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T-REGULAR PROBABILISTIC CONVERGENCE SPACES

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Abstract

A probabilistic convergence structure assigns a probability that a given filter converges to a given element of the space. The role of the t-norm (triangle norm) in the study of regularity of probabilistic convergence spaces is investigated. Given a probabilistic convergence space, there exists a finest T-regular space which is coarser than the given space, and is referred to as the 'T-regular modification'. Moreover, for each probabilistic convergence space, there is a sequence of spaces, indexed by nonnegative ordinals, whose first term is the given space and whose last term is its T-regular modification. The T-regular modification is illustrated in the example involving 'convergence with probability λ' for several t-norms. Suitable function space structures in terms of a given t-norm are also considered.

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1. Introduction

Menger [11] suggested that due to inherent uncertainties of measurements in physical situations, the distance $d(p, q)$ should be replaced by a probability distribution function $F_{pq}(x)$, where the latter is interpreted as 'the probability that the distance between p and q is less than x'. This led to the development of the area now called 'probabilistic metric spaces'. An excellent reference on the development of this subject can be found in Schweizer and Sklar [14]. Floresque [7] replaced the probabilistic metric concept by assigning a probability that a given net converges to a given element of the space. Replacing nets by filters permits a more manageable theory and has resulted in the study of probabilistic convergence spaces by Richardson and Kent [13]. Connections between probabilistic convergence spaces and generalized metric spaces introduced by E. Lowen and R. Lowen [9], and R. Lowen [10], have been given by Brock and Kent [2].

A t-norm, denoted by T, permits one to define a triangle-type inequality in the probabilistic metric space setting. Also, the t-norm has been the fundamental tool used in defining 'diagonal' and 'regular' probabilistic convergence spaces. The former was investigated in [13] and the latter was studied by Brock and Kent [3]. Both of the above notions have their origins in definitions given by Cook and Fisher [6] in the convergence space setting. The t-norm permits one to extend these ideas to the probabilistic convergence space setting where there is a collection of convergence spaces which approximates (in the probabilistic sense) the true state.

Basic definitions and notations pertaining to convergence and probabilistic convergence spaces are given in Section 2. In Section 3, the T-regular sequence for a given probabilistic convergence space is defined and is shown to terminate with the T-regular modification of the given space. The behavior of the T-regular sequence with regard to various t-norms is investigated. An example illustrating the T-regular modification is given in Section 4, and it shows the importance of selecting an appropriate t-norm. Given the t-norm T, function space structures defined in terms of T and possessing desirable properties are considered in Section 5.

2. Preliminaries

The aim of the theory of convergence spaces is to generalize traditional general topology to include convergences one often encounters in analysis without the 'local coherence condition' required of topologies. This represents a point of view of analysis rather than geometry. Intuitively, a convergence structure on a set is merely a rule indicating which filters converge to which points. More precisely, if $F(X)$ denotes the set of all filters (proper) on X, a *convergence structure* on X is a function $q : F(X) \to 2^X$ satisfying:

$(\mathrm{CS})_1$ $x \in q(\dot{x})$ for each $x \in X$, where $\dot{x}$ denotes the ultrafilter containing $\{x\}$

$(\mathrm{CS})_2$ $\mathscr{F} \leq \mathscr{G}$ (that is, $\mathscr{F} \subseteq \mathscr{G}$) implies $q(\mathscr{F}) \subseteq q(\mathscr{G})$

$(\mathrm{CS})_3$ $q(\mathscr{F}) \cap q(\mathscr{G}) \subseteq q(\mathscr{F} \cap \mathscr{G})$.

The pair (X, q) is called a convergence space and $\mathscr{F}$ is said to *q-converge to x* when $x \in q(\mathscr{F})$ and is denoted by $\mathscr{F} \xrightarrow{q} x$. Let $C(X)$ denote the set of all convergence structures on X and define a partial order by $p \leq q$ if and only if $\mathscr{F} \xrightarrow{q} x$ implies $\mathscr{F} \xrightarrow{p} x$. Then $(C(X), \leq)$ is a complete lattice whose largest (smallest) member is the discrete (indiscrete) topology, respectively.

Given a convergence space (X, q) and $A \subseteq X$, the *closure* (*interior*) operator is defined by $\mathrm{cl}_q A = \{x \in X : \mathscr{F} \xrightarrow{q} x$ for some $\mathscr{F}$ containing $A\}$ ($I_q A = \{x \in X : \mathscr{F} \xrightarrow{q} x$ implies $A \in \mathscr{F}\}$), respectively. If $\mathscr{G} \in F(X)$, then $\mathrm{cl}_q \mathscr{G}$ denotes the filter on X whose base is $\{\mathrm{cl}_q A : A \in \mathscr{G}\}$. Moreover, (X, q) is called *regular* when $\mathscr{F} \xrightarrow{q} x$ implies $\mathrm{cl}_q \mathscr{F} \xrightarrow{q} x$. Fix $x \in X$; then $\mathscr{V}_q(x) = \{V \subseteq X : x \in I_q V\}$ is called the q-

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neighborhood filter at x and $A \subseteq X$ is said to be *q-open* if $A \in \mathscr{V}_q(x)$ for each $x \in A$. The convergence structure q for X is called a *pseudo-topology* (*pretopology, topology*) if $\mathscr{F} \xrightarrow{q} x$ when each ultrafilter containing $\mathscr{F}$ q-converges to x ($\mathscr{V}_q(x) \xrightarrow{q} x$ for each $x \in X$, q is a pretopology and $\mathscr{V}_q(x)$ has a base of q-open subsets of X), respectively. A map $f : (X, q) \to (Y, p)$ between two convergence spaces is said to be *continuous* if $\mathscr{F} \xrightarrow{q} x$ implies $f\mathscr{F} \xrightarrow{p} f(x)$, where $f\mathscr{F}$ denotes the filter on Y whose base is $\{f(F) : F \in \mathscr{F}\}$. Let $CONV$ denote the category whose objects are the collection of convergence spaces and whose morphisms are the continuous functions. The collection of objects of $CONV$ is denoted by $|CONV|$.

Denote $I = [0, 1]$. A *probabilistic convergence structure* $\bar{q}$ for X is a map $\bar{q} : F(X) \times I \to 2^X$ satisfying:

(PCS)$_1$ $\bar{q}(\mathscr{F}, \lambda) = q_\lambda(\mathscr{F})$, where $(X, q_\lambda) \in |CONV|$

(PCS)$_2$ $q_\mu \le q_\lambda$ when $\mu \le \lambda$, $\mu, \lambda \in I$.

