MULTIPLY SUBADDITIVE FUNCTIONS

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1. Introduction. Let S denote a Boolean ring with elements e, that is, a distributive, relatively complemented lattice with zero element 0 [**2**, p. 153]. In this paper we study real-valued functions $\Phi(e)$, $e \in S$ which have a representation of the form

$$\Phi(e) = \sup_{\phi \epsilon C} \phi(e), \tag{1.1}$$

C being a certain class of *additive* functions on S; $\phi(e)$ is additive if $\phi(e_1 \cup e_2) = \phi(e_1) + \phi(e_2)$ for any pair $e_1, e_2 \in S$, $e_1 \cap e_2 = 0$. We find a relation between (a) the possibility of representation (1.1); (b) the possibility of extension of $\Phi(e)$ onto a vector space X containing S; (c) some simple intrinsic properties of $\Phi(e)$. For instance, one of our results (Theorem 4 in §5) is that $\Phi(e)$ possesses a representation (1.1), C being a family of addititive and positive functions $\phi(e)$, if and only if $\Phi(e)$ is increasing and has the property

$$p\Phi(e) \leqslant \sum_{\nu=1}^{n} \Phi(e_\nu) \tag{1.2}$$

whenever the e_ν cover e exactly p times (for a precise definition, see §§2,3). Functions Φ, satisfying (1.2), we call *multiply subadditive;* this property is stronger than the ordinary subadditivity expressed by the inequality

$$\Phi(e_1 \cup e_2) \leqslant \Phi(e_1) + \Phi(e_2), \qquad e_1 \cap e_2 = 0.$$

On the other hand, we shall see that (1.2), with $=$ instead of $\leqslant$, holds for any additive function $\Phi(e)$. Multiply subadditive functions constitute, therefore, an intermediary class between the subadditive and the additive functions.

The problems treated in this paper arose, in the case when S is a Boolean ring of measurable sets, in connection with the study of certain spaces of functions, see [**5**, §4].

2. The vector space $X(S)$. A natural extension of a Boolean ring S into a space $X(S) = X = \{x\}$ is obtained as follows. Let x be any finite sum

$$x = \sum_{\nu=1}^{n} a_\nu e_\nu,$$

the order of terms being by definition irrelevant, where a_ν are arbitrary real numbers and e_ν arbitrary elements of S (with repetitions allowed). We define an equivalence relation $x \equiv y$ for two sums $x = \sum a_\nu e_\nu$, $y = \sum b_\mu f_\mu$ of this kind to mean that x can be transformed into y by a finite number of changes of the

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types: (A) a term ae in the sum x is replaced by $ae' + ae''$, if $e = e' \cup e''$, $e' \cap e'' = 0$, or conversely, $ae' + ae''$ is replaced by ae; (B) $0e$ is omitted, or conversely, is added; (C) ae is replaced by $a'e + a''e$, where $a = a' + a''$, or conversely.

This equivalence relation is reflexive, symmetric, and transitive. Let X be the set of all equivalence classes and let X be provided with operations of addition and scalar multiplication as follows. If $x = \sum a_\nu e_\nu$, $y = \sum b_\mu f_\mu$, then

$$ax = \sum aa_\nu e_\nu, \quad x + y = \sum a_\nu e_\nu + \sum b_\mu f_\mu.$$

Clearly $x \equiv x_1$ and $y \equiv y_1$ imply $ax \equiv ax_1$ and $x + y \equiv x_1 + y_1$ and it follows that X is a vector space with zero of S as zero element.

