwise.*

Proof. Since D is normal crossing and $(\omega)_\infty = D$, the 1-form ω, seen as a section of $\Omega^1_{\mathbb{P}_\mathbb{C}^n}(\log D)$, has no zeros at a neighborhood of the support of D. The ampleness of D implies that the zero set of ω consists of finitely many points, as otherwise positive

dimensional components of the zero set of ω would necessarily intersect D. Therefore, $N_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(D)$ and $\deg(\mathcal{F}) = \delta_1(\mathcal{F}) = \deg D - 2$.

If $\delta_2(\mathcal{F}) < \deg(\mathcal{F}) - 1$, then $h^0(\mathbb{P}^n_{\mathbb{C}}, \Omega^1_{\mathcal{F}}(-j)) \neq 0$ for some $j > 0$. But this contradicts Proposition 10.6 and, consequently, implies that $\delta_2(\mathcal{F}) \geq \deg(\mathcal{F}) - 1$.

If $\ell \geq 3$, then the restriction of a general logarithmic 1-form with poles on D shows that $h^0(\mathbb{P}^n_{\mathbb{C}}, \Omega^1_{\mathcal{F}}) \neq 0$. Hence, when $\ell \geq 3$, $\delta_2(\mathcal{F}) = \deg(\mathcal{F}) - 1$, as claimed.

It remains to analyze the case $\ell = 2$. We will first show that $\delta_2(\mathcal{F}) \geq \deg(\mathcal{F})$. Aiming at a contradiction, assume $\delta_2(\mathcal{F}) = \deg(\mathcal{F}) - 1$. Lemma 10.1 gives a non-zero section η of $\Omega^1_{\mathcal{F}}$. The wedge product of η with ω, well-defined as explained by Lemma 10.2, gives a non-zero element of $H^0(\mathbb{P}^n_{\mathbb{C}}, \Omega^2_{\mathbb{P}^n_{\mathbb{C}}}(\log D))$. Since D has only $\ell = 2$ irreducible components by assumption, the group $H^0(\mathbb{P}^n_{\mathbb{C}}, \Omega^2_{\mathbb{P}^n_{\mathbb{C}}}(\log D)) = 0$ is zero; see, for instance, [21, Corollary 2.9]. This gives the sought contradiction showing that $\delta_2(\mathcal{F}) \geq \deg(\mathcal{F})$ when $\ell = 2$. Finally, we apply Proposition 10.3 to obtain the reverse inequality $\delta_2(\mathcal{F}) \leq \deg(\mathcal{F})$. $\square$

11. Codimension two subdistributions of small degree

11.1. Integrability and uniqueness

The proof of Corollary 10.7 shows that, in general, for a given foliation on a projective space, we do not have uniqueness for the subdistributions realizing $\delta_i(\mathcal{F})$. Our next result shows that if $\delta_2(\mathcal{F})$ is small enough, then we have not only uniqueness but also integrability of the subdistribution realizing $\delta_2(\mathcal{F})$.

Proposition 11.1. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n$, $n \geq 3$, and let $\mathcal{D}$ be a distribution of codimension two contained in $\mathcal{F}$. If $\mathcal{D}$ is non-integrable, then*

$$\deg(\mathcal{D}) \geq \frac{\deg(\mathcal{F}) + 1}{2}.$$

Moreover, if there exists another codimension subdistribution $\mathcal{D}'$ of the same degree of $\mathcal{D}$ but different from it, then

$$\deg(\mathcal{D}) \geq \frac{\deg(\mathcal{F})}{2}.$$

Proof. Since $\mathcal{D}$ is contained in $\mathcal{F}$, the proof of Lemma 10.1 shows that $\mathcal{D}$ is defined by a section η of $\Omega^1_{\mathcal{F}}(\deg(\mathcal{D}) - \deg(\mathcal{F}) + 1)$. If η is non-integrable, then

$$0 \neq \theta = \eta \wedge d_{\mathcal{F}}\eta \in H^0\left(\mathbb{P}^n, \Omega^3_{\mathcal{F}}(2(\deg(\mathcal{D}) - \deg(\mathcal{F}) + 1))\right).$$

The $\mathcal{F}$-differential 3-form θ determines a non-zero section of

$$\Omega^4_{\mathbb{P}^n}\left(\deg(\mathcal{F}) + 2 + (2(\deg(\mathcal{D}) - \deg(\mathcal{F}) + 1))\right),$$

according to Lemma 10.2. Since $h^0(\Omega^4_{\mathbb{P}^n}(k)) = 0$ for any $k \leq 4$, we deduce that $\deg(\mathcal{F}) + 2 + (2(\deg(\mathcal{D}) - \deg(\mathcal{F}) + 1)) \geq 5$. Therefore, the non-integrability of $\mathcal{D}$ implies

$$\deg(\mathcal{D}) \geq \frac{\deg(\mathcal{F}) + 1}{2},$$

proving the first claim.

To prove the second claim, let $\mathcal{D}'$ be another codimension two distribution contained in $\mathcal{F}$. If $\mathcal{D}$ and $\mathcal{D}'$ do not coincide, then $\mathcal{D} + \mathcal{D}' = \mathcal{F}$. Equation (10.1) implies that

$$\deg(\mathcal{D}) + \deg(\mathcal{D}') = \deg(\mathcal{D} \cap \mathcal{D}') + \deg(\mathcal{F}) + \deg(\text{defect}(\mathcal{D}, \mathcal{D}')).$$

Hence, if $\deg(\mathcal{D}) = \deg(\mathcal{D}')$, then $2\deg(\mathcal{D}) \geq \deg(\mathcal{F})$, implying the result. $\quad\square$

11.2. Comparison with the kernel of the *p*-curvature

We start by settling the notation used throughout this subsection.

Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_{\mathbb{C}}$. Let $\mathcal{D}$ be a codimension $q \geq 2$ distribution, contained in $\mathcal{F}$ and with degree equal to $\delta_q(\mathcal{F})$. Let us fix an integral model $(\mathscr{F}, \mathscr{D}, \mathbb{P}^n_R)$ for $(\mathcal{F}, \mathcal{D}, \mathbb{P}^n_{\mathbb{C}})$ defined over a finitely generated $\mathbb{Z}$-algebra R. We want to compare $\mathscr{D}_{\mathfrak{p}}$ with $\mathscr{V}(\mathscr{F}_{\mathfrak{p}})$ for maximal primes $\mathfrak{p}$ of R.

Lemma 11.2. *If $\mathscr{F}_{\mathfrak{p}}$ is not p-closed, then the following assertions hold true.*

(1) *The degree of $\mathscr{V}(\mathscr{F}_{\mathfrak{p}})$ is at most $\deg(\mathcal{F}) - 1$.*

(2) *If $\mathscr{D}_{\mathfrak{p}}$ is p-closed, then $\mathscr{D}_{\mathfrak{p}}$ is contained in $\mathscr{V}(\mathscr{F}_{\mathfrak{p}})$.*

(3) *If $\mathscr{D}_{\mathfrak{p}}$ is not contained in $\mathscr{V}(\mathscr{F}_{\mathfrak{p}})$, then*

$$\deg(\mathscr{V}(\mathscr{F}_{\mathfrak{p}})) \geq \deg(\mathscr{F}_{\mathfrak{p}}) - \deg(\mathscr{D}_{\mathfrak{p}}) + \delta_{q+1}(\mathscr{F}_{\mathfrak{p}}).$$

Moreover, for almost every prime of R, $\deg(\mathscr{D}_{\mathfrak{p}}) = \delta_q(\mathscr{F}_{\mathfrak{p}})$.

Proof. The first assertion follows from Proposition 4.12, whereas the second assertion follows from the definition of $\mathscr{V}(\mathscr{F}_{\mathfrak{p}})$.

Consider the restriction of the p-curvature morphism $\psi_{\mathcal{F}}$ to $\text{Frob}^* T_{\mathscr{D}_{\mathfrak{p}}}$. The kernel coincides with $\text{Frob}^* T_{\mathscr{D}_{\mathfrak{p}} \cap \mathscr{V}(\mathscr{F}_{\mathfrak{p}})}$. Comparing the cokernels of $\psi_{\mathcal{F}}$ and of $\psi_{\mathcal{F}}|_{\text{Frob}^* T_{\mathscr{D}_{\mathfrak{p}}}}$, we deduce that

$$\deg(\mathscr{D}_{\mathfrak{p}}) - \deg(\mathscr{D}_{\mathfrak{p}} \cap \mathscr{V}(\mathscr{F}_{\mathfrak{p}})) \geq \deg(\mathscr{F}_{\mathfrak{p}}) - \deg(\mathscr{V}(\mathscr{F}_{\mathfrak{p}})).$$

Since $\mathscr{D}_{\mathfrak{p}} \cap \mathscr{V}(\mathscr{F}_{\mathfrak{p}})$ has codimension $q+1$, the inequality $\deg(\mathscr{D}_{\mathfrak{p}} \cap \mathscr{V}(\mathscr{F}_{\mathfrak{p}})) \geq \delta_{q+1}(\mathscr{F}_{\mathfrak{p}})$ holds by definition. The third assertion follows.

Finally, the fact that $\deg(\mathscr{D}_{\mathfrak{p}}) = \delta_q(\mathscr{F}_{\mathfrak{p}})$ for almost every prime follows from semi-continuity. $\quad\square$

11.3. Foliations with codimension two subdistributions of degree zero.

Foliations on $\mathbb{P}^n_{\mathbb{C}}$ of degree zero and codimension q $(0 < q < n$ arbitrary$)$ are easy to describe: they are all defined by the fibers of a linear projection $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^q_{\mathbb{C}}$; see, for instance, [19, Theorem 3.8].

Proposition 11.3. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$, of degree $d \geq 1$. If $\delta_2(\mathcal{F}) = 0$, then $\mathcal{F}$ is a linear pull-back of a foliation on $\mathbb{P}^2_{\mathbb{C}}$.*

Proof. According to Proposition 11.1, there exists a subfoliation $\mathcal{G} \subset \mathcal{F}$ realizing $\delta_2(\mathcal{F}) = 0$. If $\pi : \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ is the linear projection defining $\mathcal{G}$, then [13, Lemma 3.1] implies that $\mathcal{F} = \pi^* \mathcal{H}$ for a foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$. $\quad\square$

Let $\mathsf{Lin}_d(\mathbb{P}^n_{\mathsf{k}}) \subset \mathsf{Fol}_d(\mathbb{P}^n_{\mathsf{k}})$ be the reduced subscheme whose closed points correspond to foliations on $\mathbb{P}^n_{\mathsf{k}}$ that are pull-backs of degree d foliations on $\mathbb{P}^2_{\mathsf{k}}$ under a linear projection $\pi : \mathbb{P}^n_{\mathsf{k}} \dashrightarrow \mathbb{P}^2_{\mathsf{k}}$. It is well-known that $\mathsf{Lin}_d(\mathbb{P}^n_{\mathbb{C}})$ is an irreducible component of $\mathsf{Fol}_d(\mathbb{P}^n_{\mathbb{C}})$ (considered with its reduced structure) for every $d \geq 0$ and every $n \geq 3$; see, for instance, [15, Subsection 5.1].