The pair $(X, \bar{q})$ is called a *probabilistic convergence space* and, more intuitively, $x \in q_\lambda(\mathscr{F})$ is denoted by $\mathscr{F} \xrightarrow{q_\lambda} x$, where the latter is interpreted as 'the probability that $\mathscr{F}$ q-converges to x is at least λ'. The probability that $\mathscr{F}$ $\bar{q}$-converges to x is defined to be $\sup\left\{\lambda \in I : \mathscr{F} \xrightarrow{q_\lambda} x\right\}$ and $\{q_\lambda : \lambda \in I\}$ can be viewed as a collection of convergence structures approximating q_1. A map $f : (X, \bar{q}) \to (Y, \bar{p})$ between two probabilistic convergence spaces is called *continuous* if $f : (X, q_\lambda) \to (Y, p_\lambda)$ is a morphism in $CONV$, for each $\lambda \in I$. The category whose objects consist of the collection of probabilistic convergence spaces and whose morphisms are the continuous maps is denoted by $PCONV$. The collection of objects in $PCONV$ is designated by $|PCONV|$. Let $PC(X)$ be the set of all probabilistic convergence structures on set X, and define the partial order $\bar{q} \le \bar{p}$ if and only if $q_\lambda \le p_\lambda$ for each $\lambda \in I$; then $PC(X)$ is a complete lattice. Also $\bar{q} \in PC(X)$ is said to be *constant* when $q_\lambda = p \in C(X)$ for each $0 < \lambda \le 1$. Indeed, this correspondence shows that $CONV$ is isomorphic to a subcategory of $PCONV$. It is known that $PCONV$ has initial and final structures (topological category) and suitable function space structures (Cartesian closed); in particular, products and coproducts exist in $PCONV$ as will be needed later.

A *t-norm* (triangle norm) is a binary operation $T : I^2 \to I$ which is associative, commutative, increasing in each variable, and satisfies $T(a, 1) = a$ for all $a \in I$. Given t-norms T_1 and T_2, define $T_1 \le T_2$ by $T_1(a, b) \le T_2(a, b)$ for each $a, b \in I$. There exist a smallest and a largest t-norm, and the latter is given by $T_M(a, b) = \min\{a, b\}$. Let T be a t-norm. Then $(X, \bar{q}) \in |PCONV|$ is called *T-regular* if $\mathscr{F} \xrightarrow{q_\lambda} x$ implies $\mathrm{cl}_{q_\mu} \mathscr{F} \xrightarrow{q_{T(\lambda,\mu)}} x$. Brock and Kent [3] defined the notion of a T-regular space and proved it was equivalent to the above simpler condition. Note that if $\bar{q}$ is a constant probabilistic convergence structure for X, then $(X, \bar{q})$ is T-regular if and only if (X, q_1) is a regular convergence space, where $\bar{q} = (q_\lambda)$, $\lambda \in I$.

Moreover, Brock and Kent [3] proved that when $T = T_M$, $(X, \bar{q})$ is T-regular if and only if (X, q_λ) is a regular convergence space, for each $\lambda \in I$. They also showed that 'T-regular' is an initial property, and thus for any given $(X, \bar{q}) \in |PCONV|$ there exists a finest T-regular object in $|PCONV|$ which is coarser than $(X, \bar{q})$.

3. T-regular sequences

Without further mention, all results will be relative to an assumed t-norm T. Quite often it is convenient to denote $T(\lambda, \mu)$ by $\lambda * \mu$. Given $(X, \bar{q}) \in |PCONV|$; assume that $(X, \bar{p}) \in |PCONV|$ is T-regular and $\bar{p} \leq \bar{q}$. Observe that if $\mathscr{G} \xrightarrow{q_\lambda} x$, then $\mathrm{cl}_{q_\mu} \mathscr{G} \xrightarrow{p_{\lambda*\mu}} x$, and thus $\mathrm{cl}_{q_\mu} \mathscr{G} \xrightarrow{p_\delta} x$ when $\delta \leq \lambda * \mu$. The latter follows since $p_\delta \leq p_{\lambda*\mu}$ when $\delta \leq \lambda * \mu$. More generally, $\mathrm{cl}_{q_{\mu_n}} \mathrm{cl}_{q_{\mu_{n-1}}} \ldots \mathrm{cl}_{q_{\mu_1}} \mathscr{G} \xrightarrow{p_\delta} x$ when $\delta \leq \lambda * \mu_1 * \mu_2 * \cdots * \mu_n$. The preceding is motivation for the following definition of a sequence of objects in $|PCONV|$ beginning with $(X, \bar{q})$ and terminating with the finest T-regular object which is coarser than $(X, \bar{q})$.

Fix $(X, \bar{q}) \in |PCONV|$, where $\bar{q} = (q_\lambda)$, $\lambda \in I$. The *T-regular sequence* $\{(X, \bar{q}_\alpha)\}$, where $\bar{q}_\alpha = (q_{\alpha\delta})$, $\delta \in I$, is defined recursively as follows:

(1) $\bar{q}_0 = \bar{q}$ (that is, $q_{0\delta} = q_\delta$ for each $\delta \in I$)

(2) if $\bar{q}_\beta$ has been defined for each original $0 \leq \beta < \alpha$, then $\bar{q}_\alpha = (q_{\alpha\delta})$ is defined by $\mathscr{F} \xrightarrow{q_{\alpha\delta}} x$ if and only if there exist an integer $n \geq 1$, $\beta < \alpha$ and $\mathscr{G} \xrightarrow{q_{0\lambda}} x$ for which $\mathscr{F} \geq \mathrm{cl}_{q_{\beta\mu_n}} \mathrm{cl}_{q_{\beta\mu_{n-1}}} \ldots \mathrm{cl}_{q_{\beta\mu_1}} \mathscr{G}$ and $\delta \leq \lambda * \mu_1 * \mu_2 * \cdots * \mu_n$.

In order to condense the notation in (2), $\mathrm{cl}_{q_{\beta\mu_n}} \mathrm{cl}_{q_{\beta\mu_{n-1}}} \ldots \mathrm{cl}_{q_{\beta\mu_1}} \mathscr{G}$ and $\lambda*\mu_1*\mu_2*\cdots*\mu_n$ are denoted by $\mathrm{cl}_{q_{\beta\mu_1-\mu_n}} \mathscr{G}$ and $T(\lambda; \mu_1 - \mu_n)$, respectively.

