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SURPRISING RELATIONSHIPS AMONG UNITARY REFLECTION GROUPS

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1. Introduction

The transpositions that generate a symmetric group can be represented as real reflections: symmetry operations of a regular simplex. Analogous unitary reflections serve to generate other factor groups of the braid group; they are symmetry operations of regular complex polytopes. Certain relationships among these groups have, as geometric counterparts, unexpected plane sections of the polytopes, beginning with the square sections of the regular tetrahedron. In Section 6, 5-dimensional coordinates will be used to exhibit pentagonal sections of the 4-dimensional regular simplex. The most spectacular instance of such "equatorial" sections occurs in the case of the Witting polytope in complex 4-space, so exquisitely drawn by Peter McMullen for the frontispiece of *Regular Complex Polytopes* [**6**]. This has a plane section 3{5}3 which appears thare as Fig. 4.8B on page 48. Shephard [**9**, p. 92] called it 3(360)3. Its 120 vertices will be seen to be situated "inside" 120 of the 2160 faces 3{3}3 of the Witting polytope. These "faces" are self-inscribed octagons [7, p. 290].

2. The symmetric group $\mathfrak{S}_q$ and its dihedral subgroup $\mathfrak{D}_q$

The symmetric group $\mathfrak{S}_q$ of order $q!$ is obviously generated by the $q-1$ transpositions

$$R_1=(1\ \ 2),\ R_2=(2\ \ 3),\ \ldots,\ R_{q-1}=(q-1\ \ q),$$

in terms of which it has the presentation

$$R_1^2=1, \tag{2.1}$$

$$R_vR_{v+1}R_v=R_{v+1}R_vR_{v+1} \qquad (1\leqq v\leqq q-2), \tag{2.2}$$

$$R_\mu R_v=R_vR_\mu \qquad (\mu\leqq v-2) \tag{2.3}$$

[**8**, p. 64]. By (2.2), the R_v are mutually conjugate, so it is unnecessary to specify $R_v^2=1$, from which it follows that the products

$$P=R_1R_3R_5\ldots \quad \text{and} \quad Q=R_2R_4R_6\ldots \tag{2.4}$$

are likewise involutory. The permutations

$$(1\,2)\cdot(2\,3)=(1\,3\,2),$$

$$(1\,2)(3\,4)\cdot(2\,3)=(1\,3\,4\,2),$$

$$(1\,2)(3\,4)\cdot(2\,3)(4\,5)=(1\,3\,5\,4\,2),$$

$$(1\,2)(3\,4)(5\,6)\cdot(2\,3)(4\,5)=(1\,3\,5\,6\,4\,2)$$

serve to initiate a pattern which shows that

$$(PQ)^q=1. \tag{2.5}$$

Thus the elements P and Q of $\mathfrak{S}_q$ generate a subgroup $\mathfrak{D}_q$ of order $2q$.

More generally, in any finite reflection group ("Weyl group"), the generators can be separated into two sets of mutually commutative reflections whose products generate a dihedral subgroup $\mathfrak{D}_h$, where h is the appropriate "Coxeter number" [**5**, pp. 225–234].

3. The analogous subgroup of the braid group

Since $P^2=Q^2=1$, (2.5) can be replaced by

$$PQP\ldots=QPQ\ldots \tag{3.1}$$

with q factors on each side of the equation. Without making use of the permutations, we could have derived (3.1) directly from (2.2) and (2.3). For instance, when $q=3$,

$$PQP=R_1R_2R_1=R_2R_1R_2=QPQ;$$

when $q=4$,

$$\begin{aligned}(PQ)^2&=(R_1R_3R_2)^2\\&=R_3R_1R_2R_1R_3R_2\\&=R_3R_2R_1R_2R_3R_2\\&=R_3R_2R_1R_3R_2R_3\\&=R_3R_2R_3R_1R_2R_3\\&=R_2R_3R_2R_1R_2R_3\\&=R_2R_3R_1R_2R_1R_3\\&=(R_2R_1R_3)^2=(QP)^2;\end{aligned}$$