Corollary 11.4. *Let $\Sigma \subset \mathsf{Fol}_d(\mathbb{P}^n_{\mathbb{C}})$ be an irreducible component. If $\delta_2(\Sigma) = 0$, then $\Sigma = \mathsf{Lin}_d(\mathbb{P}^n_{\mathbb{C}})$.*

11.4. Foliations with codimension two subdistributions of degree one

If $\mathcal{F}$ is a degree one foliation of codimension q on $\mathbb{P}^n_{\mathbb{C}}$, then [30, Theorem 6.2] gives the following precise description of $\mathcal{F}$.

(1) The foliation $\mathcal{F}$ is defined by a dominant rational map $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}_{\mathbb{C}}(1^q, 2)$ with irreducible general fiber determined by q linear forms and one quadratic form; or

(2) The foliation $\mathcal{F}$ is the linear pull-back of a foliation induced by a global holomorphic vector field on $\mathbb{P}^{q+1}_{\mathbb{C}}$ and has tangent sheaf isomorphic to $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-1)^{\oplus n-q-1} \oplus \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}$.

Building on this description, we can prove the following characterization of algebraically integrable foliations of degree one and arbitrary codimension.

Lemma 11.5. *Let $\mathcal{F}$ be a foliation of degree one and codimension q on $\mathbb{P}^n_{\mathbb{C}}$. The foliation $\mathcal{F}$ is algebraically integrable if and only if $\mathcal{F}$ is p-closed for almost every prime.*

Proof. Let $\mathcal{F}$ be a foliation of degree one. If the fibers of a rational map $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}_{\mathbb{C}}(1^q, 2)$ define $\mathcal{F}$, then there is nothing to prove. Assume from now on that this is not the case. The classification of degree one foliations recalled above implies that $\mathcal{F}$ is the linear pull-back of foliation on $\mathbb{P}^{q+1}_{\mathbb{C}}$ defined by a global vector field $v \in H^0(\mathbb{P}^{q+1}_{\mathbb{C}}, T_{\mathbb{P}^{q+1}_{\mathbb{C}}})$. The foliation $\mathcal{F}$ is algebraically integrable if and only if the foliation defined by v is algebraically integrable. Represent v by a degree one homogeneous vector field on $\mathbb{A}^{q+2}_{\mathbb{C}}$ with divergent zero and let $v = v_S + v_N$ be its Jordan decomposition into semi-simple and nilpotent parts. Explicit integration of v implies that the algebraicity of its orbits is equivalent to the following:

(1) the nilpotent part v_N is zero and the quotient of any two eigenvalues of the semi-simple part of v is a rational number; or

(2) the semi-simple part v_S is zero.

In the first case, the vector field is tangent to an algebraic action of the multiplicative group $\mathbb{C}^*$, whereas in the second case, the vector field is tangent to an algebraic action of the additive group $\mathbb{C}$ given by the exponential of $v = v_N$.

Let v_R be an integral model for v defined over a finitely generated $\mathbb{Z}$-algebra R contained in $\mathbb{C}$. We can assume that both v_N and v_S have integral models over R. Let $\mathfrak{p} \subset R$ be a maximal ideal and set $v_{\mathfrak{p}}$ equal to the reduction modulo $\mathfrak{p}$ of v_R. Likewise, set $v_{N,\mathfrak{p}}$ and $v_{S,\mathfrak{p}}$ as the reduction modulo $\mathfrak{p}$ of the nilpotent and semi-simple parts of v.

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Since $[v_S, v_N] = 0$, and the same holds for the reduction modulo $\mathfrak{p}$, Formula (3.2) implies that

$$(v_{S,\mathfrak{p}} + v_{N,\mathfrak{p}})^p = v_{S,\mathfrak{p}}^p + v_{N,\mathfrak{p}}^p \,.$$

Moreover, if the characteristic of the residue field $R/\mathfrak{p}$ is sufficiently large (i.e., greater than $q+2$), then $v_{N,\mathfrak{p}}^p = 0$. Thus, assuming that $p > q+2$, it follows that $v_{\mathfrak{p}}$ is p-closed if and only if

(1) the nilpotent part $v_{N,\mathfrak{p}}$ is zero and the quotient of any eigenvalues of the semi-simple part of v belongs to $\mathbb{F}_p$; or

(2) the semi-simple part $v_{S,\mathfrak{p}}$ is zero.

Combining the above observations with a classical result by Kroenecker ([14, Theorem 2.2]) that asserts that an algebraic number is rational if and only if its reduction modulo p is in $\mathbb{F}_p$ for almost every prime p, we obtain that $\mathcal{F}$ is algebraically integrable if and only if the reduction of v_R modulo $\mathfrak{p}$ is p-closed for an open set of maximal primes $\mathfrak{p}$ of $\mathrm{Spec}(R)$, as wanted. $\qquad\square$

Proposition 11.6. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}_{\mathbb{C}}^n$, $n \geq 3$. If $\deg(\mathcal{F}) \geq 3$ and $\delta_2(\mathcal{F}) = 1$, then one of the following assertions holds true.*

(1) *The foliation $\mathcal{F}$ is defined by a closed rational 1-form without codimension one zeros.*

(2) *The foliation $\mathcal{F}$ contains a codimension two algebraically integrable subfoliation $\mathcal{G}$ of degree one.*

Proof. Let $\mathcal{G} \subset \mathcal{F}$ be a codimension two subfoliation of degree one. There is no loss of generality in assuming that $\mathcal{G}$ admits a model $\mathscr{G}$ over $\mathrm{Spec}(R)$. Let $\mathfrak{p}$ be an arbitrary maximal prime of R and let $\mathscr{F}_{\mathfrak{p}}$ be the reduction of $\mathscr{F}_{\mathfrak{p}}$ of $\mathscr{F}$ modulo $\mathfrak{p}$. Reduce $\mathscr{G}$ modulo $\mathfrak{p}$ to obtain a subfoliation $\mathscr{G}_{\mathfrak{p}}$ of degree one.