The following lemma will be useful:

(2.1) *Two sums $x = \sum a_\nu e_\nu$ and $y = \sum b_\mu f_\mu$ are equivalent if and only if there are disjoint elements $g_1, \ldots, g_N$ such that every e_ν and every f_μ is a union of some of these g_ρ in such a way that, if the e_ν, f_μ are replaced by the sums of the corresponding g_ρ and the terms reduced, the two expressions $\sum a_\nu e_\nu$, $\sum b_\mu f_\mu$ become identical.*

For the proof, we write $x \backsim y$, if there are such g_ρ. Clearly, $x \backsim y$ implies $x \equiv y$. But also the converse is true. First, we have $x \backsim x$ for any x. For if $e_1, \ldots, e_n$ is a finite set of elements of S, the e_ν can be expressed as unions of suitable disjoint g_ρ. Such g_ρ are obtained by taking all possible intersections $\bigcap_{\nu=1}^n e'_\nu$, where each e'_ν is either e_ν or the complement of e_ν with respect to $\bigcup_{\mu=1}^n e_\mu$. Again, the relation $x \backsim y$ is not destroyed when any of the admissible changes (A), (B), (C) is performed on x. This shows that $x \backsim y$ is equivalent to $x \equiv y$ and proves our assertion. In particular, it follows that if $\sum e_\nu$ and $\sum f_\mu$ are equivalent then $\bigcup e_\nu = \bigcup f_\mu$. As a corollary we obtain that two elements e_1, e_2 of S which are equivalent, are identical in S.

We can now describe the relation $pe = \sum_{\nu=1}^n e_\nu$ in X, that is, the equivalence

$$\sum_{\mu=1}^{p} f_\mu \equiv \sum_{\nu=1}^{n} e_\nu,$$

where $f_1 = \ldots = f_p = e$, more directly in terms of S. Using the $g_1, \ldots, g_N$ of (2.1) it follows that

(2.2) *$pe = \sum e_\nu$ holds if and only if there are disjoint decompositions $e_\nu = \bigcup_{\mu=1}^p e_{\nu\mu}$ such that $e = \bigcup_{\nu=1}^n e_{\nu\mu}$ as a disjoint decomposition for every $\mu = 1, \ldots, p$.*

For instance, we may by induction on ν define the decompositions $e_\nu = \bigcup e_{\nu\mu}$ as follows: let $e_{\nu\mu}$ be the union of those g_ρ which satisfy $g_\rho \subset e_\nu$ and $g_\rho \subset e_\sigma$ for precisely $\mu - 1$ indices $\sigma < \nu$. If (2.2) holds, we shall say that the $e_1, \ldots, e_n$ *cover e exactly p times*. In the same way, we shall say that $e_1, \ldots, e_n$ *cover e at least p times* if there are disjoint decompositions $e_\nu = \bigcup_{\mu=1}^p e_{\nu\mu}$ with $e \subset \bigcup_{\nu=1}^n e_{\nu\mu}$ $(\mu = 1, \ldots, p)$. It is clear that this is the case if and only if there are $e'_\nu \subset e_\nu (\nu = 1, \ldots, n)$ which cover e exactly p times.

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We shall write $x \leqslant y$, $x, y \in X$ if there exist representations

$$x = \sum_1^n a_\nu e_\nu, \quad y = \sum_1^n b_\nu e_\nu$$

with $a_\nu \leqslant b_\nu (\nu = 1, \ldots, n)$. This relation is transitive by (2.1). For instance, $e_1 \subset e_2$ implies $e_1 \leqslant e_2$ in X.

3. Multiply subadditive functions. As stated in §1, a function $\Phi(e)$, $e \in S$ is multiply subadditive if $p\Phi(e) \leqslant \sum \Phi(e_\nu)$ whenever $pe = \sum e_\nu$ in X, that is, whenever the e_ν cover e exactly p times. If $\Phi(e)$ is, moreover, increasing, $\Phi(e) \leqslant \Phi(e')$ for $e \subset e'$, then the last inequality holds even if the e_ν cover e at least p times.

Writing $0 = 0 + 0$, $2 \cdot 0 = 0$, we obtain $\Phi(0) \leqslant 2\Phi(0)$, $2\Phi(0) \leqslant \Phi(0)$. Therefore, a multiply subadditive function has the property $\Phi(0) = 0$. If, in addition, Φ is increasing, it follows that $\Phi(e) \geqslant 0$, $e \in S$.

