SOME GEOMETRIC EXTREMAL PROBLEMS *

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1. Background and notation

In a recent study of generalized transfinite diameters [4, 5] some geometric extremal problems were encountered. These form the subject matter of this note.

A generalized transfinite diameter is based on an averaging process defined by a strictly monotone and continuous function $F(u)$, $0 < u$. If X is a complete metric space, E a closed bounded set in X, let $P_1, P_2, \ldots, P_n$ be n points of E and set $\delta_{jk} = d(P_j, P_k)$. The F-average of the $N = \frac{1}{2}n(n-1)$ numbers δ_{jk} is defined by

$$F(a) = \frac{1}{N} \sum_{1 \leq j < k \leq n} F(\delta_{jk}). \tag{1}$$

The first problem is to maximize this average. If for a given n the supremum of a is denoted by $F\text{-}\delta_n(E)$ then the sequence $\{F\text{-}\delta_n(E)\}$ is decreasing. Its limit $F\text{-}\delta_0(E)$ is by definition the transfinite F-diameter of E. The case $F(u) = u^p$ will figure prominently in the following. Here we write $\delta_0^{(p)}(E)$ instead.

In the following we restrict ourselves essentially to inner product spaces such as m-dimensional Euclidean space R^m and Hilbert space H. The set E will be the unit ball of such a space. We write U^m in R^m and U^H in H.

2. The basic identity

A straightforward computation shows that for any n vectors X_j in an inner product space

$$\sum_{1 \leq j < k \leq n} ||X_j - X_k||^2 + ||\sum_{j=1}^{n} X_j||^2 = n \sum_{j=1}^{n} ||X_j||^2. \tag{2}$$

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The particular case $n = 2$ with

$$||X_1+X_2||^2+||X_1-X_2||^2 = 2(||X_1||^2+||X_2||^2) \tag{3}$$

is very well known and is often used as a characterization of an inner product space.

For the case of real or complex numbers, the identity has been known for some thirty odd years and has been used in the geometry of numbers by authors such as R. Remak, J. G. van der Corput, and G. Schaake [1]. I am indebted to Professor Kurt Mahler for this information. For the case of vectors in R^m priority appears to be due to L. Fejes Thót [2] who, however, gives credit to E. Makai. Closely related work was done by J. Schopp [6]. Some of the results given below (especially Theorems 1 and 2 for $p = 1$) are included in the work of these authors. I am indebted to Professor A. Rényi for this information.

3. The mean square case

From the identity (2) we obtain

THEOREM 1. *If the X's are unit vectors in an inner product space, then*

$$\sum_{1\leq j<k\leq n} ||X_j-X_k||^2 \leqq n^2 \tag{4}$$

with equality if and only if $\sum X_j = 0$.

Let us now take $F(u) = u^2$ in formula (1). Using (4) we see that

$$\max \left(\frac{1}{N}\sum \delta_{jk}^2\right)^{\frac{1}{2}} = \sqrt{2}\sqrt{\frac{n}{n-1}}$$

and that the maximum value is attained for any choice of points P_j, $j = 1, 2, \ldots, n$, on the unit sphere under consideration such that the corresponding unit vectors $X_j = \overrightarrow{OP_j}$ satisfy $\sum_1^n X_j = 0$. Hence

$$\delta_n^{(2)}(U^m) = \delta_n^{(2)}(U^H) = \sqrt{2}\sqrt{\frac{n}{n-1}}, \tag{5}$$

and, letting $n \to \infty$,

$$\delta_0^{(2)}(U^m) = \delta_0^{(2)}(U^H) = \sqrt{2}. \tag{6}$$

Since for any admissible function F we have

$$\sqrt{2} \leqq F\text{-}\delta_0(U^H) \leqq 2, \tag{7}$$

we see that the minimum value $\sqrt{2}$ is attained for $F(u) = u^2$.

4. The pth power case, $0 < p < 2$.

Here we use Hölder's inequality to get

$$\sum_{j<k} \delta_{jk}^{p} \leqq N^{1-p/2} \big(\sum_{j<k} \delta_{jk}^{2} \big)^{p/2} \leqq N^{1-p/2} n^{p} \tag{8}$$

with equality in the first place if and only if all δ_{jk} are equal and in the second place if and only if $\sum_1^n X_j = \mathbf{0}$. For each n the maximizing configuration is unique up to a rotation. This gives

THEOREM 2. *For $0 < p < 2$ the p-th power mean of the lengths of the edges of an $(n+1)$-simplex inscribed in U^n is an absolute maximum if the simplex is regular.*