If $\mathscr{G}_{\mathfrak{p}}$ is p-closed for almost every maximal prime $\mathfrak{p}$, then Lemma 11.5 implies that $\mathcal{G}$ is algebraically integrable as claimed in Item (2).

If $\mathscr{G}_{\mathfrak{p}}$ is not p-closed for a Zariski dense set of primes, then according to the classification of degree one foliations recalled at the beginning of this subsection, we have that $T_{\mathcal{G}}$ is generated by global sections. We distinguish two possibilities. Either $\mathscr{G}_{\mathfrak{p}}$ coincides with $\mathscr{V}(\mathscr{F}_{\mathfrak{p}})$ or not. Assume first that $\mathscr{G}_{\mathfrak{p}}$ coincides with $\mathscr{V}(\mathscr{F}_{\mathfrak{p}})$. In this case, there exists a vector field $v \in H^0(\mathbb{P}_{\mathsf{k}}^n, T_{\mathscr{G}_{\mathfrak{p}}})$ such that

$$v^p \in H^0(\mathbb{P}_{\mathsf{k}}^n, T_{\mathscr{F}_{\mathfrak{p}}}) - H^0(\mathbb{P}_{\mathsf{k}}^n, T_{\mathscr{G}_{\mathfrak{p}}}) \,.$$

It follows that we have a morphism

$$T_{\mathscr{G}_{\mathfrak{p}}} \oplus \mathcal{O}_{\mathbb{P}_{\mathsf{k}}^n} \longrightarrow T_{\mathscr{F}_{\mathfrak{p}}}$$
$$w \oplus f \mapsto w + f v^p$$

generically surjective. Therefore, $\deg(\mathscr{F}_{\mathfrak{p}}) \leq 2$, contrary to our assumptions.

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It remains to treat the case where $\mathscr{G}_{\mathfrak{p}}$ does not coincide with $\mathscr{V}(\mathscr{F}_{\mathfrak{p}})$. In this case, there exists a vector field $v \in H^0(\mathbb{P}^n_{\mathsf{k}}, T_{\mathscr{G}_{\mathfrak{p}}})$ such that

$$v^p \notin H^0(\mathbb{P}^n_{\mathsf{k}}, T_{\mathscr{F}_{\mathfrak{p}}}).$$

Notice that v must have zero set of codimension at least two, as otherwise v^p would be proportional to v. If $\omega_{\mathfrak{p}} \in H^0(\mathbb{P}^n_{\mathsf{k}}, \Omega^1_{\mathbb{P}^n_{\mathsf{k}}} \otimes N_{\mathscr{F}_{\mathfrak{p}}})$ is the twisted 1-form defining $\mathscr{F}_{\mathfrak{p}}$, then Proposition 4.4 implies that the rational 1-form $\omega_{\mathfrak{p}}/\omega_{\mathfrak{p}}(v^p)$ is closed. The divisor defined by the vanishing of $\omega_{\mathfrak{p}}(v^p)$ is of degree $d+2$ (independent of p) and coincides with $\Delta_{\mathscr{F}_{\mathfrak{p}}}$. Proposition 4.13 implies that $\Delta_{\mathscr{F}_{\mathfrak{p}}}$ is $\mathscr{F}_{\mathfrak{p}}$-invariant if $p > d+2$. We can thus apply Lemma 8.4 to lift $\omega_{\mathfrak{p}}(v^p)$ to characteristic zero and deduce that we are in the situation described by Item (1). $\qquad\square$

Lemma 11.7. *The set in $\mathsf{Fol}_d(\mathbb{P}^n_{\mathbb{C}})$ corresponding to foliations defined by a closed rational 1-form without codimension one zeros is closed.*

Proof. A foliation is defined by a closed rational 1-form without codimension one zeros if and only if it admits a polynomial integrating factor; see [17, Subsection 3.3], in particular, [17, Remark 3.4]. The result is a restatement of [17, Lemma 3.6]. $\qquad\square$

Lemma 11.8. *Let $\mathcal{F}$ be a p-dense foliation $\mathcal{F}$ on a projecive surface X defined over a field of characteristic $p > 0$. If $\omega_{\mathcal{F}}$ intersects non-negatively any ample divisor, $\mathcal{F}$ is defined by a closed rational 1-form without codimension one zeroes, and the divisor $\Delta_{\mathcal{F}}$ is free from p-th powers then $\omega_{\mathcal{F}}^{\otimes p} = \mathcal{O}_X$.*

Proof. If α is any rational 1-form defining $\mathcal{F}$, then the normal bundle of $\mathcal{F}$ is isomorphic to $\mathcal{O}_X((\alpha)_{\infty} - (\alpha)_0)$. By assumption, there exists a closed rational 1-form η, with $(\eta)_0 = 0$, defining $\mathcal{F}$. Therefore, $N_{\mathcal{F}} = \mathcal{O}_X((\eta)_{\infty})$.