PROPOSITION 3.1. *Assume that $\{(X, \bar{q}_\alpha)\}$ is the T-regular sequence for $(X, \bar{q}) \in |PCONV|$. Then*

(a) *$(X, \bar{q}_\alpha) \in |PCONV|$.*

(b) *$\bar{q}_\alpha \leq \bar{q}_\beta$ when $\alpha \geq \beta$.*

(c) *$\bar{q}_{\alpha+1} = \bar{q}_\alpha$ implies that $(X, \bar{q}_\alpha)$ is the finest T-regular object in $|PCONV|$ which is coarser than $(X, \bar{q})$.*

PROOF. Verification of (a) and (b) is straightforward. Assume that $\bar{q}_{\alpha+1} = \bar{q}_\alpha$. First, consider the case when $\alpha = 0$. Observe that $\mathscr{F} \xrightarrow{q_{0\lambda}} x$ implies that $\mathrm{cl}_{q_{0\mu}} \mathscr{F} \xrightarrow{q_{1\lambda*\mu}} x$ and since $q_{0.\lambda*\mu} = q_{1.\lambda*\mu}$, $\mathrm{cl}_{q_{0\mu}} \mathscr{F} \xrightarrow{q_{0\lambda*\mu}} x$. Hence $(X, \bar{q}_0)$ is T-regular. Next, assume that $\alpha > 0$ and $\mathscr{F} \xrightarrow{q_{\alpha r}} x$; it must be shown that $\mathrm{cl}_{q_{\alpha s}} \mathscr{F} \xrightarrow{q_{\alpha,r*s}} x$. Since $\mathscr{F} \xrightarrow{q_{\alpha r}} x$, there exist $\mathscr{G} \xrightarrow{q_{0\lambda}} x$, $n \geq 1$, $\beta < \alpha$ for which $\mathscr{F} \geq \mathrm{cl}_{q_{\beta:\mu_1-\mu_n}} \mathscr{G}$ and $r \leq T(\lambda; \mu_1 - \mu_n)$. It follows that $\mathrm{cl}_{q_{\alpha s}} \mathscr{F} \geq \mathrm{cl}_{q_{\alpha:s.\mu_1-\mu_n}} \mathscr{G} \xrightarrow{q_{\alpha+1.r*s}} x$ since $T(r, s) \leq T(\lambda; s, \mu_1 - \mu_n)$ and, according to the hypothesis, $\mathrm{cl}_{q_{\alpha s}} \mathscr{F} \xrightarrow{q_{\alpha.r*s}} x$. Therefore $(X, \bar{q}_\alpha)$ is T-regular.

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Suppose that $(X, \bar{p})$ is T-regular and $\bar{p} \le \bar{q}$; it must be shown that $\bar{p} \le \bar{q}_\alpha$. Transfinite induction is used to verify this result. Note that above is valid when $\alpha = 0$ since $\bar{p} \le \bar{q} = \bar{q}_0$. Assume that $\bar{p} \le \bar{q}_\beta$ for each $\beta < \alpha$ and that $\mathscr{F} \xrightarrow{q_{\alpha\delta}} x$. Then $\mathscr{F} \ge \mathrm{cl}_{q_{\beta:\mu_1-\mu_n}} \mathscr{G}$ for some $\beta < \alpha$, $n \ge 1$, and $\mathscr{G} \xrightarrow{q_{0\lambda}} x$, where $\delta \le T(\lambda; \mu_1 - \mu_n)$. Then $\mathscr{F} \xrightarrow{p_\delta} x$ follows from the induction hypothesis and the T-regularity of $\bar{p}$, and thus $\bar{p} \le \bar{q}_\alpha$. Hence $\bar{q}_\alpha$ is the finest T-regular object in $|PCONV|$ which is coarser than $\bar{q}$.

The least original number α for which Proposition 3.1(c) is valid is called the *length of the T-regular sequence* and is denoted by $\ell_{R_T}(\bar{q}) = \alpha$. Given a convergence space (X, σ); a *regular sequence* $\{(X, r_\alpha\sigma)\}$ for (X, σ) was defined by Richardson and Kent [12] as follows:

(1) $r_0\sigma = \sigma$

(2) If $r_\beta\sigma$ has been defined for each $\beta < \alpha$, then $\mathscr{F} \xrightarrow{r_\alpha\sigma} x$ if and only if $\mathscr{F} \ge \mathrm{cl}^n_{r_\beta\sigma} \mathscr{G}$, for some $\mathscr{G} \xrightarrow{\sigma} x$, $n \ge 1$, and $\beta < \alpha$.

Assume that $(X, \bar{q}) \in |PCONV|$ is constant with $q_\lambda = \sigma$ for $0 < \lambda \le 1$. Then $(X, \bar{q})$ is T-regular if and only if (X, σ) is regular and, moreover, each term in the T-regular sequence $\{(X, \bar{q}_\alpha)\}$ of $(X, \bar{q})$ is constant; indeed, $q_{\alpha\delta} = r_\alpha\sigma$ when $0 < \delta \le 1$. It follows that $\ell_{R_T}(\bar{q}) = \ell_R(\sigma)$, where the latter denotes the length of the regular series for the convergence space (X, σ). Recall that $T_M(a, b) = \min\{a, b\}$ is the largest possible t-norm.

PROPOSITION 3.2. *Suppose that $(X, \bar{q})$ has a $T(T_1)$-regular sequence $\{(X, \bar{q}_\alpha)\}$ $(\{(X, \bar{p}_\alpha)\})$, respectively. Then*

(a) *$T \le T_1$ implies $\bar{q}_\alpha \ge \bar{p}_\alpha$.*

(b) *$T = T_m$ implies $\bar{q}_\alpha = r_\alpha\bar{q}$ and $\ell_{R_T}(\bar{q}) = \sup\{\ell_R(q_\delta) : 0 \le \delta \le 1\}$, where $r_\alpha\bar{q} := (r_\alpha q_{0\delta})$, $0 \le \delta \le 1$.*

(c) *$\bar{q}_\alpha \ge r_\alpha\bar{q}$.*

(d) *$\bar{q}_\alpha = r_\alpha\bar{q}$ when $\bar{q}$ is constant, where $q_\lambda = \sigma$, $0 < \lambda \le 1$, and, moreover, $\ell_{R_T}(\bar{q}) = \ell_R(\sigma)$.*

PROOF. Transfinite induction is used to verify (a)–(b).

(a): Assume that $\bar{q}_\beta \ge \bar{p}_\beta$ for each $\beta < \alpha$, where $\alpha > 0$, and let $\mathscr{F} \xrightarrow{q_{\alpha\delta}} x$. Then $\mathscr{F} \ge \mathrm{cl}_{q_{\beta:\mu_1-\mu_n}} \mathscr{G}$ for some $\mathscr{G} \xrightarrow{q_{0\lambda}} x$, $n \ge 1$,$\beta < \alpha$, where $\delta \le T(\lambda; \mu_1 - \mu_n)$. Since $T \le T_1$, $\delta \le T_1(\lambda; \mu_1 - \mu_n)$ and the induction hypothesis implies that $\mathscr{F} \xrightarrow{p_{\alpha\delta}} x$. Hence $\bar{q}_\alpha \ge \bar{p}_\alpha$.