and when $q=5$,

$$\begin{aligned}
PQPQP &= R_1R_3R_2R_4R_1R_3R_2R_4R_1R_3 \\
&= R_3R_1R_2R_1R_4R_3R_4R_2R_3R_1 \\
&= R_3R_1R_2R_1R_3R_4R_3R_2R_3R_1 \\
&= R_3R_2R_1R_2R_3R_4R_2R_3R_2R_1 \\
&= R_3R_2R_1R_2R_3R_2R_4R_3R_2R_1 \\
&= R_3R_2R_1R_3R_2R_3R_4R_3R_2R_1 \\
&= R_3R_2R_3R_1R_2R_4R_3R_4R_2R_1 \\
&= R_2R_3R_2R_1R_2R_4R_3R_4R_2R_1 \\
&= R_2R_3R_1R_2R_1R_4R_3R_4R_2R_1 \\
&= R_2R_1R_3R_4R_2R_1R_3R_4R_2R_1 \\
&= R_2R_1R_3R_4R_2R_3R_1R_2R_1R_4 \\
&= R_2R_1R_3R_4R_2R_3R_2R_1R_2R_4 \\
&= R_2R_1R_3R_4R_3R_2R_3R_1R_2R_4 \\
&= R_2R_1R_4R_3R_4R_2R_3R_1R_2R_4 \\
&= R_2R_4R_1R_3R_2R_4R_1R_3R_2R_4 = QPQPQ.
\end{aligned}$$

It is significant that this procedure is independent of (2.1). Hence,

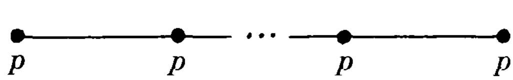
In the infinite braid group defined by (2.2) *and* (2.3) [**1**, p. 245; **8**, p. 62], *the elements* (2.4) *generate a subgroup which has the presentation* (3.1).

4. Factor groups of the braid group

Combining (2.2) and (2.3) with a new relation

$$R_1^p = 1, \tag{4.1}$$

where p may be greater than 2, we obtain a factor group which has been denoted by

$p[3]p[3]\dots p[3]p$ or 

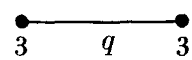
[**3**, p. 114; **6**, pp. 117, 152]. The graphical symbol contains $q-1$ dots, one for each generator R_ν. The group so presented is finite only if

$$(p-2)(q-2) \leqq 3,$$

that is, only if $\{p, q\}$ is a regular polyhedron or spherical tessellation. In the case when $p=3$ and $q \leqq 5$, the order is

$$(6/6-q)^{q-1} q!,$$

and the elements P and Q generate a subgroup

3[q]3 or 

of order $72q/(6-q)^2$ [**6**, p. 95]. Thus, in the group 3[3]3[3]3 or order $3^3 4! = 648$ [**6**, pp. 119–124] the elements R_1R_3 and R_2 generate a subgroup 3[4]3 of order 72, which is the direct product of the binary tetrahedral group and the group of order 3 [**8**, p. 77; **6**, p. 100]; in the group 3[3]3[3]3[3]3 of order $6^4 5! = 155520$ [**6**, pp. 132–134], R_1R_3 and R_2R_4 generate a subgroup 3[5]3 of order 360, which is the direct product of the binary icosahedral group and the group of order 3 [**8**, pp. 71, 78; **6**, p. 102]; and in the infinite group 3[3]3[3]3[3]3[3]3 [**6**, pp. 135–137], $R_1R_3R_5$ and R_2R_4 generate an infinite subgroup 3[6]3 [**6**, p. 111].

Similarly with $p=q=4$, in the infinite group 4[3]4[3]4 [**6**, p. 144], $R_1R_3R_5$ and $R_2R_4R_6$ generate an infinite subgroup 4[4]4.

5. The Hessian polyhedron

In unitary 3-space, the group 3[3]3[3]3 of order 648 is generated by reflections of period of 3 in the three mirrors

$$u_1 = 0, \quad u_1 + u_2 + u_3 = 0, \quad u_3 = 0.$$

When expressed as matrices, these unitary reflections are

$$R_1 = \begin{bmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_2 = -\frac{i\omega^2}{\sqrt{3}} \begin{bmatrix} \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{bmatrix},$$

where $\omega = (-1 + i\sqrt{3})/2$. Hence the subgroup 3[4]3 is generated by

$$P = \begin{bmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{bmatrix}, \quad Q = -\frac{i\omega^2}{\sqrt{3}} \begin{bmatrix} \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \end{bmatrix},$$

which satisfy the presentation $P^3 = Q^3 = 1$, $(PQ)^2 = (QP)^2$. By Wythoff's construction [**6**,

