Polar Loci.

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This paper contains two parts :—

I. *An endeavour to remove the present confusion in polar diagrams.*

II. *On the curves derived from a given curve by increasing or diminishing the vectorial angle in a constant ratio.*

I.

1. In order to cover the whole plane of xy, x and y must both vary from $-\infty$ to $+\infty$; but the same plane is covered while r varies from $-\infty$ to $+\infty$, and θ through any range π. Hence when we allow θ any larger amplitude than this, we create confusion. The curve $r=a\theta$ appears to give an infinite number of values of r for each value of θ, and the curve $\frac{l}{r}=1+e\cos\theta$ seems to give two, while the equation in each case gives only one; while for the curve $r^2=a^2\cos 3\theta$, the region between $\frac{5\pi}{6}$ and π appears, on considering $\frac{5\pi}{6}<\theta<\pi$, to be unoccupied, and on considering $-\frac{\pi}{6}<\theta<0$, to be occupied.

We can remove the confusion by two simple suggestions.

(i) Confining ourselves for the moment to positive values of r, conceive an infinite number of planes, one above another, and each slit up from 0 to ∞ along the initial line. Let the under lip of each slit be joined along its length to the upper lip on the sheet just above; this produces a surface on which θ can vary from indefinitely large negative values on the lowest planes, to indefinitely large positive values on the highest.

(ii) Now considering any single plane, conceive that the radius vector, in passing through the origin, passes to the *under side of the plane*, and so is drawn *on the back* of the paper. Thus the two sides of each sheet are utilised (the upper for positive, the under for negative, values of r), and we have a *unifacial* surface (see

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Forsyth T. F., § 165) as a proper field for the representation of the locus $f(r, \theta) = 0$.

2. Now each plane (both sides included) corresponds to variation of θ through a range 2π. Hence if $f(r, \theta)$ admits the period 2π with respect to θ, the curves on the different planes will be identical. This will be the case, not only for all curves algebraic in x and y, but also for many types of transcendental curves.

Or $f(r, \theta)$ may admit some period $2m\pi$, where m is an integer greater than unity. In this case the curve will begin to repeat itself after having described m planes of the surface.

Again, the period of $f(r, \theta)$ may be an *irrational* multiple of 2π; in which case the curve will repeat itself, but identical portions will not come under one another.

Lastly, $f(r, \theta)$ may not be periodic in θ at all.

3. In general, there will be portions of the locus on both sides of each plane; those on the upper side being described by positive, and those on the under side by negative, radii vectores. Thus in the hyperbola $\frac{l}{r} = 1 + e\cos\theta$, the more remote branch is described altogether by negative radii vectores, and is therefore to be conceived as drawn on the *back* of our paper. We may draw it in as a *dotted* line to remind ourselves of this fact.

Again, the circle $r = a\sin\theta$ is described twice while θ varies through a range 2π; once with positive, and once with negative, radii vectores. It thus appears identically on both sides of the paper, and we may represent this by a dotted circle immediately inside a continuous one (Fig. 33). In each case when a curve passes through the origin, it changes to the other side of the paper, so that dotted and continuous lines are described alternately.

This mode of connection between the two sides of a plane is exactly what we would obtain from an hyperboloid of one sheet, with its generators drawn, if we were to flatten it into the plane of its principal elliptic section, and at the same time to contract that section into a pinhole. The surface would then consist of the two sides of the plane, and each generator would change to the other face of the plane in passing through the pinhole.

4. The curves $r = a\theta$, $r\theta = a$, appear in the ordinary diagram to have an indefinite number of double points. But our new convention enables us to discriminate.

(α) A *bona fide* double point, *i.e.*, one in which two branches of the locus actually meet on our surface, must satisfy the equations

$$f(r,\,\theta)=0,$$
$$\frac{\partial}{\partial r}f(r,\,\theta)=0.$$

(β) But a point which satisfies the equations

$$f(r,\,\theta)\qquad=0,$$
$$f(r,\,\theta+2n\pi)=0,$$

or the equations

$$f(r,\,\theta)\qquad=0,$$
$$f(-r,\,\theta+\overline{2n+1}\pi)=0,$$

where n is a positive or negative integer,

will in ordinary diagrams be mistaken for a double point. In the former case, the branches are on different planes of the surface; in the latter, they may be on the same or different planes, but are on opposite *sides* of the surface. We may call them *pseudo*-double points of the first and second kinds respectively. The apparent double points on the two spirals mentioned are *pseudo*-double points of the second kind.

II.