(b): Assume that $\alpha > 0$ and $\bar{q}_\beta = r_\beta\bar{q}$ for each $\beta < \alpha$. If $\mathscr{F} \xrightarrow{q_{\alpha\delta}} x$, then $\mathscr{F} \ge \mathrm{cl}_{q_{\beta:\mu_1-\mu_n}} \mathscr{G}$ for some $\mathscr{G} \xrightarrow{q_{0\lambda}} x$, $n \ge 1$, $\beta < \alpha$, where $\delta \le T_M(\lambda; \mu_1 - \mu_n)$. It follows that $\delta \le \lambda$ and $\delta \le \mu_i$, $1 \le i \le n$, and thus $\mathscr{F} \ge \mathrm{cl}^n_{r_\beta q_{0\delta}} \mathscr{G}$. Hence

$\mathscr{F} \xrightarrow{r_\alpha q_{\alpha\delta}} x$. The converse is proved similarly because of the special nature of T_M and thus $\bar{q}_\alpha = r_\alpha \bar{q}$ is valid. The second assertion of (b) clearly follows from the first part.

(c): Since $T \leq T_M$, (a) implies that $\bar{q}_\alpha \geq \bar{p}_\alpha$, where $\{(X, \bar{p}_\alpha)\}$ is the T_M-regular sequence for $(X, \bar{q})$. However (b) implies that $\bar{p}_\alpha = r_\alpha \bar{q}$ and thus $\bar{q}_\alpha \geq r_\alpha \bar{q}$.

(d): Validity of this result follows by the remarks made preceding this proposition.

Recall that $(C(X), \leq)$ and $(PC(X), \leq)$ are complete lattices. Indeed, if $p = \sup_{j \in J} q_j$ in $C(X)$, then $\mathscr{F} \xrightarrow{p} x$ if and only if $\mathscr{F} \xrightarrow{q_j} x$ for each $j \in J$. Likewise, $\bar{p} = \sup_{j \in J} \bar{q}_j$ in $PC(X)$ if and only if $p_\lambda = \sup_{j \in J} q_{j\lambda}$ in $C(X)$, for each component $\lambda \in I$. Moreover, $(X, \bar{q}) \in |PCONV|$ is called *left-continuous* provided $\sup_{\mu < \lambda} q_\mu = q_\lambda$ in $C(X)$, for each $0 < \lambda \leq 1$. Let $LCPCONV$ ($R_T PCONV$, $R_T LCPCONV$) denote the full subcategory of $PCONV$ whose objects are left-continuous (T-regular, T-regular and left-continuous), respectively. Given $(X, \bar{q}) \in |PCONV|$, define $\bar{\sigma} := c\bar{q}$ as follows:

(1) $\sigma_\lambda = \sup_{\mu < \lambda} q_\mu, 0 < \lambda \leq 1$

(2) $\sigma_0 =$ indiscrete topology.

PROPOSITION 3.3. *Suppose that $(X, \bar{q}) \in |PCONV|$ and let $\sigma = c\bar{q}$ be as defined above. Then*

(a) *$(X, c\bar{q})$ is the finest object in $|LCPCONV|$ which is coarser than $(X, \bar{q})$.*

(b) *$(X, \bar{q}) \in |R_T PCONV|$ implies $(X, c\bar{q}) \in |R_T PCONV|$ when T is a left-continuous t-norm.*

(c) *$R_T PCONV$, $LCPCONV$ ($R_T LCPCONV$) is a bireflective subcategory of $PCONV$ (provided T is left-continuous), respectively. More generally, continuity is preserved under the corresponding T-regular series as well as with respect to taking left-continuous modifications.*

PROOF. (a): It follows easily that $(X, c\bar{q}) \in |LCPCONV|$. Assume that $(X, \bar{q}) \in |LCPCONV|$ and $\bar{p} \leq \bar{q}$. Then, for $0 < \lambda \leq 1$, $p_\lambda = \sup_{\mu < \lambda} p_\mu \leq \sup_{\mu < \lambda} q_\mu = \sigma_\lambda$ and thus $\bar{p} \leq c\bar{q}$. Hence (a) is valid.

(b): Suppose that $\mathscr{F} \xrightarrow{\sigma_\lambda} x$; it is shown that $\mathrm{cl}_{\sigma_\mu} \mathscr{F} \xrightarrow{\sigma_{\lambda * \mu}} x$. The latter clearly holds when $\lambda * \mu = 0$. Assume that $\lambda * \mu > 0$; it suffices to show that $\mathrm{cl}_{\sigma_\mu} \mathscr{F} \xrightarrow{q_\beta} x$ for each $0 \leq \beta < \lambda * \mu$. Fix $0 \leq \beta < \lambda * \mu$, then since T is left-continuous, there exists $\delta > 0$ for which $\beta \leq (\lambda - \delta) * (\mu - \delta)$. However, $\mathscr{F} \xrightarrow{q_{\lambda-\delta}} x$, $\mathrm{cl}_{\sigma_\mu} \mathscr{F} \geq \mathrm{cl}_{q_{\mu-\delta}} \mathscr{F}$, and $(X, \bar{q}) \in |R_T PCONV|$ implies that $\mathrm{cl}_{\sigma_\mu} \mathscr{F} \to x$ in $(X, q_{(\lambda-\delta)*(\mu-\delta)})$. Since $\beta \leq (\lambda - \delta) * (\mu - \delta)$, $\mathrm{cl}_{\sigma_\mu} \mathscr{F} \xrightarrow{q_\beta} x$ and thus $(X, c\bar{q}) \in |R_T PCONV|$.

(c): Let $f : (X, \bar{q}) \to (Y, \bar{p})$ be a continuous map between two objects in $|PCONV|$. It is shown that $f : (X, c\bar{q}) \to (Y, c\bar{p})$ and $f : (X, \bar{q}_\alpha) \to (Y, \bar{p}_\alpha)$ are also continuous, where $\{(X, \bar{q}_\alpha)\}$ and $\{(Y, \bar{p}_\alpha)\}$ denote the T-regular sequences for $(X, \bar{q})$ and $(Y, \bar{p})$. The first part is easily verified; transfinite induction is used to prove the second assertion. The hypothesis guarantees the result is valid when $\alpha = 0$. Assume that $f : (X, \bar{q}_\beta) \to (Y, \bar{p}_\beta)$ is continuous for each $\beta < \alpha$ and suppose that $\mathscr{F} \xrightarrow{q_{\alpha\delta}} x$. Then $\mathscr{F} \geq \mathrm{cl}_{q_{\beta:\mu_1-\mu_n}} \mathscr{G}$ for some $n \geq 1$, $\beta < \alpha$, and $\mathscr{G} \xrightarrow{q_{0\lambda}} x$, where $\delta \leq T(\lambda; \mu_1 - \mu_n)$. The induction hypothesis implies that $f\mathscr{F} \geq \mathrm{cl}_{p_{\beta:\mu_1-\mu_n}} f\mathscr{G}$, $f\mathscr{G} \xrightarrow{p_{0\lambda}} f(x)$, and thus $f\mathscr{F} \xrightarrow{p_{\alpha\delta}} f(x)$. Hence $f : (X, \bar{q}_\alpha) \to (Y, \bar{p}_\alpha)$ is a continuous and the second assertion is valid. These results combined with (b) give the desired conclusion.