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p. 104], we can obtain the 24 vertices of a regular complex polygon 3{4}3 [**6**, p. 47] lying in the plane $u_1=u_3$ (orthogonal to the plane $u_1+u_2+u_3=0$), as the orbit of a suitable point, namely, either a point such as $(1,-2,1)$ on the mirror for $Q(=R_2)$ or a point such as $(0,i\sqrt{3},0)$ on the axis $u_1=u_3=0$ of the "rotation" P. Choosing the latter, we observe that

$$(0,i\sqrt{3},0)Q=(\omega^2,\omega,\omega^2),\quad (\omega^2,\omega,\omega^2)P=(1,\omega,1),$$

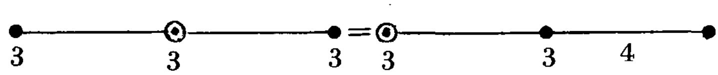
and so on. In fact, the $9+6+9$ vertices of 3{4}3 are

$$(\omega^\mu,\omega^\lambda,\omega^\mu),\quad (0,\pm i\omega^\lambda\sqrt{3},0),\quad (-\omega^\mu,-\omega^\lambda,-\omega^\mu).$$

The orbit of the same point $(0,i\sqrt{3},0)$ for the whole group 3[3]3[3]3 consists of the $27+18+27=72$ points

$$(\omega^\lambda,\omega^\mu,\omega^\nu),\quad (\pm i\omega^\lambda\sqrt{3},0,0)\text{ permuted,}\quad \text{and } (-\omega^\lambda,-\omega^\mu,-\omega^\nu)$$

which, being the vertices of the complex polyhedron



$=3\{3\}3\{4\}2,$

are also the edge-centres of 3{3}3{3}3 [**6**, p. 127]. Thus the occurence of 3[4]3 as a subgroup of 3[3]3[3]3 has, as its geometric counterpart:

The complex polygon 3{4}3 *arises as a plane section of the Hessian polyhedron* 3{3}3{3}3 [**2**, p. 469 (5.3)]. More precisely, it is the section by a plane through the centres of 24 of its 72 edges, such as the centre $(0,1,0)$ of the edge $(0,1,-\omega^\nu)$.

This is analogous to the square section of the regular tetrahedron $\{3,3\}=\alpha_3$, which is also an equatorial polygon or "van Oss polygon" of the octahedron $\left\{\begin{smallmatrix}3\\3\end{smallmatrix}\right\}=\beta_3=\{3,4\}$ [**6**, pp. 13, 141]. In preparation for what is to come, we may conveniently call this square the section of the tetrahedron α_3 by an *equatorial* plane. In fact, the geometric counterpart for the occurrence of $\mathfrak{D}_q$ as a subgroup of $\mathfrak{S}_q$ is the occurrence of the regular q-gon $\{q\}=2\{q\}2$ as the section of the regular simplex

$$\alpha_{q-1}=\{3,3,\ldots,3\}=2\{3\}2\ldots\{3\}2$$

by an "equatorial plane".

6. The regular 4-simplex and its pentagonal sections

Returning for a moment to the case when $q=4$, we may represent the 4 vertices of the regular tetrahedron α_3 in 4 dimensions by the points

$$(1,0,0,0),\quad (0,1,0,0),\quad (0,0,1,0),\quad (0,0,0,1)$$

in the 3-space $x_1+x_2+x_3+x_4=1$. The equatorial plane corresponding to this cyclic order is $x_1+x_3=x_2+x_4=\frac{1}{2}$, and the vertices of the square section are

$$(\tfrac{1}{2},\tfrac{1}{2},0,0),\quad (0,\tfrac{1}{2},\tfrac{1}{2},0),\quad (0,0,\tfrac{1}{2},\tfrac{1}{2}),\quad (\tfrac{1}{2},0,0,\tfrac{1}{2});$$

for instance, $(\frac{1}{2},\frac{1}{2},0,0)$ is the common point of the equatorial plane with the plane $x_1+x_2=1, x_3=x_4=0$ which contains the edge $(1,0,0,0)(0,1,0,0)$ of the tetrehedron.