If $\Phi(e)$ is additive on S, we obtain an extension $F(x)$ of ϕ onto X by putting $F(x) = \sum a_\nu \phi(e_\nu)$ if $x = \sum a_\nu e_\nu$. Since the first sum is invariant under changes (A), (B), (C) of §2, $F(x)$ is a function defined on X. Clearly $F(x)$ is additive. In particular, we obtain

$$p\phi(e) = \sum a_\nu \phi(e_\nu), \qquad pe = \sum a_\nu e_\nu, \tag{3.1}$$

so that any additive function ϕ on S is multiply subadditive with equality in (1.1). If, in addition, ϕ is positive, $\phi(e) \geqslant 0$, $e \in S$, then

$$\sum a_\nu \phi(e_\nu) \leqslant \sum b_\nu \phi(e_\nu), \qquad \sum a_\nu e_\nu \leqslant \sum b_\nu e_\nu. \tag{3.2}$$

We finally remark that the condition

$$\Phi(e) \leqslant \sum_{\nu=1}^n a_\nu \Phi(e_\nu) \qquad \text{whenever } e = \sum a_\nu e_\nu,\ a_\nu \geqslant 0, \tag{3.3}$$

is equivalent to multiple subadditivity. If the a_ν are all rational, we write $a_\nu = k_\nu/k$ with positive integers k_ν, k, and repeating each e_ν exactly k_ν times, deduce (3.3) from (1.2). In the general case we see, using (2.1), that, for fixed e_ν, e, the relation $e = \sum a_\nu e_\nu$ is equivalent to a system of linear equations, with integral coefficients, for the a_ν. Solutions $a_1, \ldots, a_n$ of this system can be approximated by positive rational solutions $a_1^{(m)}, \ldots, a_n^{(m)}$. Then $a_\nu^{(m)} \to a_\nu$ for $m \to \infty$ and $e = \sum a_\nu^{(m)} e_\nu$. Making $m \to \infty$ in

$$\Phi(e) \leqslant \sum a_\nu^{(m)} \Phi(e_\nu),$$

we obtain (3.3).

4. Extension of functions from S onto X. In this section we connect the possibility of representation of the form

$$\Phi(e) = \sup_{\phi \in C} \phi(e), \tag{4.1}$$

$\phi(e)$ additive, with the possibility of extension of $\Phi(e)$ onto $X(S)$.

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THEOREM 1. $\Phi(e)$ *has a representation*

$$\Phi(e) = \sup_{\phi \in C} |\phi(e)| \tag{4.2}$$

if and only if $\Phi(e)$ *has an extension* $P(x)$ *onto* X *which satisfies the conditions*

$$\text{(i)} \qquad P(x+y) \leqslant P(x) + P(y),$$

$$\text{(ii)} \qquad P(ax) = aP(x), \qquad a \geqslant 0,$$

$$\text{(iii)} \qquad P(x) \geqslant 0,$$

$$\text{(iv)} \qquad P(-x) = P(x).$$

Proof. If (4.2) holds, we define

$$P(x) = \sup_{\phi \in C} |\sum a_\nu \phi(e_\nu)|, \qquad x = \sum a_\nu e_\nu, \tag{4.3}$$

the value of $\sum a_\nu \phi(e_\nu)$ being independent of the choice of the representation $x = \sum a_\nu e_\nu$. Then $P(x)$ is finite, since

$$0 \leqslant P(x) \leqslant \sum |a_\nu| \Phi(e_\nu) < + \infty.$$

Also, $P(x)$ satisfies conditions (i)–(iv). Moreover, $P(x) = \Phi(e)$ for $x = e \in S$.

If, on the other hand, $\Phi(e)$ has an extension $P(x)$ of the required kind, we apply the Hahn-Banach theorem [**1**] and obtain, for each $e_0 \in S$, a linear functional $F(x)$ on X satisfying $F(e_0) = P(e_0) = \Phi(e_0)$ and $F(x) \leqslant P(x)$, $x \in X$. Then $F(x) \geqslant -P(-x) = -P(x)$, that is, $|F(x)| \leqslant P(x)$, $x \in X$. If C is the class of all functions $\phi(e) = F(e)$, $e \in S$ for all $F(x)$ of this kind, then (4.1) holds.