The behavior of the T-regular sequence relative to the basic constructions of retractions, products, and coproducts is listed below. The proofs are similar to those given in [12] for the regular sequence of a convergence space and are omitted.

PROPOSITION 3.4. (a) *The property of $f : (X, \bar{q}) \to (Y, \bar{p})$ being a retraction is preserved under the corresponding T-regular sequences.*
(b) $(X, \bar{q}_\alpha) \geq \prod_{j\in J}(X_j, \bar{q}_{j\alpha})$, *where* $(X, \bar{q}) = \prod_{j\in J}(X_j, \bar{q}_j)$, *and* $\{(X, \bar{q}_\alpha)\}$ *and* $\{(X_j, \bar{q}_{j\alpha})\}$ *denote the T-regular sequences for $(X, \bar{q})$ and $(X_j, \bar{q}_j)$, respectively, $j \in J$. Moreover, $\ell_{R_T}(\bar{q}) \geq \ell_{R_T}(\bar{q}_j)$ for each $j \in J$.*
(c) $(X, \bar{q}_\alpha) = \coprod_{j\in J}(X_j, \bar{q}_{j\alpha})$, *where* $(X, \bar{q}) = \coprod_{j\in J}(X_j, \bar{q}_j)$ *denotes the coproduct (disjoint union) of $(X_j, \bar{q}_j)$, $j \in J$, and $\ell_{R_T}(\bar{q}) = \sup\{\ell_{R_T}(\bar{q}_j) : j \in J\}$.*

4. Example

A probabilistic convergence space describing convergence with probability λ is considered below. The example fails to be T_M-regular but is T-regular for an appropriately chosen T. The T-regular modification is found for both $T = T_M$ and the product T-norm, defined by $T_{PR}(a, b) = ab$.

Let $I = [0, 1]$, P denote Lebesgue measure on I, and let X be the set of all real-valued, Lebesgue measurable functions defined on I. Define $\bar{q} = (q_\lambda)$ as follows: q_0 is the indiscrete topology and, for $0 < \lambda \leq 1$, $\phi \xrightarrow{q_\lambda} f$ if and only if for each $0 < \delta < \lambda$ and $a > 0$, there exists $A \in \phi$ for which $g \in A$ implies $P\{|g - f| \leq a\} \geq \delta$. Then $(X, \bar{q}) \in |PCONV|$ and it is shown in [13] that $(X, \bar{q})$ is a pretopology with neighborhood filter $\mathscr{V}_{q_\lambda}(f)$ generated by $\{\mathscr{V}_{\delta\alpha}(f) : 0 < \delta < \lambda, a > 0\}$, where $\mathscr{V}_{\delta\alpha}(f) = \{g \in X : P\{|g - f| \leq a\} \geq \delta\}$, $0 < \lambda \leq 1$.

LEMMA 4.1. *Let $\mu, \lambda \in I$ and $f \in X$. Then*
(a) $\lambda + \mu \leq 1$ *implies* $\mathrm{cl}_{q_\mu} \mathscr{V}_{q_\lambda}(f) = \dot{X} = \{X\}$.
(b) $\lambda + \mu > 1$ *implies* $\mathrm{cl}_{q_\mu} \mathscr{V}_{q_\lambda}(f) = \mathscr{V}_{q_{\lambda+\mu-1}}(f)$.

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PROOF. (a): Observe that (a) is valid when either $\mu = 0$ or $\lambda = 0$ since q_0 is the indiscrete topology. Assume that $0 < \mu, \lambda \leq 1$ and let $0 < \delta < \lambda, a > 0$. It suffices to show that $\mathrm{cl}_{q_\mu} V_{\delta a}(f) = X$. Suppose that $g \in X$ and let A be any subset of I for which $P(A) = \delta$. Define $h = f \cdot 1_A + g \cdot 1_{A^c}$, where $A^c = I - A$ and 1_A denotes the indicator (characteristic) function for A. Then $P\{|h - f| \leq a\} \geq P(A) = \delta$ and, for $b > 0$, $P\{|h - g| \leq b\} \geq P(A^c) = 1 - \delta \geq 1 - \lambda \geq \mu$ since $\lambda + \mu \leq 1$. It follows that $h \in V_{\delta a}(f) \cap V_{\rho b}(g)$ for each $0 < \rho < \mu$ and thus $g \in \mathrm{cl}_{q_\mu} V_{\delta a}(f)$. The desired conclusion follows since g is any element in X.

(b): Assume that $\lambda + \mu > 1$; it must be shown that $\mathrm{cl}_{q_\mu} \mathscr{V}_{q_\lambda}(f) = \mathscr{V}_{q_{\lambda+\mu-1}}(f)$. A typical base member of $\mathscr{V}_{q_\lambda}(f)$ ($\mathscr{V}_{q_{\lambda+\mu-1}}(f)$) is $V_{\delta a}(f)$ ($V_{\mu+\delta-1,a}(f)$), respectively, where $0 < \delta < \lambda$ and $a > 0$. It suffices to show that $\mathrm{cl}_{q_\mu} V_{\delta a}(f) = V_{\mu+\delta-1,a}(f)$. Suppose that $g \in \mathrm{cl}_{q_\mu} V_{\delta a}(f)$ and $0 < \rho < \mu$ and $b > 0$. Then there exists $h \in V_{\rho b}(g) \cap V_{\delta a}(f)$, where h depends on ρ and b; hence $P\{|h - g| \leq b\} \geq \rho$ and $P\{|h - f| \leq a\} \geq \delta$. Without loss of generality, assume that $\rho + \delta - 1 > 0$, and thus $-b \leq h - g \leq b$ and $-a \leq f - h \leq a$ on a set of probability measures at least $\rho + \delta - 1$. It follows that $P\{|f - g| \leq a + b\} \geq \rho + \delta - 1$, for each $b > 0$. Since $\{|f - g| \leq a + b\} \downarrow \{|f - g| \leq a\}$ as $b \downarrow 0$, $P\{|f - g| \leq a\} \geq \rho + \delta - 1$, for each $0 < \rho < \mu$. Hence $P\{|f - g| \leq a\} \geq \mu + \delta - 1$ and thus $g \in \mathrm{cl}_{q_\mu} V_{\mu+\delta-1,a}(f)$; therefore $\mathrm{cl}_{q_\mu} V_{\delta a}(f) \subseteq V_{\mu+\delta-1,a}(f)$.