Formula (8) gives upon letting $n \to \infty$

$$\delta_0^{(p)}(U^H) = \sqrt{2} \tag{9}$$

but gives no information about U^m except the trivial

$$\delta_0^{(p)}(U^m) \leqq \sqrt{2}.$$

5. The pth power case, $2 < p$

Here the discussion is based on the observation that $u^p \leqq u^2$ for $0 \leqq u \leqq 1$ with equality if and only if $u = 0$ or 1. This leads to the following sequence of inequalities

$$\begin{aligned}\sum_{j<k} \delta_{jk}^{p} &= 2^{p} \sum_{j<k} (\tfrac{1}{2}\delta_{jk})^{p} \leqq 2^{p} \sum_{j<k} (\tfrac{1}{2}\delta_{jk})^{2} \\ &= 2^{p-2} \sum_{j<k} \delta_{jk}^{2} \leqq 2^{p-2} n^{2}.\end{aligned} \tag{10}$$

Here equality holds in the first place if and only if all δ_{jk} are 0 or 2 and in the second place if and only if $\sum_1^n X_j = \mathbf{0}$. If n is even, $n = 2\nu$, both conditions can be satisfied by choosing an arbitrary unit vector X and setting $X_j = X$ for $1 \leqq j \leqq \nu$, and $X_j = -X$ for $\nu+1 \leqq j \leqq 2\nu$. It follows that

$$\delta_{2\nu}^{(p)}(U^m) = \delta_{2\nu}^{(p)}(U^H) = 2^{1-1/p} \left(\frac{2\nu}{2\nu-1} \right)^{1/p}.$$

Hence

$$\delta_0^{(p)}(U^m) = \delta_0^{(p)}(U^H) = 2^{1-1/p},\ 2 < p. \tag{11}$$

The last result has a bearing on the problem of finding $\delta_0^{(p)}(U_p)$ where U_p is the unit ball in the space (l_p) of sequences (ξ_j) such that

$$\|x\| = \big(\sum |\xi_j|^p\big)^{1/p} < \infty.$$

If in the preceding argument we choose $X = (1, 0, 0, \ldots)$, then this is a unit vector in (l_p) and we see that

$$\delta_{2\nu}^{(p)}(U_p) \geqq 2^{1-1/p}\left(\frac{2\nu}{2\nu-1}\right)^{1/p},$$

so that

$$\delta_0^{(p)}(U_p) \geqq 2^{1-1/p},\ p > 2. \tag{12}$$

Here it is not unlikely that the sign of equality holds, but the argument does not show it.

6. Convex F-functions

The results obtained above hold also for certain classes of convex functions $F(u)$.

THEOREM 3. *If $F(u)$ is decreasing and strictly convex then*

$$F\text{-}\delta_0(U^H) = \sqrt{2}. \tag{13}$$

Moreover, the analogue of Theorem 2 holds for such averages.

PROOF. We have

$$F(a) = \frac{1}{N}\sum_{j<k} F(\delta_{jk}) \geqq F\left(\frac{1}{N}\sum \delta_{jk}\right) \geqq F\left(\sqrt{2}\sqrt{\frac{n}{n-1}}\right),$$

so that

$$a \leqq \sqrt{2}\sqrt{\frac{n}{n-1}}$$

with equality if and only if $\sum_1^n X_j = 0$ and δ_{jk} are equal. This means that for each n the maximizing configuration is given by the regular n-simplex which is unique up to a rotation. Passing to the limit with n we get formula (13). This completes the proof.

The same type of proof gives

THEOREM 4. *If $F(u)$ is increasing and strictly concave, then formula* (13) *holds as well as the analogue of Theorem* 2.

This includes the case $F(u) = u^p$, $0 < p < 1$, as well as $F(u) = \log u$ which leads to the geometric mean. Professor Basil Rennie of the RAAF Academy has called my attention to the fact that the theorem remains valid if we replace $F(u)$ by $F(\sqrt{u})$. It then includes $F(u) = u^p$ for $0 < p < 2$. The proof of the extension is immediate.

7. Other cases

We have a less favorable situation if $F(u)$ is increasing and strictly concave. The special case $F(u) = u^p$, $2 < p$, indicates that the transfinite F-diameter of U^H depends effectively on F. In the general case all we can get are inequalities.