THEOREM 2. $\Phi(e)$ *has a representation*

$$\Phi(e) = \sup_{\phi \in C} \phi(e), \qquad \phi(e) \geqslant 0, \tag{4.4}$$

where C *is a class of positive additive functions* ϕ *if and only if* $\Phi(e)$ *has an extension* $P(x)$ *onto* X *with properties* (i)–(iv) *and*

$$\text{(v)} \qquad P(e_1) \leqslant P(e_2), \qquad e_1 \subset e_2.$$

Proof. If $\Phi(e)$ satisfies (4.4), then $P(x)$, defined by (4.3), has the properties (i)–(v), so that they are necessary.

On the other hand, if $\Phi(e)$ has a continuation $P(x)$, then the proof of Theorem 1 establishes (4.4) where, however, the functions $\phi \in C$ are not necessarily positive. Let

$$\phi_1(e) = \sup_{e' \subset e} \phi(e') \geqslant 0$$

be the positive variation of $\phi \in C$. It is easy to see that ϕ_1 is additive and moreover (since $\Phi(e)$ increases by (v))

$$\Phi(e) = \sup_{e' \subset e} \Phi(e') = \sup_{\phi \in C} [\sup_{e' \subset e} \phi(e')] = \sup_{\phi_1 \in C_1} \phi_1(e),$$

which establishes (4.4) with $C_1 = \{\phi_1\}$ instead of C.

5. Representation of multiply subadditive functions. In this section we give the main results of this paper which connect the possibility of representation of a function $\Phi(e)$ in the form $\Phi(e) = \sup \phi(e)$ with the multiple subadditivity of $\Phi(e)$.

THEOREM 3. *A function $\Phi(e)$ on S has a representation*

$$\Phi(e) = \sup_{\phi \in C} |\phi(e)| \tag{5.1}$$

if and only if $\Phi(e)$ satisfies the condition

$$\Phi(e) \leqslant \sum_{\nu=1}^{n} |a_\nu| \Phi(e_\nu) \qquad \textit{whenever } e = \sum a_\nu e_\nu. \tag{5.2}$$

Proof. We begin by remarking that (5.1) and (5.2) both imply $\Phi(e) \geqslant 0$, the latter condition by putting $e = e - e + e$. If (5.1) holds and $e = \sum a_\nu e_\nu$, then

$$|\phi(e)| = |\sum a_\nu \phi(e_\nu)| \leqslant \sum |a_\nu| \Phi(e_\nu)$$

and (5.2) follows. Conversely, if this condition is fulfilled, we set

$$P(x) = \inf \sum |a_\nu| \Phi(e_\nu) \tag{5.3}$$

where the infimum is taken for all representations $x = \sum a_\nu e_\nu$. Then $0 \leqslant P(x) < +\infty$ and, by (5.2), $P(e) = \Phi(e)$, $e \in S$. As $P(x)$ satisfies (i)–(iv), we obtain (5.1) by Theorem 1.

Remark. As in the proof of (3.3), we may show that (5.2) is equivalent to the condition

$$p\Phi(e) \leqslant \sum_{\nu=1}^{n} \Phi(e_\nu) \qquad \text{whenever } pe = \sum \pm e_\nu. \tag{5.4}$$

THEOREM 4. *A function $\Phi(e)$ on S admits a representation*

$$\Phi(e) = \sup_{\phi \in C} \phi(e), \qquad \phi(e) \geqslant 0, \tag{5.5}$$

if and only if $\Phi(e)$ is increasing and multiply subadditive.

Proof. The necessity of the conditions is obvious. Conversely, let $\Phi(e)$ be increasing and multiply subadditive, we show that (5.2) holds. By the Remark, it is sufficient to prove (5.4). But if $pe = \sum \pm e_\nu$ then the e_ν cover e at least p times (see §2) and therefore, by §3, we obtain (5.4) for the function $\Phi(e)$. As in Theorem 3, (5.3) gives an extension of $\Phi(e)$ onto X satisfying (i)–(iv). Also (v) is satisfied; hence our result follows from Theorem 2.