Conversely, suppose that $g \in V_{\mu+\delta-1,a}(f)$ and let us show that $g \in \mathrm{cl}_{q_\mu} V_{\delta a}(f)$. It must be shown that $V_{\rho b}(g) \cap V_{\delta a}(f) \neq \phi$ for each $0 < \rho < \mu$ and $b > 0$. Since $V_{\mu b}(g) \subseteq V_{\rho b}(g)$, it suffices to show that $V_{\mu b}(g) \cap V_{\delta a}(f) \neq \phi$ when $b > 0$. Now $g \in V_{\mu+\delta-1,a}(f)$ implies that $P\{|g - f| \leq a\} \geq \mu + \delta - 1 = \alpha > 0$ and let $A = \{|g - f| \leq a\}$. There exist $B \subset A$ for which $P(B) = \alpha$ and disjoint subsets C and D such that $I - B = C \cup D$, $P(C) = \mu - \alpha$, and $P(D) = \delta - \alpha$. Note that $\{B, C, D\}$ is a partition of I. Define $h = f \cdot 1_D + g \cdot 1_{B \cup C}$. Then $P\{|h - f| \leq a\} \geq P(B) + P(D) = \alpha + (\delta - \alpha) = \delta$ and $P\{|h - g| \leq b\} \geq P(B) + P(C) = \alpha + (\mu - \alpha) = \mu$. Hence $h \in V_{\mu b}(g) \cap V_{\delta a}(f)$ and thus $g \in \mathrm{cl}_{q_\mu} V_{\delta a}(g)$; therefore $\mathrm{cl}_{q_\mu} V_{\delta a}(f) = V_{\mu+\delta-1,a}(f)$ and the desired result follows.

Define the t-norm $T_P(a, b) = \max\{a + b - 1, 0\}$ and recall $T_{PR}(a, b) = ab$, $a, b \in I$.

PROPOSITION 4.2. *Let $(X, \bar{q})$ be the example described above, where $\bar{q} = (q_\lambda)$. Then*

(a) *$(X, \bar{q})$ is T_P-regular.*

(b) *the T_M-regular modification of $(X, \bar{q})$ is $(X, \bar{p})$, where $\bar{p} = (p_\lambda)$ is defined by $p_\lambda = q_1$ when $\lambda = 1$; otherwise p_λ is the indiscrete topology. Moreover, $\ell_R(\bar{q}) = 1$.*

(c) *$(X, \bar{\sigma})$ is the finest left-continuous, T_{PR}-regular object in $|PCONV|$ which is coarser than $(X, \bar{q})$, where $\bar{\sigma} = (\sigma_\lambda)$ is defined by $\sigma_\lambda = q_{1+\ell_n\lambda}$ when $\lambda \in [e^{-1}, 1]$;*

otherwise, it is the indiscrete topology.

(d) *the topological modification of* $(X, \bar{q})$ *is also* $(X, \bar{p})$.

PROOF. (a): This result has been proved directly by Brock and Kent ([3, Example 3.6]) but it is also follows from Lemma 4.1 above.

(b): Since $T = T_M$, the T-regular modification reduces to finding the regular modification component-wise for each convergence space. If $0 \leq \lambda \leq \frac{1}{2}$, then by Lemma 4.1(a), $\mathrm{cl}_{q_\lambda} \mathscr{V}_{q_\lambda}(f) = \dot{X}$. Next, assume that $\frac{1}{2} < \lambda \leq 1$; then by Lemma 4.1(b), $\mathrm{cl}_{q_\lambda} \mathscr{V}_{q_\lambda}(f) = \mathscr{V}_{q_{2\lambda-1}}(f)$ and, in particular, $\mathrm{cl}_{q_\lambda} \mathscr{V}_{q_\lambda}(f) = \mathscr{V}_{q_\lambda}(f)$ when $\lambda = 1$. Suppose that $\frac{1}{2} < \lambda < 1$. If $3\lambda - 1 \leq 1$, then $\mathrm{cl}^2_{q_\lambda} \mathscr{V}_{q_\lambda}(f) = \mathrm{cl}_{q_\lambda} \mathscr{V}_{q_{2\lambda-1}}(f) = \dot{X}$ by Lemma 4.1(a); otherwise, $\mathrm{cl}^2_{q_\lambda} \mathscr{V}_{q_\lambda}(f) = \mathscr{V}_{q_{3\lambda-2}}(f)$ according to Lemma 4.1(b). Let k denote the smallest positive integer for which $(k+1)\lambda - (k-1) \leq 1$; then $\mathrm{cl}^k_{q_\lambda} \mathscr{V}_{q_\lambda}(f) = \dot{X}$. It follows that $(X, q_{1\lambda})$ is the indiscrete topology when $\frac{1}{2} < \lambda < 1$. Hence the T_M-regular sequence of $(X, \bar{q})$ terminates with $(X, \bar{q}_1)$ (one iteration), where $\bar{q}_1 = (q_{1\lambda})$ and $q_{1\lambda} = q_1$ when $\lambda = 1$ and, otherwise, $q_{1\lambda}$ is the indiscrete topology. Indeed, it is well-known that (X, q_1) is metrizable when one identifies elements equal P-almost surely ([4, p. 67]).

(c): Observe that for $\delta \in I$, $X \xrightarrow{q_{1\delta}} f$ if and only if for some $n \geq 1$, there exist $\lambda, \mu_i \in I$, $1 \leq i \leq n$, such that the following are satisfied:

(i) $A_n := \lambda\mu_1\mu_2 \cdots \mu_n \geq \delta$

(ii) $B_n := \lambda + \mu_1 + \mu_2 + \cdots + \mu_n - n \leq 0$.

