Bull. Aust. Math. Soc. **92** (2015), 228–232
doi:10.1017/S0004972715000520

FIXED POINT CHARACTERISATION FOR EXACT AND AMENABLE ACTION

Z. DONG and Y. Y. WANG✉

(Received 6 March 2015; accepted 12 March 2015; first published online 16 June 2015)

Abstract

Let G be a finitely generated group acting on a compact Hausdorff space X. We give a fixed point characterisation for the action being amenable. As a corollary, we obtain a fixed point characterisation for the exactness of G.

2010 *Mathematics subject classification*: primary 20F65.

Keywords and phrases: amenable action, fixed point, exactness.

1. Introduction

Amenability is a property of groups introduced by Von Neumann in his investigation of the Banach–Tarski paradox. A group is amenable if it admits an invariant mean. There are many equivalent formulations of amenability. One of the well-known characterisations is Day's fixed point theorem [4]: a discrete group G is amenable if and only if any affine action of G on a nonempty compact convex subset of a locally convex Hausdorff space has a fixed point.

The notion of an amenable action of a group on a topological space was discussed by Anantharaman-Delaroche and Renault [2]. It is a generalisation of amenability and arises naturally in many areas of mathematics. For example, a group acts amenably on a point if and only if it is amenable and every hyperbolic group acts amenably on its Gromov boundary [1].

Another generalisation of amenability was given by Kirchberg and Wassermann [6] with the definition of exactness for groups in terms of properties of the minimal tensor product of the reduced group C*-algebras. As with amenability, exactness has equivalent characterisations, which are of interest in different areas of mathematics. Higson and Roe [5] and Ozawa [8] proved a remarkable result that unifies the two approaches: a finitely generated discrete group is exact if and only if the action on its Stone–Čech compactification βG is amenable.

Project partially supported by the National Natural Science Foundation of China (No. 11271321).

Coarse geometric versions of classical notions or results in group theory can sometimes be obtained by considering the problem with coefficients in $\ell_\infty(G)$. With this point of view, Brodzki *et al.* [3] introduced a notion of an invariant mean for a topological action and proved that the existence of such a mean characterises the amenability of an action and its exactness. In a similar fashion, we will give a fixed point characterisation for amenable actions and exactness. Our characterisation is a partial generalisation of Day's fixed point theorem.

2. Fixed point characterisation

First, we recall some notation and definitions from [3]. Let $\mathcal{X}$ be a compact Hausdorff topological space and let $C(\mathcal{X})$ denote the space of real-valued continuous functions on $\mathcal{X}$. For a function $f : G \to C(\mathcal{X})$, we denote by f_g the continuous function on $\mathcal{X}$ obtained by evaluating f at $g \in G$. We define the sup-ℓ_1-norm of f by

$$\|f\|_{\infty,1} = \sup_{x \in X} \sum_{g \in G} |f_g(x)|$$

and we denote by $\mathcal{V}$ the Banach space of all functions on G with values in $C(\mathcal{X})$ that have finite norm.

Definition 2.1 [3]. Let $W_{00}(G, \mathcal{X})$ be the subspace of $\mathcal{V}$ consisting of all functions $f : G \to C(\mathcal{X})$ which have finite support and such that for some $c \in \mathbb{R}$, depending on f, $\sum_{g \in G} f_g = c1_{\mathcal{X}}$, where $1_{\mathcal{X}}$ denotes the constant function 1 on $\mathcal{X}$. The closure of this space in the sup-ℓ_1-norm will be denoted $W_0(G, \mathcal{X})$.

Let $\pi : W_{00}(G, \mathcal{X}) \to \mathbb{R}$ be defined by $\sum_{g \in G} f_g = \pi(f)1_{\mathcal{X}}$. The map π is continuous with respect to the sup-ℓ_1-norm and so extends to the closure $W_0(G, \mathcal{X})$.