6. Special classes of multiply subadditive functions. Examples of multiply subadditive functions may be obtained by considering

$$\Phi(e) = F(\psi(e)), \tag{6.1}$$

where $\psi(e)$ is a fixed positive additive function on S and $F(u)$ a function of the real variable $u \geqslant 0$.

We shall assume that S is *ψ-nonatomic*, that is, if $\psi(e) = \delta$ for some $e \in S$ and $0 \leqslant \delta_1 \leqslant \delta$, there is an $e_1 \subset e$ with $\psi(e_1) = \delta_1$. Clearly, with this condition, Φ is increasing if and only if F is increasing. Moreover, we have

THEOREM 5. *A function* (6.1) *with an increasing* F, $F(0) = 0$ *is multiply subadditive on a* ψ*-nonatomic Boolean ring* S *if and only if* F *has the property*

(6.2) $kF(\delta) \leqslant F(k\delta)$ *for* $0 \leqslant k \leqslant 1$ *and all values* $\delta = \psi(e)$, $e \in S$.

Proof. If $pe = \sum_{\nu=1}^{n} e_\nu$, then $e_\nu \subset e$, and putting $\delta_\nu = \psi(e_\nu)$, $\delta = \psi(e)$, we see that $0 \leqslant \delta_\nu \leqslant \delta$, $p\delta = \sum \delta_\nu$. If (6.2) holds and $\Phi(e)$ is defined by (6.1), we have, therefore, for $\delta > 0$,

$$p\Phi(e) = pF(\delta) = \sum \frac{\delta_\nu}{\delta} F(\delta) \leqslant \sum F(\delta_\nu) = \sum \Phi(e_\nu).$$

For $\delta = 0$ this inequality holds since $F(0) = 0$, so that $\Phi(e)$ is multiply subadditive.

Conversely, suppose that Φ has this property and that $\psi(e) = \delta$ for some $e \in S$; further, let $0 \leqslant k' = p/n \leqslant 1$ be a rational number and p, n be relatively prime. We decompose e into a disjoint union $e = \bigcup_{j=1}^{n} \bar{e}_j$ of elements $\bar{e}_j$ with $\psi(\bar{e}_j) = \delta/n$. For any integer $1 \leqslant i \leqslant pn$ let $\bar{e}_i = \bar{e}_j$, where j is the residue of i modulo n in the interval $1 \leqslant j \leqslant n$. Then

$$e_\nu = \bigcup_{(\nu-1)p < i \leqslant \nu p} \bar{e}_i$$

is a disjoint union and the e_ν cover e exactly p times. Moreover, $\psi(e_\nu) = p\delta/n = k'\delta$. Therefore,

$$pF(\delta) = p\Phi(e) \leqslant \sum \Phi(e_\nu) = \sum_{\nu=1}^{n} F(k'\delta) = nF(k'\delta),$$

or

$$k'F(\delta) \leqslant F(k'\delta).$$

If now k is a real number $0 \leqslant k \leqslant 1$, we take an increasing sequence of rationals $k'_n \to k$ and deduce $k'_n F(\delta) \leqslant F(k'_n \delta) \leqslant F(k\delta)$, which gives (6.2).

A function $F(u)$ satisfying (6.2) is easily seen to be continuous. Conversely, any positive, continuous, and *concave* function $F(u)$ satisfies (6.2). For it is known that F with $F(0) = 0$ has these properties if and only if

$$F(u) = \int_0^u f(x)dx, \tag{6.3}$$

f positive and decreasing, and this implies (6.2). There are functions of the type (6.1) which are subadditive, but not multiply subadditive. Let S be the Boolean algebra of measurable sets $e \subset (0,1)$ and $\psi(e)$ be the Lebesgue measure of the set $e \subset (0,1)$. Set $F(u) = \frac{3}{2}u$ in $(0, \frac{1}{3})$, $F(u) = \frac{1}{2}$ in $(\frac{1}{3}, \frac{2}{3})$, and $F(u) = \frac{3}{2}u - \frac{1}{2}$ in $(\frac{2}{3}, 1)$. Then the function (6.1) is subadditive because $F(u)$ has the property $F(u_1 + u_2) \leqslant F(u_1) + F(u_2)$. However, condition (6.2) is not satisfied, for $\frac{2}{3} = \frac{2}{3} F(1) > F(\frac{2}{3}) = \frac{1}{2}$.