Analogously in 5 dimensions, we may identify the 5 vertices *1*, *2*, *3*, *4*, *5* of the regular simplex α_4 with points

$$(1,0,0,0,0),\quad (0,1,0,0,0),\quad (0,0,1,0,0),\quad (0,0,0,1,0),\quad (0,0,0,0,1)$$

in the hyperplane $x_1+x_2+x_3+x_4+x_5=1$. An equatorial plane corresponding to this cyclic order might be given by 3 equations of the form

$$x_1-x_4=X(x_2-x_3),\quad x_2-x_5=X(x_3-x_4),\quad x_3-x_1=X(x_4-x_5),$$

provided these equations would imply

$$x_4-x_2=X(x_5-x_1),\quad x_5-x_3=X(x_1-x_2)$$

so as to complete the cyclic symmetry. This redundancy of the 5 equations is easily seen to require

$$X^2-X-1=0,$$

so that $X=\tau$ or $-\tau^{-1}$, where

$$\tau=(1+\sqrt{5})/2.$$

Thus the cyclic permutation of the 5 coordinates is a "double rotation" whose two invariant planes are

$$x_1-x_4=\tau(x_2-x_3),\quad x_2-x_5=\tau(x_3-x_4),\quad x_3-x_1=\tau(x_4-x_5) \tag{6.1}$$

and

$$x_2-x_3=\tau(x_4-x_1),\quad x_3-x_4=\tau(x_5-x_2),\quad x_4-x_5=\tau(x_1-x_3). \tag{6.2}$$

The face *2 3 4* of the simplex α_4 lies in the plane

$$x_2+x_3+x_4=1,\quad x_1=x_5=0$$

whose common point with (6.1) is

$$(0,\tau^{-1}/\sqrt{5},1/\sqrt{5},\tau^{-1}/\sqrt{5},0).$$

Permuting the 5 coordinates, we obtain the vertices of a regular pentagon in the plane (6.1), which is thus seen to be an equational section of α_4. Another such section is the completely orthogonal plane (6.2), yielding a congruent pentagon whose typical vertex

$$(\tau^{-1}/\sqrt{5}, 0, 1/\sqrt{5}, 0, \tau^{-1}/\sqrt{5})$$

is the common point of (6.2) with the plane

$$x_1 + x_3 + x_5 = 1, \quad x_2 = x_4 = 0$$

which carries the face *135*. Hence

For each subgroup $\mathfrak{D}_5$ of $\mathfrak{S}_5$, the regular simplex α_4 has two completely orthogonal equatorial planes, each providing a pentagonal section.

We observe that each vertex of either pentagon is situated in the appropriate triangular face in a position which is given by barycentric coordinates $(1, \tau, 1)$, so that it divides a median (from vertex to opposite side) in the ratio $2:\tau$. This ratio is, of course, maintained in Figure 1, which is the standard orthogonal projection [**5**, pp. 120, 245].