The G-action on $\mathcal{X}$ gives an isometric action of G on $C(\mathcal{X})$ in the usual way: for $g \in G$ and $f \in C(\mathcal{X})$, we have $(g \cdot f)(x) = f(g^{-1}x)$. The group G also acts isometrically on the space $\mathcal{V}$ in a natural way: for $g, h \in G$, $f \in \mathcal{V}$ and $x \in \mathcal{X}$, we have $(g \cdot f)_h(x) = f_{g^{-1}h}(g^{-1}x) = (g \cdot f_{g^{-1}h})(x)$.

Definition 2.2 [3]. Let $\mathcal{E}$ be a Banach space. We say that $\mathcal{E}$ is a $C(\mathcal{X})$-module if it is equipped with a contractive unital representation of the Banach algebra $C(\mathcal{X})$. If $\mathcal{X}$ is a G-space, then a $C(\mathcal{X})$-module $\mathcal{E}$ is said to be a G-$C(\mathcal{X})$-module if the group G acts on $\mathcal{E}$ by linear isometries and the representation of $C(\mathcal{X})$ is G-equivariant, that is, for every $g \in G$, $f \in \mathcal{E}$ and $t \in C(\mathcal{X})$, we have $g(tf) = (gt)(gf)$.

Let $\mathcal{E}$ be a G-$C(\mathcal{X})$-module, let $\mathcal{E}^*$ be the Banach dual of $\mathcal{E}$ and let $\langle -, - \rangle$ be the pairing between the two spaces. The induced actions of G and $C(\mathcal{X})$ on $\mathcal{E}^*$ are defined as follows. For $\alpha \in \mathcal{E}^*$, $g \in G$, $f \in C(\mathcal{X})$ and $v \in \mathcal{E}$, we let

$$\langle g\alpha, v \rangle = \langle \alpha, g^{-1}v \rangle, \quad \langle f\alpha, v \rangle = \langle \alpha, fv \rangle.$$

Given a Banach space $\mathcal{E}$, define $\ell_\infty(G, \mathcal{E})$ to be the space of functions $f : G \to \mathcal{E}$ such that $\sup_{g \in G} \|f(g)\| < \infty$. If G acts on $\mathcal{E}$, then the action of the group G on the

https://doi.org/10.1017/S0004972715000520 Published online by Cambridge University Press

space $\ell_\infty(G, \mathcal{E})$ is defined in an analogous way to the action of G on $\mathcal{V}$, using the induced action of G on $\mathcal{E}$:

$$(g\tau)_h = g(\tau_{g^{-1}h})$$

for $\tau \in \ell_\infty(G, \mathcal{E})$ and $g \in G$.

Definition 2.3. A G-invariant subset $\mathcal{K}$ in a $C(\mathcal{X})$-module $\mathcal{E}$ is called $C(\mathcal{X})$-convex if given any finite collection of positive elements $f_1, \dots, f_n \in C(\mathcal{X})$ such that $\sum_{i=1}^n f_i = 1_{\mathcal{X}}$, we have $\sum_{i=1}^n f_i k_i \in \mathcal{K}$ for any $k_1, \dots, k_n \in \mathcal{K}$.