Indeed, in this case, $\mathrm{cl}_{q_{\mu_1 - \mu_n}} \mathscr{V}_{q_\lambda}(f) = \dot{X} \xrightarrow{q_{1\delta}} f$ since $A_n \geq \delta$ and $B_n \leq 0$. If conditions (i)–(ii) are valid, then

$$\delta^{1/(n+1)} \leq A_n^{1/(n+1)} \leq \frac{\lambda + \mu_1 + \mu_2 + \cdots + \mu_n}{n+1} \leq \frac{n}{n+1}$$

since the geometric mean is no larger than the arithmetic mean. Hence $\delta \leq (n/(n+1))^{n+1} \uparrow e^{-1}$ as $n \to \infty$. Fix $\delta \in [0, e^{-1})$ and choose $\lambda = \mu_i = n/(n+1)$, $1 \leq i \leq n$. It follows from the above result that conditions (i)–(ii) are satisfied when n is sufficiently large, and thus $q_{1\delta}$ is the indiscrete topology when $\delta \in [0, e^{-1})$. Also, this implies that $cq_{1\delta}$ is the indiscrete topology when $\delta = e^{-1}$; hence $\sigma_\delta = cq_{1\delta}$ provided $\delta \in [0, e^{-1}]$.

Fix $\delta \in (e^{-1}, 1]$. First, it is shown that $\sigma_\delta \geq cq_{1\delta}$. Let $\rho \in [0, \delta)$, $\epsilon = 1 + \ln \delta$, and choose $\lambda = \mu_i = (n+\epsilon)/(n+1)$, $1 \leq i \leq n$. Since

$$\left(\frac{n+\epsilon}{n+1}\right)^{n+1} = \left(1 - \frac{1-\epsilon}{n+1}\right)^{n+1} \uparrow e^{\epsilon - 1} = \delta \quad \text{as} \quad n \to \infty,$$

it follows that $\mathrm{cl}_{q_{\mu_1 - \mu_n}} \mathscr{V}_{q_\lambda}(f) = \mathscr{V}_{q_\epsilon}(f)$, $A_n \geq \rho$ for n sufficiently large, and thus $\mathscr{V}_{q_\epsilon}(f) \xrightarrow{q_{1\rho}} f$. This shows that $\mathscr{V}_{\sigma_\delta}(f) \xrightarrow{q_{1\rho}} f$ for each $0 \leq \rho < \delta$ and thus

$\mathscr{V}_{\sigma_\delta}(f) \xrightarrow{cq_{1\delta}} f$. It follows that $\sigma_\delta \geq cq_{1\delta}$ when $\delta \in (e^{-1}, 1]$. Next, it is shown that $(X, \bar{\sigma})$ is T_{PR}-regular; that is, $\mathrm{cl}_{\sigma_\mu} \mathscr{V}_{\sigma_\lambda}(f) \geq \mathscr{V}_{\sigma_{\lambda\mu}}(f)$. Since $\sigma_{\lambda\mu}$ is the indiscrete topology when $\lambda\mu \leq e^{-1}$, assume that $\lambda\mu > e^{-1}$. Recall that $\sigma_{\lambda\mu} = q_{1+\ln(\lambda\mu)}$, and a typical base member of $\mathscr{V}_{\sigma_{\lambda\mu}}(f)$ is $V_{\gamma a}(f) = \{g \in X : P\{|g - f| \leq a\} \geq \gamma\}$ for some $a > 0$ and $0 < \gamma < 1 + \ln(\lambda\mu)$. Choose $b > 0$ for which $b < 1 + \ln(\lambda\mu) - \gamma$ and let ρ be such that $1 + \ln\lambda - b/2 < \rho < 1 + \ln\lambda$. Then $V_{\rho a/2}(f) \in \mathscr{V}_{\sigma_\lambda}(f)$ and it can be verified that $\mathrm{cl}_{\sigma_\mu} V_{\rho a/2}(f) \subseteq V_{\gamma a}(f)$ and thus $\mathrm{cl}_{\sigma_\mu} \mathscr{V}_{\sigma_\lambda}(f) \geq \mathscr{V}_{\sigma_{\lambda\mu}}(f)$. Hence $(X, \bar{\sigma})$ is T_{PR}-regular.

Fix $\lambda \in (e^{-1}, 1]$. Since $1 + \ln\lambda \leq \lambda$, $\sigma_\lambda = q_{1+\ln\lambda} \leq q_\lambda$, and thus $\bar{\sigma} \leq \bar{q}$. The T-regularity of $(X, \bar{\sigma})$ implies that $\bar{\sigma} \leq \bar{q}_1$. Moreover, $\bar{\sigma}$ is defined in terms of $\bar{q}$ and hence the left-continuity of $\bar{\sigma}$ follows from that of $\bar{q}$. Therefore $\bar{\sigma} \leq c\bar{q}_1$ and from the earlier result, $\bar{\sigma} = c\bar{q}_1$.

(d): As mentioned above (X, q_λ) is pseudo-metrizable when $\lambda = 1$. The details are deleted here but it can be verified that when $0 \leq \lambda < 1$, X is the only nonempty closed subset and thus q_λ is the indiscrete topology in this case.

The example $(X, \bar{q})$ above is also a T_P-probabilistic convergence vector space over R in the sense that for each fixed $\lambda, \mu \in I$, addition is a continuous operation from $(X, q_\mu) \times (X, q_\lambda) \to (X, q_{\lambda * \mu})$ and scalar multiplication is a continuous map from $R \times (X, q_\lambda) \to (X, q_\lambda)$. Furthermore, Proposition 4.2(c) can be improved at the expense of a slightly more technical argument. The results are listed below.

REMARK 4.3. Suppose that $(X, \bar{q})$ is the same space as in the example above and let $\{(X, \bar{q}_\alpha)\}$ be its T_{PR}-regular sequence. Then $q_{1\delta}$ is the indiscrete topology when $\delta \in [0, e^{-1})$, $q_{1\delta} = q_1$ when $\delta = 1$, and $\phi \xrightarrow{q_{1\delta}} f$ if and only if $\phi \geq \mathscr{V}_{q_{1+\ln\mu}}(f)$ for some $\mu > \delta$ when $\delta \in [e^{-1}, 1)$. Moreover, $q_{2\delta} \neq q_{1\delta}$ when $\delta \in [e^{-1}, 1)$ and thus $\ell_{R_T}(\bar{q}) > 1$ when $T = T_{PR}$.

5. Function space structures

A fundamental paper in the search for suitable topologies for function spaces is due to Arens [1]. Cook and Fisher [5] showed that more satisfactory results can be obtained in the larger category of convergence spaces. Given a t-norm T, appropriate function space structures in the category of probabilistic convergence spaces are considered in this section.