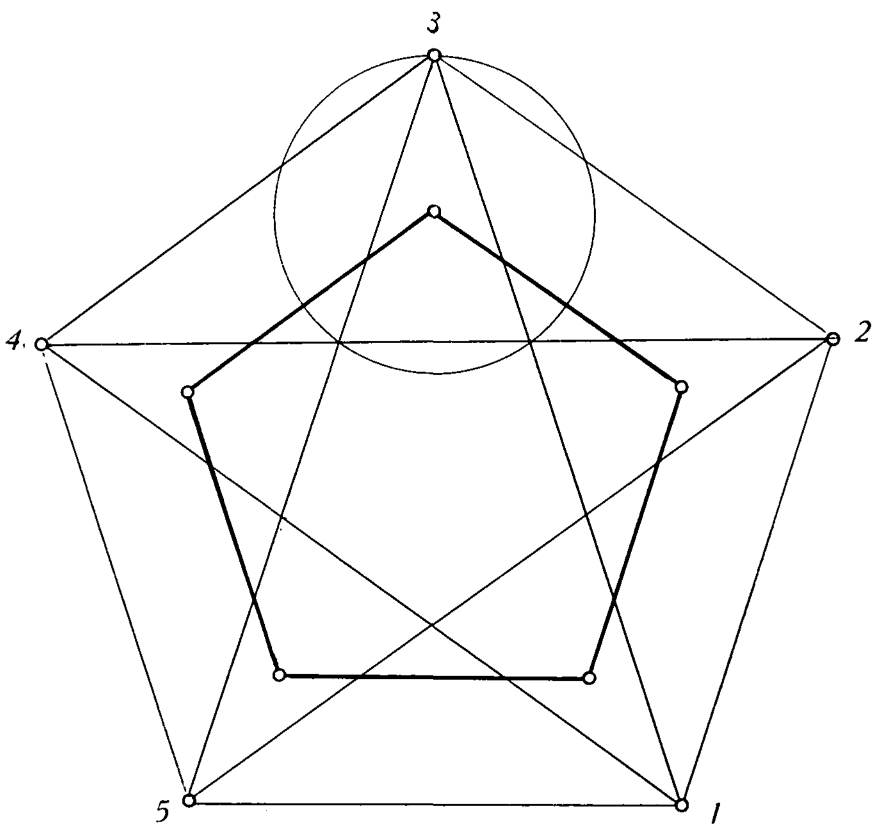


Figure 1. $\{3,3,3\}$ and its equatorial section $\{5\}=2\{5\}2$.

Here *1 2 3 4 5* appears as a regular pentagon with an inscribed pentagram *1 3 5 2 4*. Since the circumradius and inradius of a regular pentagon are in the ratio $2:\tau$, the relevant point in the isosceles triangle *1 3 5* coincides with the centre of the pentagon. In the isosceles triangle *2 3 4*, the point divides the median from *3* in the same ratio, enabling us to draw the section (in heavy lines) in Figure 1. Thus

In the standard projection of α_4, one of the two equatorial pentagons is foreshortened into a single point (the centre) while the other is represented faithfully.

7. The Witting polytope

Passing now from the simplex

$$\alpha_4 = \{3,3,3\} = 2\{3\}2\{3\}2\{3\}2,$$

with its section $\{5\} = 2\{5\}2$, to the Witting polytope

$$3\{3\}3\{3\}3\{3\}3,$$

we recall that, in unitary 4-space, the group 3[3]3[3]3[3]3, of order 155520, is generated by the reflections

$$R_1 = \begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_2 = -\frac{i\omega^2}{\sqrt{3}} \begin{bmatrix} \omega^2 & 1 & 1 & 0 \\ 1 & \omega^2 & 1 & 0 \\ 1 & 1 & \omega^2 & 0 \\ 0 & 0 & 0 & i\omega\sqrt{3} \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_4 = -\frac{i\omega^2}{\sqrt{3}} \begin{bmatrix} i\omega\sqrt{3} & 0 & 0 & 0 \\ 0 & \omega^2 & -1 & 1 \\ 0 & -1 & \omega^2 & -1 \\ 0 & 1 & -1 & \omega^2 \end{bmatrix}$$

[6, p. 132], which combine to form

$$P = R_1 R_3 = \begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = R_2 R_4 = -\frac{i\omega^2}{\sqrt{3}} \begin{bmatrix} \omega^2 & 1 & 1 & 0 \\ 1 & -\omega & 0 & 1 \\ 1 & 0 & -\omega & -1 \\ 0 & 1 & -1 & \omega^2 \end{bmatrix}.$$

These two "rotations" satisfy the presentation $P^3 = Q^3 = 1$, $PQPQP = QPQPQ$. Their axial planes are

$$u_1 = u_3 = 0 \quad \text{and} \quad u_1 + u_2 + u_3 = u_2 - u_3 + u_4 = 0.$$

Being real, the planes may be treated like lines with Plücker coordinates

$$(0,1,0;0,0,0) \quad \text{and } (-2,1,1;1,1,1)$$

in elliptic 3-space [4, pp. 88–92]. The two orthogonal traversals (or "common perpendiculars") of these lines may be found with the help of the remark that the condition for two lines

$$(p_{14}, p_{24}, p_{34}; p_{23}, p_{31}, p_{12}) \quad \text{and} \quad (q_{14}, q_{24}, q_{34}; q_{23}, q_{31}, q_{12})$$

to intersect orthogonally is

$$p_{14}q_{23}+p_{24}q_{31}+p_{34}q_{12}+p_{23}q_{14}+p_{31}q_{24}+p_{12}q_{34}$$

$$=p_{14}q_{14}+p_{24}q_{24}+p_{34}q_{34}+p_{23}q_{23}+p_{31}q_{31}+p_{12}q_{12}=0.$$

In the present case the two orthogonal transversals are thus seen to be

$$(1,0,\tau;1,0,-\tau^{-1}) \quad (1,0,-\tau^{-1};1,0,\tau).$$

In other words, the group generated by P and Q leaves invariant the two planes

$$\tau u_1-u_3=\tau u_2+u_4=0 \quad \text{and} \quad u_1+\tau u_3=u_2-\tau u_4=0.$$

These meet the axial planes of P and Q along lines joining the origin to the four points

$$(0,-1,0,\tau), \quad (1,-\tau^2,\tau,\tau^3), \quad (0,\tau,0,1), \quad (-\tau^3,\tau,\tau^2,1)$$

any one of which will yield the 120 vertices of a complex polygon 3{5}3 (see Figure 2) under the action of the group generated by P and Q.