Remark 2.4. From [3], we know that $W_0(G, \mathcal{X})$ is a G-module and $W_0(G, \mathcal{X})$ is not invariant under the action of $C(\mathcal{X})$. So, $W_0(G, \mathcal{X})$ is not a $C(\mathcal{X})$-submodule of $\mathcal{V}$. If we define $W_{00}^1(G, \mathcal{X}) = \{f \in W_{00}(G, \mathcal{X}) : \sum_{g\in G} f_g = 1_{\mathcal{X}}\}$ and $W_0^1(G, \mathcal{X})$ to be the closure of $W_{00}^1(G, \mathcal{X})$, then $W_0^1(G, \mathcal{X})$ is a G-module in $\mathcal{V}$ and is $C(\mathcal{X})$-convex. Indeed, for any $h \in G$, $f \in W_{00}^1(G, \mathcal{X})$ and $x \in X$, $\sum_{g\in G}(h \cdot f)_g(x) = \sum_{g\in G} f_{h^{-1}g}(h^{-1}x) = 1$. So, $h \cdot f \in W_{00}^1(G, \mathcal{X})$. This implies that $W_{00}^1(G, \mathcal{X})$ is a G-module and so is its closure $W_0^1(G, \mathcal{X})$. For any $\{f_i\}_{i=1}^n \subseteq C(\mathcal{X})$ with $f_i \geq 0$, $\sum_{i=1}^n f_i = 1_{\mathcal{X}}$ and $\{k_i\}_{i=1}^n \subseteq W_{00}^1(G, \mathcal{X})$, we have $\operatorname{supp}(\sum_{i=1}^n f_i k_i) \subseteq \bigcup_{i=1}^n \operatorname{supp} k_i$ and

$$\begin{aligned}
\sum_{g\in G}\Big(\sum_{i=1}^n f_i k_i\Big)_g(x) &= \sum_{g\in G}\sum_{i=1}^n f_i(x) k_{i,g}(x) \\
&= \sum_{i=1}^n f_i(x) \sum_{g\in G} k_{i,g}(x) \\
&= \sum_{i=1}^n f_i(x) = 1, \quad \forall x \in \mathcal{X}.
\end{aligned}$$

This implies that $\sum_{i=1}^n f_i k_i \in W_{00}^1(G, \mathcal{X})$ and $W_{00}^1(G, \mathcal{X})$ is $C(\mathcal{X})$-convex and so is its closure $W_0^1(G, \mathcal{X})$.

Definition 2.5 [3]. Let $\mathcal{E}$ be a Banach space and a $C(\mathcal{X})$-module. We say that v_1 and v_2 in $\mathcal{E}$ are disjointly supported if there exist $f_1, f_2 \in C(\mathcal{X})$ with disjoint supports such that $f_1 v_1 = v_1$ and $f_2 v_2 = v_2$. We say that the module $\mathcal{E}$ is ℓ_1-geometric if for every two disjointly supported v_1 and v_2 in $\mathcal{E}$, $\|v_1 + v_2\| = \|v_1\| + \|v_2\|$.

Definition 2.6 [3]. The action of G on $\mathcal{X}$ is amenable if and only if there exists a sequence of elements $f^n \in W_0(G, \mathcal{X})$ such that:

(1) $f_g^n \geq 0$ in $C(\mathcal{X})$ for every $n \in \mathbb{N}$ and $g \in G$;
(2) $\pi(f^n) = 1$ for every n;
(3) for each $g \in G$, we have $\|f^n - gf^n\|_{\mathcal{E}} \to 0$.

Theorem 2.7. *Let G be a finitely generated group acting by homeomorphisms on a compact Hausdorff space $\mathcal{X}$. This action is amenable if and only if, for any ℓ_1-geometric G-$C(\mathcal{X})$-module $\mathcal{E}$, any nonempty weak*-compact $C(\mathcal{X})$-convex G-invariant subset $\mathcal{K} \subseteq \mathcal{E}^*$ contains a G-fixed point.*

https://doi.org/10.1017/S0004972715000520 Published online by Cambridge University Press

Proof. *Necessity.* Suppose that the action of G on $\mathcal{X}$ is amenable. By [3, Theorem A], there exists an invariant mean $\mu \in W_0(G,\mathcal{X})^{**}$ for the action on $\mathcal{X}$. Since $\mu \in W_0(G,\mathcal{X})^{**}$ and $W_{00}(G,\mathcal{X})$ is norm-dense in $W_0(G,\mathcal{X})$, μ is the weak*-limit of a bounded net of elements $f^\lambda \in W_{00}(G,\mathcal{X})$ with $f^\lambda_h \geq 0$ in $C(\mathcal{X})$ for any $h \in G$ and $\sum_{h\in G} f^\lambda_h = \pi(f^\lambda) = 1_{\mathcal{X}}$. We define $\ell_\infty(G,\mathcal{E}^*)$ to be the space of functions $\tau : G \to \mathcal{E}^*$ such that $\sup_{g\in G}\|\tau_g\|_{\mathcal{E}^*} < \infty$. Choose $\tau \in \ell_\infty(G,\mathcal{E}^*)$ and $v \in \mathcal{E}$ and define a linear functional $\sigma_{\tau,v} : W_{00}(G,\mathcal{X}) \to \mathbb{R}$ by