According to Proposition 4.12, the line-bundles $\mathcal{O}_X(\Delta_{\mathcal{F}})$ and $N_{\mathcal{F}} \otimes \omega_{\mathcal{F}}^{\otimes p}$ are isomorphic. According to Proposition 4.14, the divisors $(\eta)_{\infty}$ and $\Delta_{\mathcal{F}}$ coincide modulo p. Thus, we can write $(\eta)_{\infty} = \Delta_{\mathcal{F}} + pD$ for some divisor D such that $\mathcal{O}_X(pD) = \omega_{\mathcal{F}}^{\otimes p}$. Since both $(\eta)_{\infty}$ and $\Delta_{\mathcal{F}}$ are effective and $\Delta_{\mathcal{F}}$ is free from p-th powers, we deduce that $pD = 0$ (i.e., $\omega_{\mathcal{F}}^{\otimes p} = \mathcal{O}_X$), as claimed. $\qquad\square$

11.5. Irreducible components with $\delta_2(\Sigma) = 1$

Proposition 11.6 could be easily proved without using reduction to positive characteristic. Instead of considering the p-th powers of vector fields tangent to a subfoliation of degree one, one could consider the Zariski closure of the subgroup generated by the flow of these vector fields as is done in [17, Subsection 3.4].

Our next result seems to be of a different nature. It generalizes to arbitrary degree an irreducible component of $\mathsf{Fol}_3(\mathbb{P}^n_{\mathbb{C}})$ found in [17]. In degree three, the original proof relies on the structure theorem for degree three foliations on projective spaces established in [32] ($n = 3$) and [17, Theorem A] ($n \geq 3$). For arbitrary degrees, we are not aware of any proof that does not rely on the reduction to positive characteristic. We have reasons to believe that the standard arguments of the subject (stability of local singular type under deformation as in [12] or infinitesimal methods as in [15, 16, 12]) are not sufficient to prove

this result. Concerning the stability of singular type, we point out that general foliation described by Theorem 11.9 below has singularities of arbitrary algebraic multiplicity along two skew lines, making it hard to believe that local arguments would show the stability of them. Concerning infinitesimal methods, computer-aided calculations show that the corresponding irreducible component of $\mathsf{Fol}_3(\mathbb{P}^n_\mathbb{C})$ is generically non-reduced. We believe this is also the case for every degree, as predicted by [17, Conjecture 6.1].

Theorem 11.9. *For every $n \geq 3$ and every positive integer $a \geq 2, b \geq 3$, there exists an irreducible component of $\mathsf{Fol}_{a+b-2}(\mathbb{P}^n_\mathbb{C})$ whose general elements correspond to the pull-back of a foliation $\mathcal{G}$ on $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$, with normal bundle $N_\mathcal{G} = \mathcal{O}_{\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}}(a,b)$ under a rational map $\pi : \mathbb{P}^n_\mathbb{C} \dashrightarrow \mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ of the form*

$$\pi(x_0 : \ldots : x_n) = ((\ell_1 : \ell_2),(\ell_3 : \ell_4)),$$

where $\ell_1, \ldots, \ell_4 \in H^0(\mathbb{P}^n_\mathbb{C}, \mathcal{O}_{\mathbb{P}^n_\mathbb{C}}(1))$ are linear forms in general position.

Proof. Let $\mathcal{G}$ be a foliation on $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ with normal bundle $N_\mathcal{G} = \mathcal{O}_{\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}}(a,b)$. Assume that the 1-form $\omega \in H^0(\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}, \Omega^1_{\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}}(a,b))$ is not contained in any proper subvariety of $H^0(\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}, \Omega^1_{\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}}(a,b))$ defined over $\mathbb{Q}$. If we write down ω in bihomogeneous coordinates, this means that the coefficients of ω generate a purely transcendental extension of $\mathbb{Q}$ with transcendence degree equal to $h^0(\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}, \Omega^1_{\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}}(a,b))$.

Let $\Sigma \subset \mathsf{Fol}_d(\mathbb{P}^n_\mathbb{C})$ be an irreducible component containing $\mathcal{F} = \pi^*\mathcal{G}$. Let R be a finitely generated $\mathbb{Z}$-algebra such that both $\mathcal{G}$ and Σ admit integral models defined over R. Our assumptions on ω imply that the reduction of the integral model $\mathcal{G}$ of $\mathcal{G}$ modulo $\mathfrak{p}$ is a generic foliation for a Zariski dense set of maximal primes $Z \subset \operatorname{Spec}(R)$ and, Theorem 5.10 implies that the degeneracy divisor of its p-curvature is reduced.

Since we are assuming that $a \geq 2$ and $b \geq 3$, the line-bundle $\omega_\mathcal{G} = \mathcal{O}_{\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}}(a-2, b-2)$ is not torsion. Therefore, Lemma 11.7 guarantees that the foliation $\mathscr{G}_\mathfrak{p}$ is not defined by a closed rational 1-form without codimension one zeros for $\mathfrak{p} \in Z$.

Proposition 5.3 implies that the degeneracy divisor of the p-curvature of $\mathscr{F}_\mathfrak{p} = \pi^*\mathscr{G}_\mathfrak{p}$ is also reduced for the Zariski dense set Z of primes. For every maximal prime $\mathfrak{p}$ in Z, the kernel of the p-curvature of $\mathscr{F}_\mathfrak{p}$ is the p-closed foliation of degree one defined by the reduction modulo $\mathfrak{p}$ of the rational map π. We apply Lemma 6.3 to deduce that the kernel of the p-curvature for the reduction modulo $\mathfrak{p}$ of a general foliation in Σ also has degree one (i.e., $\delta_2(\Sigma) = 1$). Proposition 11.6 shows that a general element of Σ corresponds to a holomorphic foliation containing a unique algebraically integrable codimension two subfoliation of degree one. It remains to verify that this codimension two subfoliation is conjugated to the foliation defined by the fibers of π.