Assume $(X, \bar{q}), (Y, \bar{p}) \in |PCONV|$; define $\mathrm{Hom}(X, Y)$ to be the set of all continuous maps from $(X, \bar{q})$ into $(Y, \bar{p})$. If $(Z, \bar{r}) \in |PCONV|$, then a map $f : (X, \bar{q}) \times (Y, \bar{p}) \to (Z, \bar{r})$ is *T-continuous* provided $f(\mathscr{F} \times \mathscr{G}) \xrightarrow{r_{\lambda*\mu}} f(x, y)$ when $\mathscr{F} \xrightarrow{q_\lambda} x$ and $\mathscr{G} \xrightarrow{p_\mu} y$. Observe that T-continuity of f reduces to continuity when $T = T_M$. A

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probabilistic convergence structure $\bar{\theta}$ for $\mathrm{Hom}(X, Y)$ is called *T-splitting* if for every $(Z, \bar{r}) \in |PCONV|$, T-continuity of $f : (X, \bar{q}) \times (Z, \bar{r}) \to (Y, \bar{p})$ implies continuity of $f^* : (Z, \bar{r}) \to (\mathrm{Hom}(X, Y), \bar{\theta})$, where $f^*(z)(x) = f(x, z)$. Moreover, $\bar{\theta}$ is said to be *T-admissible* if the evaluation map $e : (X, \bar{q}) \times (\mathrm{Hom}(X, Y), \bar{\theta}) \to (Y, \bar{p})$, defined by $e(x, f) = f(x)$, is *T-continuous*.

PROPOSITION 5.1. *Let $(X, \bar{q}), (Y, \bar{p}) \in |PCONV|$; then there exists $\bar{\theta}$ which is both the finest T-splitting and coarsest T-admissible probabilistic convergence structure for* $\mathrm{Hom}(X, Y)$.

PROOF. Define $\bar{\theta} = (\theta_\lambda)$, $\lambda \in I$, as follows: $\phi \xrightarrow{\theta_\lambda} f$ if and only if when $\mathscr{F} \xrightarrow{q_\mu} x$, $\mu \in I$, $\phi(\mathscr{F}) \xrightarrow{p_{\lambda * \mu}} f(x)$, where $\phi(\mathscr{F})$ has a base of subsets of the form $A(F) = \{g(z) : g \in A, z \in F\}$, $A \in \phi$, $F \in \mathscr{F}$. It is straightforward to show that $(\mathrm{Hom}(X, Y), \bar{\theta}) \in |PCONV|$ and $\bar{\theta}$ has the desired properties.

Given the t-norm T; let us conclude by giving some basic properties of $(Y, \bar{p})$ which are inherited by $(\mathrm{Hom}(X, Y), \bar{\theta})$.

PROPOSITION 5.2. (a) $(\mathrm{Hom}(X, Y), \bar{\theta})$ *is T-regular (left-continuous, pseudo-topological) when $(Y, \bar{p})$ is T-regular (left-continuous, pseudo-topological), respectively.*
(b) $(\mathrm{Hom}(X, Y), \bar{\theta})$ *is constant when $(Y, \bar{p})$ is constant, provided $ab > 0$ implies* $T(a, b) > 0$.

PROOF. (a) Suppose that $\phi \xrightarrow{\theta_\lambda} f$; it must be shown that $\mathrm{cl}_{\theta_\mu} \phi \xrightarrow{\theta_{\lambda * \mu}} f$. Assume that $\mathscr{F} \xrightarrow{q_\delta} x$ for some $\delta \in I$; then $\phi(\mathscr{F}) \xrightarrow{p_{\lambda * \delta}} f(x)$ and since $(Y, \bar{p})$ is T-regular, $\mathrm{cl}_{p_\mu} \phi(\mathscr{F}) \xrightarrow{p_{\lambda * \mu * \delta}} f(x)$. It is shown that $(\mathrm{cl}_{\theta_\mu} \phi)(\mathscr{F}) \geq \mathrm{cl}_{p_\mu} \phi(\mathscr{F})$. Indeed, if $g \in \mathrm{cl}_{\theta_\mu} A$, then there exists $\psi \xrightarrow{\theta_\mu} g$ with $A \in \psi$. However, $\dot{x} \xrightarrow{q_1} x$ and thus $\psi(\dot{x}) \xrightarrow{p_\mu} g(x)$; it follows that $g(x) \in \mathrm{cl}_{p_\mu} A(x)$ and $g(F) \subseteq \mathrm{cl}_{p_\mu} A(F)$. Hence $(\mathrm{cl}_{\theta_\mu} A)(F) \subseteq \mathrm{cl}_{p_\mu} A(F)$ and thus $(\mathrm{cl}_{\theta_\mu} \phi)(\mathscr{F}) \geq \mathrm{cl}_{p_\mu} \phi(\mathscr{F})$. Therefore $(\mathrm{cl}_{\theta_\mu} \phi)(\mathscr{F}) \xrightarrow{p_{\lambda * \mu * \delta}} f(x)$ and $\mathrm{cl}_{\theta_\mu} \phi \xrightarrow{\theta_{\lambda * \mu}} f$; hence $(\mathrm{Hom}(X, Y), \bar{\theta})$ is T-regular.

Assume that $(Y, \bar{p})$ is left-continuous. It is shown that $\sup_{\mu < \lambda} \theta_\mu = \theta_\lambda$ when $0 < \lambda \leq 1$. Suppose that $\phi \xrightarrow{\theta_\mu} f$ for each $\mu < \lambda$ and $\mathscr{F} \xrightarrow{q_\delta} x$. Then $\phi(\mathscr{F}) \xrightarrow{p_{\delta * \mu}} f(x)$ for each $\mu < \lambda$ and the left-continuity of $(Y, \bar{p})$ implies that $\phi(\mathscr{F}) \xrightarrow{p_{\delta * \lambda}} f(x)$. Hence $\phi \xrightarrow{\theta_\lambda} f$ and it follows that $(\mathrm{Hom}(X, Y), \bar{\theta})$ is also left-continuous. Proof of the last part of (a) is deleted since it closely resembles that given in the convergence space setting [8].

(b) Suppose that $\bar{p} = (p_\lambda)$ is constant. It must be shown that if $0 < \mu < \lambda \leq 1$, then $\theta_\mu = \theta_\lambda$. Assume that $\phi \xrightarrow{\theta_\mu} f$ and $\mathscr{F} \xrightarrow{q_\delta} x, \delta \in I$. If $\delta = 0$, then $\phi(\mathscr{F}) \xrightarrow{p_0} f(x)$

since p_0 is the indiscrete topology. Suppose that $\delta > 0$; then $\phi(\mathscr{F}) \xrightarrow{p_{\mu*\delta}} f(x)$ and, by hypothesis, $\mu * \delta > 0$ and thus $\phi(\mathscr{F}) \xrightarrow{p_{\lambda*\delta}} f(x)$ since $\bar{p}$ is constant. Hence $\phi \xrightarrow{\theta_\lambda} f$ and therefore $\bar{\theta}$ is constant.

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