$$\sigma_{\tau,v}(f) = \Big\langle \sum_{h\in G} f_h\tau_h | v\Big\rangle, \quad \forall f \in W_{00}(G,\mathcal{X}).$$

It follows from [3, Lemma 14] that the linear functional $\sigma_{\tau,v}$ extends to a continuous linear functional on $W_0(G,\mathcal{X})$. We also denote the extension by $\sigma_{\tau,v}$ and so $\sigma_{\tau,v} \in W_0(G,\mathcal{X})^*$ for any $\tau \in \ell_\infty(G,\mathcal{E}^*)$ and $v \in \mathcal{E}$.

So, for $\tau \in \ell_\infty(G,\mathcal{E}^*)$ and $v \in \mathcal{E}$,

$$\begin{aligned}\mu(\sigma_{\tau,v}) &= \lim_\lambda \sigma_{\tau,v}(f^\lambda)\\ &= \lim_\lambda \Big\langle \sum_h f^\lambda_h \tau_h | v\Big\rangle\\ &= \lim_\lambda \langle x_\lambda | v\rangle,\end{aligned}$$

where $x_\lambda = \sum_\lambda f^\lambda_h \tau_h \in \mathcal{E}^*$. Since $f^\lambda \geq 0$ and $\sum_h f^\lambda_h = \pi(f^\lambda) = 1$, $\|x_\lambda\| \leq \|\tau\|$. By the Alaoglu–Bourbaki theorem, there exists a convergent subnet of $\{x_\lambda\}$, which we denote again by $\{x_\lambda\}$, and we define $x_0 = \lim_\lambda x_\lambda$. Then

$$\mu(\sigma_{\tau,v}) = \langle x_0 | v\rangle. \tag{2.1}$$

For any $g \in G$, [3, Lemma 15] and the invariance of μ show that

$$\langle gx_0 | v\rangle = \langle x_0 | g^{-1}v\rangle = \mu(\sigma_{\tau,g^{-1}v}) = \mu(g\sigma_{\tau,g^{-1}v}) = \mu(\sigma_{g\tau,v}). \tag{2.2}$$

Given a weak*-compact $C(\mathcal{X})$-convex G-module $\mathcal{K} \subset \mathcal{E}^*$, we choose $k_0 \in \mathcal{K}$ and define $\tau : G \to \mathcal{E}^*$ by

$$\tau : h \to hk_0, \quad \forall h \in G.$$

Thus, $\tau \in \ell_\infty(G,\mathcal{E}^*)$ and $g\tau = \tau$. Indeed, for any $h \in G$, $(g\cdot\tau)(h) = g\cdot\tau(g^{-1}h) = g(g^{-1}hk_0) = hk_0 = \tau(h)$. Since $\mathcal{K}$ is a G-module, $\tau_h \in \mathcal{K}$ for all $h \in G$. Since $\mathcal{K}$ is weak*-closed and $C(\mathcal{X})$-convex, $x_\lambda = \sum_h f^\lambda_h \tau_h \in \mathcal{K}$ and so $x_0 \in \mathcal{K}$. For this special $\tau \in \ell_\infty(G,\mathcal{E}^*)$ and any $v \in \mathcal{E}$, it follows from (2.1) and (2.2) that

$$\langle x_0 | v\rangle = \mu(\sigma_{\tau,v}) = \langle gx_0 | v\rangle.$$

This implies that $gx_0 = x_0$ for any $g \in G$.