Let us denote by $0 \in \Sigma$ the point corresponding to $\mathcal{F} = \mathcal{F}_0$, and by $\mathcal{F}_\epsilon$ the foliation corresponding to a point $\epsilon \in \Sigma$ close to 0. Let also $\mathcal{A}_\epsilon$ be the unique codimension two foliation of $\mathcal{F}_\epsilon$ of degree one. If $n = 3$, then $\mathcal{A}_\epsilon$ is defined by a global holomorphic vector field v_ϵ such that $H^0(\mathbb{P}^3_\mathbb{C}, T_{\mathcal{F}_\epsilon}) = \mathbb{C} \cdot v_\epsilon$. As we did in the proof of Lemma 11.5, we can represent v_ϵ by a homogeneous vector field of degree one and divergence zero on $\mathbb{A}^4_\mathbb{C}$. Since $h^0(\mathbb{P}^3_\mathbb{C}, T_{\mathcal{F}_\epsilon})$ is constant for ϵ in a neighborhood of 0, we can choose homogeneous representatives of v_ϵ varying holomorphically with ϵ. The vector field v_0 is semi-simple

and, after multiplication by a suitable constant, has two eigenvalues equal to 1 and two other eigenvalues equal to -1. The quotient of any two of them is an integer. Since the orbits of v_ϵ are algebraic, v_ϵ must remain semi-simple with rational quotient of eigenvalues. Since v_ϵ varies holomorphically with ϵ, the quotient of eigenvalues must remain equal to the ones of v_0. This shows that $\mathcal{A}_\epsilon$ is conjugated to $\mathcal{A}$ when $n = 3$ as claimed. If $n > 3$, then the restrictions of $\mathcal{A}_\epsilon$ and $\mathcal{A}_0$ to general $\mathbb{P}^3_\mathbb{C}$ are conjugated. It follows from the classification of degree one foliations, recalled at the beginning of Subsection 11.4, that the same holds true for $\mathcal{A}_\epsilon$ and $\mathcal{A}_0$ before taking the restrictions. To conclude, apply [13, Lemma 3.1] to guarantee that the general element of Σ is pull-back from $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ through a rational map of the claimed form. $\qquad\square$

Theorem 11.10. *For every $n \geq 3$ and every $d \geq 3$, there exists an irreducible component Σ of $\mathsf{Fol}_d(\mathbb{P}^n_\mathbb{C})$ whose general element corresponds to the pull-back of a general foliation $\mathcal{G}$ on $\mathbb{P}(1,1,2)$ with normal sheaf $N_\mathcal{G} = \mathcal{O}_{\mathbb{P}_\mathbb{C}(1,1,2)}(d+2)$ under a rational map $\pi : \mathbb{P}^n_\mathbb{C} \dashrightarrow \mathbb{P}_\mathbb{C}(1,1,2)$ defined by two linear forms and one quadratic form.*

Proof. Observe that the blow-up of $\mathbb{P}_\mathbb{C}(1,1,2)$ at its unique singular point is isomorphic to a Hirzebruch surface having a section with self-intersection -2. From Theorem 5.10, we deduce the existence of foliations $\mathcal{G}$ on $\mathbb{P}_\mathbb{C}(1,1,2)$ with reduced degeneracy divisor of their p-curvatures for a Zariski dense set of primes whenever $\omega_\mathcal{G} = \omega_{\mathbb{P}_\mathbb{C}(1,1,2)} \otimes N_\mathcal{G} = \mathcal{O}_{\mathbb{P}_\mathbb{C}(1,1,2)}(-4+d+2)$ is nef (i.e., whenever $d \geq 2$). If we set Y equal to the smooth locus of $\mathbb{P}_\mathbb{C}(1,1,2)$ and X equal to the pre-image of Y under π, then Corollary 5.4 implies that the p-degeneracy of the pull-back of $\mathcal{G}$ under $\pi\big|_X$ is reduced since $\mathrm{Ram}(\pi\big|_X) = 0$. This shows that when $d \geq 2$, we have the existence of p-dense foliations $\mathcal{G}$ such that the degeneracy divisor of the p-curvature of $\pi^*\mathcal{G}$ is reduced for a Zariski dense set of primes.

Lemma 11.7 implies that a general $\mathcal{G}$ on $\mathbb{P}_\mathbb{C}(1,1,2)$ with $N_\mathcal{G} = \mathcal{O}_{\mathbb{P}_\mathbb{C}(1,1,2)}(d+2)$ is not defined by a closed rational 1-form without codimension one zeros when $d \geq 3$.

Arguing as in the proof of Theorem 11.9, we deduce that sufficiently small deformations of $\pi^*\mathcal{G}$ carry an algebraically integrable codimension two subfoliation of degree one defined by rational maps to $\mathbb{P}_\mathbb{C}(1,1,2)$ given by two linear forms and one quadratic form, and we conclude using [13, Lemma 3.1]. $\qquad\square$

Minor variations on the arguments used to prove Theorems 11.9 and 11.10 give the following result.

Proposition 11.11. *Let $1 \leq a \leq b \leq c$ be positive integers without a common factor and such that $(a,b,c) \neq (1,1,2)$, let $\mathcal{G}$ be a codimension one foliation on $\mathbb{P}_\mathbb{C}(a,b,c)$, let $\ell_1, \ldots, \ell_4 \in H^0(\mathbb{P}^n_\mathbb{C}, \mathcal{O}_{\mathbb{P}^n_\mathbb{C}}(1))$ be linear forms in general position and let $\pi : \mathbb{P}^n_\mathbb{C} \dashrightarrow \mathbb{P}_\mathbb{C}(a,b,c)$ be the rational map defined as*

$$\pi(x_0 : \ldots : x_n) = (\ell_1 \cdot \ell_4^{a-1} : \ell_2 \cdot \ell_4^{b-1} : \ell_3 \cdot \ell_4^{c-1}).$$

Assume that

(1) *the foliation $\mathcal{G}$ is p-dense and the degeneracy divisor of the p-curvature of $\pi^*\mathcal{G}$ is reduced for a Zariski dense set of primes; and*

(2) *the foliation $\pi^*\mathcal{G}$ is not defined by a closed rational 1-form without codimension one zeros.*

Then the foliation $\mathcal{F} = \pi^*\mathcal{G}$ belongs to a unique irreducible component Σ of $\mathsf{Fol}_d(\mathbb{P}^n_\mathbb{C})$, where $d = \deg(\mathcal{F})$, such that the general element of Σ admits the same description as $\mathcal{F}$.