Since $(1,-1,0,1)+\tau(0,-1,1,2)=(1,-\tau^2,\tau,\tau^3)$, the line joining the origin to $(1, -\tau^2,\tau,\tau^3)$ lies in the plane joining the origin to the vertex $(1,-1,0,1)$ and the centre $\frac{1}{2}(0,-1,1,2)$ of the face 3{3}3 whose 8 vertices are

$$(0,\omega,-\omega^2,1), \quad (0,\omega^2,-\omega,1), \quad (-\omega^\lambda,0,1,1), \quad (\omega^\lambda,-1,0,1)$$

[6, pp. 119 (12.35), 120 (Figure 12.3A), 132]. Thus the orbit of $(1,-\tau^2,\tau,\tau^3)$ for the group generated by P and Q consists of the 120 vertices of a 3{5}3 lying in the equatorial plane

$$\tau u_1-u_3=0, \quad \tau u_2+u_4=0;$$

and a typical vertex of the 3{5}3 is the point of intersection of this plane with the plane

$$u_3-u_2=u_4=1$$

which carries a face of the Witting polytope. In other words,

The section of 3{3}3{3}3{3}3 *by an equatorial plane is* 3{5}3.

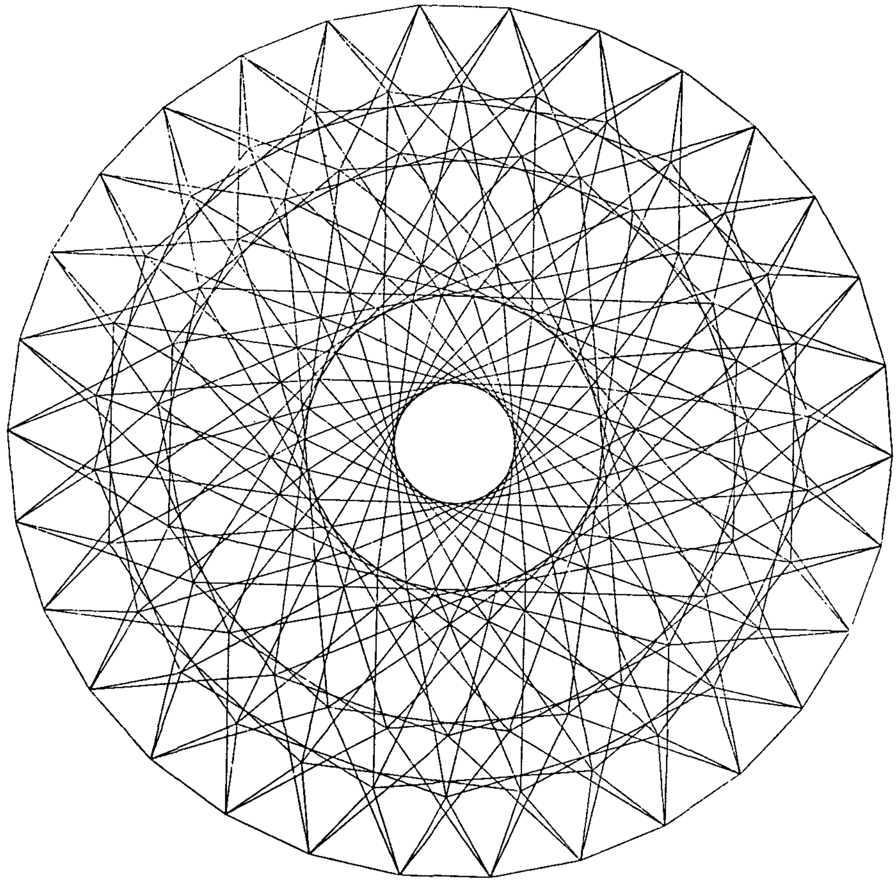


Figure 2. The equatorial section 3{5}3 of 3{3}3{3}3{3}3.

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