Sufficiency. Let $\mathcal{M}$ denote the set of all means for the action of G on $\mathcal{X}$. By Goldstine's theorem [7], if $\mu \in \mathcal{M} \subseteq W_0^{**}(G,\mathcal{X}\mathcal{X})$, μ is the weak*-limit of a bounded set of elements $f^\lambda \in W_0(G,\mathcal{X})$. We can choose $f^\lambda \in W_0^1(G,\mathcal{X})$. Indeed, given f^λ with $\pi(f^\lambda) = c_\lambda \to \mu(\pi) = 1$, we replace f^λ by $f^\lambda + (1-c_\lambda)\delta_e$, where $\delta_e \in W_{00}(G,\mathcal{X})$,

https://doi.org/10.1017/S0004972715000520 Published online by Cambridge University Press

$\delta_e(h) = 1$ if $h = e$ and 0 otherwise. Since $(1 - c_\lambda)\delta_e \to 0$ in norm in $W_0(G, \mathcal{E})$, μ is the weak*-limit of the net $f^\lambda + (1 - c_\lambda)\delta_e$, as required. Since $W_0^1(G, \mathcal{X})$ is a G-module and $C(\mathcal{X})$-convex, so is $\mathcal{M}$. The set $\mathcal{M}$ is not empty: for example, the point evaluation is a mean on $W_0(G, \mathcal{X})^*$. There is a continuous affine action $m \to gm$ of G on $\mathcal{M}$ given by $gm(\varphi) = m(g\varphi)$ for all $g \in G$ and $\varphi \in W_0(G, \mathcal{X})^*$. Theorem A in [3] shows that the action of G on $\mathcal{X}$ is amenable if and only if this action of G on $\mathcal{M}$ has an invariant mean. So, the sufficiency is clear from the hypothesis. □

If $\mathcal{X}$ is the Stone–Čech compactification βG of the group, then $C(\beta G)$ can be identified with $\ell_\infty(G)$, and we obtain the following result.

Corollary 2.8. *A finitely generated group G is exact if and only if every G-affine action of G on a bounded weak*-compact nonempty $\ell_\infty(G)$-convex G-module $\mathcal{K}$ of $\mathcal{E}^*$ has a fixed point for any ℓ_1-geometric G-$\ell_\infty(G)$-module $\mathcal{E}$.*

References

[1] S. Adams, 'Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups', *Topology* **33** (1994), 765–783.
[2] C. Anantharaman-Delaroche and J. Renault, *Amenable Groupoids*, Monographies de L'Enseignement Mathématique, 36 (L'Enseignement Mathématique, Geneva, 2000).
[3] J. Brodzki, G. A. Niblo, P. Nowak and N. Wright, 'Amenable actions, invariant means and bounded cohomology', *J. Topol. Anal.* **321** (2012), 321–334.
[4] M. M. Day, 'Fixed point theorems for compact convex sets', *Illinois J. Math.* **5** (1961), 585–590; Correction: *Illinois J. Math.* **8** (1964), 713.
[5] N. Higson and J. Roe, 'Amenable group actions and the Novikov conjecture', *J. reine angew. Math.* **519** (2000), 143–153.
[6] E. Kirchberg and S. Wassermann, 'Exact groups and continuous bundles of C*-algebras', *Math. Ann.* **315** (1999), 169–203.
[7] R. E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics, 183 (Springer, New York, 1998).
[8] N. Ozawa, 'Amenable actions and exactness for discrete groups', *C. R. Acad. Sci. Paris* **330** (2000), 691–695.

Z. DONG, Department of Mathematics, Zhejiang University,
Hangzhou 310027, PR China
e-mail: dongzhe@zju.edu.cn

Y. Y. WANG, Department of Mathematics, Zhejiang University,
Hangzhou 310027, PR China
e-mail: wngyny@163.com

https://doi.org/10.1017/S0004972715000520 Published online by Cambridge University Press