THEOREM 5. *If F_1, F_2, F are increasing and strictly convex and if for* $0 \leqq u \leqq 2$

$$F_1(u) \leqq F(u) \leqq F_2(u),$$

then

$$F_1\text{-}\delta_0(U^H) \leqq F\text{-}\delta_0(U^H) \leqq F_2\text{-}\delta_0(U^H). \tag{14}$$

Since the graph of $v = F(u)$ lies below the chord joining $[0, F(0)]$ with $[2, F(2)]$ we get

$$\begin{aligned}\frac{1}{N}\sum_{j<k} F(\delta_{jk}) &\leqq F(0)+\tfrac{1}{2}[F(2)-F(0)]\,\frac{1}{N}\sum_{j<k}\delta_{jk}\\ &\leqq F(0)+\tfrac{1}{2}[F(2)-F(0)]\left(2\,\frac{n}{n-1}\right)^{\frac{1}{2}}\\ &\to F(0)+\tfrac{1}{2}\sqrt{2}[F(2)-F(0)]\end{aligned}$$

whence

$$F\text{-}\delta_0(U^H) \leqq F^{-1}\{(1-\tfrac{1}{2}\sqrt{2})F(0)+\tfrac{1}{2}\sqrt{2}F(2)\}. \tag{15}$$

Using the convexity of F we see that the right member exceeds $\sqrt{2}$ and since F is increasing we have

$$F\text{-}\delta_0(U^H) < 2 \tag{16}$$

for this class of means.

In the opposite direction we observe that

$$F\text{-}\delta_0(U^H) > \sqrt{2} \tag{17}$$

if $F(\sqrt{u})$ is increasing and strictly convex in $(0, 4)$. This implies that

$$F(\sqrt{2}) < \tfrac{1}{2}[F(0)+F(2)] \tag{18}$$

and this is the only implication of the convexity that we need. To establish (17) we proceed as in the proof of formula (11). We take $n = 2\nu$ and choose ν unit vectors X and ν unit vectors $-X$. In formula (1) we have $\nu^2-\nu$ terms $F(0)$ and ν^2 terms $F(2)$ while $N = \nu(2\nu-1)$. It follows that $F(a)$ is at least equal to

$$\frac{\nu-1}{2\nu-1}\,F(0)+\frac{\nu}{2\nu-1}\,F(2) \to \tfrac{1}{2}[F(0)+F(2)] > F(\sqrt{2})$$

and this implies (17).

8. Remarks on other polyhedra

It was found in Section 4 that the extremal configuration in U^H for a given n is produced by n unit vectors whose sum is 0 and whose end points are equidistant from each other. This is a regular n-simplex which can be inscribed in U^{n-1}. In particular, for $n=4$ we are dealing with extremal properties of a regular tetrahedron: Of all 4-simplexes inscribed in U^3 the regular tetrahedron shows the largest pth power average for the lengths of its sides. This holds for $p < 2$. For $p = 2$ the regular tetrahedron gives $\sum \delta_{jk}^2$ its maximum value 16 but now there are infinitely many 4-simplexes which give the same square sum.

This raises the question whether or not other regular polyhedra have extremal properties with respect to sums of squares of lengths of the sides. In the case of the octahedron the square sum is 24 and for the cube 16 if the circum radius is 1. Here it turns out that the regular polyhedra are far from maximizing the square sums. There exist degenerate "octahedra" with square sum 32 and degenerate "hexahedra" with square sum 48. The latter are of some interest and are obtained by the following construction. In an ordinary cube let the eight vertices be given by the unit vectors X_1 to X_8. Let the twelve edges be denoted by the symbols

$$\begin{array}{cccccc} (1,2), & (2,3), & (3,4), & (4,1), & (1,6), & (3,8) \\ (5,6), & (6,7), & (7,8), & (8,5), & (4,5), & (2,7). \end{array} \tag{19}$$

Here (j, k) is the line joining the endpoints of the vectors X_j and X_k. Here each side has the length $\frac{2}{3}\sqrt{3}$ and the square sum is $12 \cdot \frac{4}{3} = 16$. Let us now deform the cube by moving X_1, X_3, X_5, X_7 into coincidence, say with the vector $(0, 0, 1)$, and by moving X_2, X_4, X_6, X_8 into the antipodal position $(0, 0, -1)$. Let (j, k) still denote the "edges" of the new "hexahedron" where (j, k) runs over the list (19). Each of these edges will now have the length 2 and the square sum is 48. This shows that

$$\max \sum \delta_{jk}^2 \geqq 48. \tag{20}$$

Moreover

$$\max \sum \delta_{jk} \geqq 24 \tag{21}$$

while the regular cube would give only $8\sqrt{3}$. Thus the regular cube does not even maximize the sum of the lengths of the edges of hexahedra inscribed in a fixed sphere. I have not examined corresponding properties of dodecahedra and icosahedra.

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