We can also describe functions of type (6.1) by means of their representations. Assume for simplicity that $\Phi(e) = me$ is the Lebesgue measure of a measurable set $e \subset (0, 1)$. Let T denote one-to-one measure-preserving transformations of

(0, 1) into itself, so that $e' = T(e)$ has the same measure as e for any measurable e. Then we have:

(6.4) *An increasing multiply subadditive function* $\Phi(e)$ *is of the form* $\Phi(e) = F(me)$ *if and only if* Φ *has a representation*

$$\Phi(e) = \sup_{\phi \epsilon C} \phi(e), \tag{6.5}$$

where the class C contains with any $\phi(e)$ *also any function* $\phi(T(e))$.

If Φ has a representation of this kind, $\Phi(e)$ depends only on me, since, for any two sets e, e' with $me = me'$, there is a T with $e' = T(e)$. Therefore, $\Phi(e)$ is of the form $F(me)$. On the other hand, if a multiply subadditive and increasing function (6.5) depends only on me, we may replace C by the class C_1 of all additive functions $\phi(T(e))$, $\phi \in C$, T arbitrary, and have again

$$\Phi(e) = \sup_{\phi \epsilon C_1} \phi(e).$$

A special case of the above class is described as follows. Let S be as before; we define the rearrangement of a set-function

$$\phi(e) = \int_e g(x)dx, \qquad e \in S$$

to be any function

$$\bar{\phi}(e) = \int_e \bar{g}dx,$$

where $\bar{g}(x)$ is a rearrangement of $g(x)$ (for rearrangements of a point-function see [**4**, p. 276]).

(6.6) *In order that* $\Phi(e)$ *be of the form* $\Phi(e) = \sup_C \phi(e)$, *where C is the class of all rearrangements of a single, absolutely continuous positive function* $\phi_0(e)$, *it is necessary and sufficient that* $\Phi(e) = F(me)$ *where* $F(u)$ *is continuous, increasing and concave.*

If $\Phi(e) = \sup \phi(e)$ with the stated specification, and

$$\phi_0(e) = \int_e gdx, \qquad g \geqslant 0,$$

then we have

$$\Phi(e) = \int_0^{me} g^*(x)dx,$$

where g^* is the decreasing rearrangement of g. Thus $\Phi(e) = F(me)$, where

$$F(u) = \int_0^u g^* dx$$

is continuous, increasing and concave. Conversely, if $\Phi(e) = F(me)$ and

$$F(u) = \int_0^u gdx$$

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with an integrable, positive and decreasing g, then $\Phi(e) = \sup \phi(e)$, where $\phi(e)$ are all rearrangements of

$$\phi_0(e) = \int_0^{me} g dx.$$

We finally indicate a generalization of the Hahn decomposition theorem for subadditive functions. Let S be a Boolean σ-ring with zero element [**2**] and $\Phi(e)$ a *subadditive* function on S (compare §1). An element $e \in S$ is called Φ-positive, Φ-negative, or Φ-zero if $\Phi(e') \geqslant 0$, $\Phi(e') \leqslant 0$, or $\Phi(e') = 0$, respectively, for each $e' \subset e$, $e' \in S$. Then the following statement holds:

(6.7) *If a bounded subadditive function $\Phi(e)$ on S has the property*

$$\lim_{n\to\infty} \Phi(e_n) = 0, \qquad e_1 \supset e_2 \supset \ldots, \bigcap e_n = 0, \tag{6.8}$$

and takes values of different sign, then there are disjoint elements e^-, e_α^+, $\alpha \in A$ of S such that e^- is Φ-negative, each e_α^+ is Φ-positive, $\Phi(e_\alpha^+) > 0$, and each $e \in S$ disjoint with all e^-, e_α^+ is Φ-zero.

The proof is similar to the usual proof of Hahn's theorem [**3**, p.121], but requires transfinite induction for Φ-positive elements.

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