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ON THE RECOGNITION OF RIGHT-ANGLED ARTIN GROUPS

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Abstract. There does not exist an algorithm that can determine whether or not a group presented by commutators is a right-angled Artin group.

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1. Introduction. In [4], Day and Wade introduced an elegant new homology theory for subspace arrangements and related it to recognition problems concerning *right-angled Artin groups* (RAAGs). In setting the context for their work, they asked if there is an algorithmic procedure for recognizing RAAGs among groups given by presentations whose only relations are commutators ([4] Question 1.2) and speculated that the answer was likely to be no. The purpose of this note is to confirm this speculation.

THEOREM 1.1. *There does not exist an algorithm that can determine whether or not a group presented by commutators is a RAAG.*

In more detail, there is no algorithm that given 22 words u_i in the free group $F(a_1, a_2, a_3, a_4)$ can determine whether or not the group with presentation

$$\langle a_1, a_2, a_3, a_4, t \mid [a_1, a_3], [a_1, a_4], [a_2, a_3], [a_2, a_4], [t, u_1], \ldots, [t, u_{22}]\rangle$$

is a RAAG. Nor is there an algorithm that can determine whether or not such a group is commensurable with a RAAG or quasi-isometric to a RAAG.

2. Fibre products and triviality for two-generator groups. In the aftermath of the construction by Novikov [9] and Boone [2] of finitely presented groups with unsolvable word problem, many other decisions problems for groups were proved to be unsolvable through subtle work by many authors. We shall appeal to two results that come from the work of C.F. Miller III.

Let K be a group given by a presentation with n generators and m relations. Following Miller, one can associate to each word w in the generators of K a presentation with two generators and $n + m + 3$ relations – this is derived from the presentation in Lemma 3.6 of [7] by making Tietze moves to remove unnecessary generators. The group given by this presentation is trivial if $w = 1$ in K, but it contains K if $w \neq 1$.

The most concise finite presentation that is known for a group with unsolvable word problem is the one constructed by Borisov [3] half a century ago – it has 5 generators and 12 relations. By applying Miller's construction to words in the generators of Borisov's example, we obtain a recursive sequence of two-generator presentations $\mathcal{P}_n = \langle a_1, a_2 \mid S_n\rangle$ with $|S_n| = 20$ such that the group presented is either trivial or else has an unsolvable word problem, and there is no algorithm that can determine the set of integers n for which each alternative holds.

We exploit these examples in the manner of Mihailova [8] and Miller ([6] p.39).

LEMMA 2.1. *Let F be a free group of rank 2 with generators $\{a_1, a_2\}$. If $k \geq 20$, then there exists a recursive sequence (S_n) of subsets of F, each of cardinality k, such that there is no algorithm to determine whether or not $F \times F$ is generated by $U_n = \{(a_1, a_1), (a_2, a_2), (s, 1) : s \in S_n\}$. Moreover, if $\langle U_n \rangle$ is a proper subgroup, then $H_2(\langle U_n \rangle, \mathbb{Z})$ is not finitely generated and there is no algorithm to determine which words in the generators of $F \times F$ determine elements of $\langle U_n \rangle$.*

Proof. Associated with any two-generator finite presentation $\mathcal{P} = \langle a_1, a_2 \mid r_1, \ldots, r_M \rangle$, one has the fibre product $P < F \times F$ consisting of pairs (u, v) such that $u = v$ in the group $G(\mathcal{P})$ presented by $\mathcal{P}$. It is easy to check that P is generated by $U = \{(a_1, a_1), (a_2, a_2), (r_1, 1), \ldots, (r_M, 1)\}$. If $G(\mathcal{P})$ is trivial, $P = F \times F$. But if $G(\mathcal{P})$ is infinite, Baumslag and Roseblade's Theorem A [1] shows that $H_2(P, \mathbb{Z})$ is not finitely generated. Moreover, if the word problem is unsolvable in $G(\mathcal{P})$, the membership problem of $P < F \times F$ is unsolvable, because deciding if $(w, 1) \in P$ is equivalent to deciding if $w = 1$ in $G(\mathcal{P})$. Consideration of the presentations $\mathcal{P}_n = \langle a_1, a_2 \mid S_n \rangle$ from the discussion preceding the lemma completes the proof. □

We shall apply the following lemma with $D = F \times F$ and $C = \langle U_n \rangle$.

LEMMA 2.2. *For any HNN extension of the form $\Gamma = D{*}_C$, if $H_2(D, \mathbb{Z})$ is finitely generated but $H_2(C, \mathbb{Z})$ is not, then $H_3(\Gamma, \mathbb{Z})$ is not finitely generated.*

Proof. The Mayer–Vietoris sequence for the HNN extension contains the exact sequence

$$H_3(\Gamma, \mathbb{Z}) \to H_2(C, \mathbb{Z}) \to H_2(D, \mathbb{Z}).$$

□

3. Proof of Theorem 1.1. Given 20 words $S = \{r_1, \ldots, r_{20}\}$ in the free group $F = F(a_1, a_2)$, we denote by $\Gamma(S)$ the group with generators a_1, a_2, a_3, a_4, t and relations

$$[a_1, a_3], [a_1, a_4], [a_2, a_3], [a_2, a_4], [t, a_1 a_3], [t, a_2 a_4], [t, r_1], \ldots, [t, r_{20}].$$

Note that this presentation is of the type described in Theorem 1.1. The group $\Gamma(S)$ is an HNN extension of $F \times F$ with a stable letter t that commutes with the fibre product $P < F \times F$ associated with the presentation $\mathcal{P} = \langle a, b \mid S \rangle$.

As in the proof of Lemma 2.1, we have a dichotomy: if $G(\mathcal{P}) = 1$, then $\Gamma(S) = F \times F \times \mathbb{Z}$ is a RAAG; but if $G(\mathcal{P})$ is infinite, then $\Gamma(S)$ is not a RAAG, because RAAGs have finite classifying spaces [5], whereas $H_3(\Gamma(S), \mathbb{Z})$ is not finitely generated, by Lemma 2.2, because $H_2(P, \mathbb{Z})$ is not finitely generated.

Lemma 2.1 tells us that there is no algorithm to determine which of the possibilities in this dichotomy holds. Moreover, by choosing sets (S_n) as in the lemma, we can arrange that when $\Gamma(S_n)$ is not equal to $F \times F \times \mathbb{Z}$, it will have an unsolvable word problem: given a word w in the generators a_1, a_2, Britton's Lemma implies that $[t, w] = 1$ in $\Gamma(S_n)$ if and only if $w = 1$ in $G(\mathcal{P}_n)$, and this is undecidable.

Since being of type FP_3 and having a solvable word problem are both invariants of commensurability and quasi-isometry, there is no algorithm that can determine whether $\Gamma(S_n)$ is commensurable with or quasi-isometric to a RAAG. □

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