Corollary 11.12. *For every $n \geq 3$, $m \geq 3$ and every $d \geq m+1$, there exists an irreducible component Σ of $\mathsf{Fol}_d(\mathbb{P}^n_\mathbb{C})$ whose general element corresponds to the pull-back of a general foliation $\mathcal{G}$ on $\mathbb{P}(1,1,m)$, leaving the rational curve $\{z = 0\}$ invariant and with normal sheaf $N_\mathcal{G} = \mathcal{O}_{\mathbb{P}_\mathbb{C}(1,1,m)}(d+m-2)$.*

Proof. The existence of $\mathcal{G}$ satisfying the assumptions of Proposition 11.11 follows from Theorem 5.10 and Remark 5.11 combined with Corollary 5.4. □

12. Foliations without subdistributions of small degree

12.1. Foliations with $\delta_2(\mathcal{F}) = \deg(\mathcal{F})$.

If $\delta_2(\mathcal{F}) = \deg\mathcal{F}$ (i.e., $h^0(\mathbb{P}^n_\mathsf{k}, \Omega^1_\mathcal{F}) = 0$) and the characteristic of k is positive, then $\mathcal{F}$ is p-closed, as otherwise the kernel of the p-curvature would define a subdistribution of degree strictly smaller than $\deg(\mathcal{F})$, as implied by Proposition 4.12. We conjecture that over the complex numbers, $\delta_2(\mathcal{F}) = \deg(\mathcal{F})$ implies that $\mathcal{F} \in \mathsf{Log}_{(d_1,d_2)}(\mathbb{P}^n_\mathbb{C})$ for some positive integers d_1 and d_2.

Conjecture 12.1. *Let $\Sigma \subset \mathsf{Fol}_d(\mathbb{P}^n_\mathbb{C})$ be an irreducible component. If $\delta_2(\Sigma)$ is equal to d, then there exists integer d_1 and d_2 such that $\Sigma = \mathsf{Log}_{(d_1,d_2)}(\mathbb{P}^n_\mathbb{C})$.*

We will say that codimension one foliation is virtually transversely additive if and only if Bott's partial connection extends to a flat logarithmic connection with finite monodromy.

Proposition 12.2. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_\mathbb{C}$, $n \geq 3$. If $\mathcal{F}$ leaves invariant an algebraic hypersurface, then $\mathcal{F}$ is virtually transversely additive or $\delta_2(\mathcal{F}) \leq \deg(\mathcal{F}) - 1$.*

Proof. The proof is analogous to the proof of Proposition 10.3. Let H be an invariant hypersurface and consider the flat logarithmic connection ∇ on $\mathcal{O}_X(H)$ with residue divisor $-H$ and trivial monodromy. It induces a flat logarithmic connection on $N_\mathcal{F}$ with residue divisor $-\frac{\deg(N_\mathcal{F})}{\deg H}H$, which we still denote by ∇. The $\mathcal{F}$-invariance of H implies that the restriction of ∇ to $T_\mathcal{F}$ is a holomorphic partial connection on $N_\mathcal{F}$. If it coincides with Bott's connection, then $\mathcal{F}$ is virtually transversely aditive. If instead it does not coincide, then the difference of ∇ and ∇^B is a non-zero section of $\Omega^1_\mathcal{F}$. It follows that $\delta_2(\mathcal{F}) \leq \deg(\mathcal{F}) - 1$. □

Our next result provides some evidence, admittedly weak, toward Conjecture 12.1.

Proposition 12.3. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_\mathbb{C}$, $n \geq 3$. If $\mathcal{F}$ admits two distinct invariant algebraic hypersurfaces and $\delta_2(\mathcal{F}) = \deg\mathcal{F}$, then there exist integers d_1, d_2 such that $\mathcal{F} \in \mathsf{Log}_{(d_1,d_2)}(\mathbb{P}^n_\mathbb{C})$.*

Proof. Let F and G be two distinct hypersurfaces invariant by $\mathcal{F}$. The proof of Proposition 12.2 implies the existence of two distinct flat logarithmic connections on the normal bundle of $\mathcal{F}$. Their difference is a non-zero closed logarithmic 1-form η such that $\eta\big|_{T_{\mathcal{F}}}$ is a holomorphic section of $\Omega^1_{\mathcal{F}}$. Since $\delta_2(\mathcal{F}) = \deg(\mathcal{F})$, we have that $h^0(\mathbb{P}^n_{\Bbbk}, \Omega^1_{\mathcal{F}}) = 0$. It follows that η is a closed logarithmic 1-form defining $\mathcal{F}$. Since the residues η are rational, it follows that $\mathcal{F}$ is algebraically integrable.