1. Consider the straight line $r\sin\theta=a$. If we diminish the vectorial angle of each point in the ratio $1:m$, we obtain a Cotes' Spiral $r\sin m\theta=a$; and if we *increase* the vectorial angles in the ratio $m:1$, we obtain the curve $r\sin\dfrac{\theta}{m}=a$. The same two processes applied to the circle $r=a\sin\theta$, the parabola $r=\dfrac{4a\cos\theta}{\sin^2\theta}$, the conic $\dfrac{l}{r}=1+e\cos\theta$, the folium of Descartes $r=\dfrac{3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}$, the cubical parabola $r^2=\dfrac{a^2\cos\theta}{\sin^3\theta}$, the semicubical $r=\dfrac{a\cos^2\theta}{\sin^3\theta}$, or any other known curve, will likewise produce in each case two families of curves, each family with notable characteristics. Curves of the first family in each case will consist of repetitions, symmetrically placed about the origin, of a narrow (open or closed) loop. The second family will be marked by wide overlapping loops, with an abundance of *pseudo*-double points.

Conversely, by altering the vectorial angles in a constant ratio, we are able to reduce many types of polar loci to simple algebraic curves.

2. An algebraic curve is, from the polar point of view, a locus with period 2π in θ. It will in general (applying the foregoing theory) require both sides of a plane for its representation, and, when put upon our surface, will be identically repeated on each of our infinite number of sheets. When we alter the vectorial angles in a given ratio, we are simply opening or closing our surface like a fan, and the various layers of our locus become separated. When the ratio in which the angles have been altered can be expressed as the ratio of one whole number to another, the curve will repeat itself after a definite number of sheets.

Below are some of the simplest of the curves thus derived from the circle $r = a\sin\theta$.

A branch drawn in a continuous line is one which occupies the *upper* side of its sheet, and corresponds to *positive* radii vectores; while a *dotted* line denotes a branch on the *under* side of its sheet, and corresponding to *negative* radii vectores. When a branch is described in both of these ways, the continuous and dotted lines are drawn alongside one another.

It will be noticed that a curve, on passing through the origin, always *changes sides* on its sheet.

3. First, consider those produced by what we may call "contraction" of the surface. Their general equation is

$$r = a\sin m\theta$$

and we shall take m an integer. In all cases there are m loops above and m below; but since, when m is odd,

$$\sin m\theta = -\sin m(\theta + \overline{2n+1}\pi)$$

for all values of θ, each point on the curve satisfies the condition for a *pseudo*-double point of the second kind, so that each under loop will be covered by an upper; while, for m even, they will be found in separate regions. The enveloping circle $r = a$ is drawn in each case, and the loops are numbered in the order of their description.

Figures are drawn for $m = 1, 2, 3, 4$. (Figures 34, 35, 36, 37.)

4. The general equation of curves derived from the circle

$$r = a\sin\theta$$

by what we may call "extension" of the surface is

$$r = a\sin\frac{\theta}{m}.$$

As before, we consider m integral. The complete variation of r takes place while θ goes from 0 to $2m\pi$; there will thus be m sheets before the curve begins to repeat.

Figures are drawn for $m=2, 3, 4$. (Figures 38, 39, 40.) The following short notes indicate the order in which the different branches are described; and each branch is numbered in the figures according to the sheet in which it lies.

(i) $r=a\sin\frac{\theta}{2}$ (Fig. 38).

$0<\theta<2\pi$: continuous line OAO, upper surface of first sheet;
$2\pi<\theta<4\pi$: dotted line OBO, under surface of second sheet.

(ii) $r=a\sin\frac{\theta}{3}$ (Fig. 39).

$0<\theta<2\pi$: OAB continuous, first sheet;
$2\pi<\theta<3\pi$: BCO continuous, second sheet;
$3\pi<\theta<4\pi$: OCA dotted, second sheet;
$4\pi<\theta<6\pi$: ABO dotted, third sheet.

(iii) $r=a\sin\frac{\theta}{4}$ (Fig. 40).

$0<\theta<2\pi$: OAC continuous, first sheet;
$2\pi<\theta<4\pi$: CAO continuous, second sheet;
$4\pi<\theta<6\pi$: OBD dotted, third sheet;
$6\pi<\theta<8\pi$: DBO dotted, fourth sheet.

Similarly for higher values of m.

It is useful to note that

$$\tan\phi=\frac{rd\theta}{dr}=m\tan\frac{\theta}{m}.$$

Since, for m odd, $\sin\frac{\theta}{m}=-\sin\frac{\theta+\overline{2n+1}\pi}{m}$ for $n=\frac{m-1}{2}$ and all values of θ, the same curve, just as in § 3, will be described by negative radii vectores as by positive; but not so for m even. This is illustrated by the examples drawn.

The point to note is, that those curves, becoming more and more involved as m increases, are simple "extensions" of the circle; and, finally, that without the conception of a many-sheeted surface, their diagrams would be a mass of confusion.

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