Let $f : \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ be a rational first integral for $\mathcal{F}$ with irreducible general fiber. A classical theorem by Halphen (see [30, Theorem 3.3]) says that any rational map with irreducible general fiber has, at most, two multiple fibers. Moreover, if $r(\mathcal{F})$ is the number of fibers of f with non-irreducible support, then the proof [30, Theorem 1.2] implies that $r(\mathcal{F}) = 0$. Hence, f can be written as the quotient of powers of two irreducible polynomials. Looking at its logarithmic differential, it becomes clear that $\mathcal{F} \in \mathsf{Log}_{(d_1, d_2)}(\mathbb{P}^n_{\mathbb{C}})$ for some d_1 and d_2 such that $\deg(\mathcal{F}) = d_1 + d_2 - 2$. $\qquad\square$

12.2. Foliations with $\delta_2(\mathcal{F}) = \deg(\mathcal{F}) - 1$

Let $\mathcal{F}$ be a logarithmic foliation on $\mathbb{P}^n_{\Bbbk}$ defined by $\omega \in H^0(\mathbb{P}^n_{\Bbbk}, \Omega^1_{\mathbb{P}^n_{\Bbbk}}(\log D))$, where $D = \sum_{i=1}^R H_i$ is a simple normal crossing divisor and H_i is a hypersurface of degree d_i. Assume that the zero set of ω has codimension at least two and that the polar divisor of ω is equal to D. Under these assumptions, $\mathcal{F}$ is a foliation of degree $d = \sum_{i=1}^r d_i - 2$ belonging to $\mathsf{Log}_{(d_1, \ldots, d_r)}(\mathbb{P}^n_{\Bbbk})$. The $\mathcal{F}$-invariance of the support of D implies that the restrictions of sections of $\Omega^1_{\mathbb{P}^n_{\Bbbk}}(\log D)$ to $T_{\mathcal{F}}$ have no poles and, therefore, defines a morphism

$$H^0(\mathbb{P}^n_{\Bbbk}, \Omega^1_{\mathbb{P}^n_{\Bbbk}}(\log D)) \longrightarrow H^0(\mathbb{P}^n_{\Bbbk}, \Omega^1_{\mathcal{F}})$$

with kernel generated by ω. Consequently, $h^0(\mathbb{P}^n_{\Bbbk}, \Omega^1_{\mathcal{F}}) \geq r - 2$ and, if $r \geq 3$, we obtain that $\delta_2(\mathcal{F}) \leq \deg(\mathcal{F}) - 1$.

Theorem 12.4. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$. Assume that $\mathcal{F}$ is p-dense for a Zariski dense set of primes. If $\delta_2(\mathcal{F}) = \deg(\mathcal{F}) - 1$, then $\mathcal{F}$ belongs to one of the logarithmic components $\mathsf{Log}_{(d_1, d_2, \ldots, d_r)}(\mathbb{P}^n_{\mathbb{C}})$, $r \geq 3$.*

Proof. Let $(\mathscr{X}, \mathscr{F})$ be an integral model for $\mathcal{F}$ defined over a finitely generated $\mathbb{Z}$-algebra R contained in $\mathbb{C}$ (i.e., $\mathscr{X} = \mathbb{P}^n_R$ and $\mathscr{F} \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{F}$). Let $\mathfrak{p} \in \mathrm{Spec}(R)$ be a maximal prime such that the reduction $\mathscr{F}_{\mathfrak{p}}$ of $\mathscr{F}$ is not p-closed. By semi-continuity, we may assume that $\delta_2(\mathscr{F}_p) = \deg(\mathcal{F}) - 1$. Therefore, the kernel of p-curvature of $\mathscr{F}_{\mathfrak{p}}$ is a subdistribution (indeed subfoliation according Theorem 7.3) realizing $\delta_2(\mathscr{F}_{\mathfrak{p}})$. Consequently, the divisor $\Delta_{\mathscr{F}_{\mathfrak{p}}}$ has degree $\deg(N_{\mathcal{F}})$ according to Proposition 4.12, and $\mathscr{F}_{\mathfrak{p}}$ is defined by a closed rational 1-form with polar divisor equal $\Delta_{\mathscr{F}_{\mathfrak{p}}}$ and without zero divisors. We apply Lemma 8.4 to deduce that the same holds true for $\mathcal{F}$. To conclude, we apply [17, Proposition 3.7]. $\qquad\square$

Conjecture 12.5. *Let $\Sigma \subset \mathsf{Fol}_d(\mathbb{P}^n_{\mathbb{C}})$ be an irreducible component. If $\delta_2(\Sigma) = \deg(\mathcal{F}) - 1$, then either every foliation $\mathcal{F} \in \Sigma$ is algebraically integrable or there exist integers $d_1, \ldots, d_r$ such that $\Sigma = \mathsf{Log}_{(d_1, \ldots, d_r)}(\mathbb{P}^n_{\mathbb{C}})$.*

Remark 12.6. For any $d \in \{2,3\}$ and any $n \geq 3$, there are examples of irreducible components Σ of $\mathsf{Fol}_d(\mathbb{P}^n_{\mathbb{C}})$ with $\delta_2(\Sigma) = d-1$ with Σ not of the form $\mathsf{Log}_{(d_1,\ldots,d_r)}(\mathbb{P}^n_{\mathbb{C}})$. For $d = 2$, there is exactly one irreducible component with this property, the so-called exceptional component. For $d = 3$, there are at least two irreducible components with this property: the special logarithmic components $\mathsf{SLog}_{(2,5)}(\mathbb{P}^n_{\mathbb{C}})$ and $\mathsf{SLog}_{(3,4)}(\mathbb{P}^n_{\mathbb{C}})$ which parameterize algebraically integrable foliations defined by logarithmic 1-forms with codimension one zeros. All the known examples with this property satisfy Conjecture 12.5.

A confirmation of Conjecture 8.3 (by Ekedahl, Shepherd-Barron and Taylor) combined with Proposition 12.2 and Theorem 12.4 would confirm both Conjectures 12.1 and 12.5.

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