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Infinite Classes of Covering Numbers

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Abstract. Let D be a family of k-subsets (called blocks) of a v-set $X(v)$. Then D is a (v, k, t) covering design or covering if every t-subset of $X(v)$ is contained in at least one block of D. The number of blocks is the size of the covering, and the minimum size of the covering is called the covering number. In this paper we consider the case $t = 2$, and find several infinite classes of covering numbers. We also give upper bounds on other classes of covering numbers.

1 Introduction

First we discuss some facts and notation that will be used throughout the paper. Let $D = \{B_1, B_2, \ldots, B_b\}$ be a finite family of k-subsets (called *blocks*) of a v-set $X(v) = \{1, 2, \ldots, v\}$ (with elements called *points*). Then D is a (v, k, t) *covering design* or *covering* if every t-subset of $X(v)$ is contained in at least one block of D. The number of blocks, b, is the *size* of the covering, and the minimum size of the covering is called the *covering number*, denoted $C(v, k, t)$. If every t-subset of $X(v)$ is contained in exactly one block of D, then D is a *Steiner system*, denoted $S(v, k, t)$. A Steiner system is said to be *resolvable* if there exists a partition of its set of blocks into subsets called *resolution classes* each of which in turn partitions the set $X(v)$. A *transversal design* of group-size n, and block-size k, denoted $\mathrm{TD}(k, n)$, is a triple (V, G, D), where V is a set of kn elements; G is a partition of V into k classes (the *groups*), each of size n; B is a collection of k-subsets of V (the *blocks*), and every unordered pair of elements from V is either contained in exactly one group or exactly one block, but not both. There exist transversal designs $\mathrm{TD}(n + 1, n)$ for all prime powers n. The existence of a $\mathrm{TD}(k, n)$ implies the existence of a $\mathrm{TD}(k - 1, n)$. Since, in some of our constructions, we use a $\mathrm{TD}(n-1, n)$, it is worth noting a result of Shrikhande's [7], namely that a $\mathrm{TD}(n-1, n)$ can always be extended to a $\mathrm{TD}(n+1, n)$ (with the sole exception of one of the two $\mathrm{TD}(3, 4)$ designs). Let K and G be sets of positive integers and let λ be a positive integer. A *group divisible design* (of index λ and order v) is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$, where $\mathcal{V}$ is a finite set of cardinality v, $\mathcal{G}$ is a partition of $\mathcal{V}$ into parts (*groups*) whose sizes lie in G, and $\mathcal{B}$ is a family of subsets (*blocks*) of $\mathcal{V}$ which satisfies the properties:

1. If $B \in \mathcal{B}$, then $|B| \in K$.
2. Every pair of distinct elements of $\mathcal{V}$ occurs in exactly λ blocks or one group, but not both.
3. $|\mathcal{G}| > 1$.

A general lower bound on $C(v, k, t)$ is due to Schönheim [6].

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Theorem 1.1

$$C(v,k,t) \geq \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \cdots \left\lceil \frac{v-t+1}{k-t+1} \right\rceil \cdots \right\rceil \right\rceil .$$

Let $X^{(k)}(v)$ denote the set of all k-subsets of $X(v)$. A t-$(v, \{k_1, k_2, \dots, k_n\}, \lambda)$ *design* is a pair $\big(X(v), D\big)$, where $X(v) = \{1, 2, \dots, v\}$ is a set of points and D is a subset of $X^{(k_1)}(v) \cup X^{(k_2)}(v) \cup \cdots \cup X^{(k_n)}(v)$ with elements called blocks (of size $k_1, k_2, \dots, k_n$) so that every t-set of $X(v)$ is contained in exactly λ blocks.

There is an extensive literature on the covering numbers $C(v,k,t)$. For excellent surveys on the known results we refer to [3] and [5]. In this work we continue the search for classes of covering numbers started in [2].

2 Main results

Theorem 2.1 *If n is a power of an odd prime, then*

$$C(n^2 - n, n-1, 2) = n^2 + 2n.$$

Proof We start with an affine plane of order n; that is, a resolvable Steiner system $S(n^2, n, 2)$. It has $n+1$ parallel classes, $P_1, P_2, \dots, P_{n+1}$, each containing n lines. Let

$$P_i = \{B_{ij} \mid j = 1, 2, \dots, n\}, \quad i = 1, 2, \dots, n+1$$

be the parallel classes, where the B_{ij} are the lines. Now let us remove a line, say $B_{n+1,n}$, and all of its points from the remaining blocks. The line $B_{n+1,n}$ intersects each of the lines of any of the parallel classes $P_1, P_2, \dots, P_n$ in exactly one point. Consider the union of the set of blocks

$$\mathcal{A} = \{B_{ij} \setminus B_{n+1,n} \mid 1 \leq i, j \leq n\}$$

and $P_{n+1} \setminus B_{n+1,n}$. This is a 2-$(n^2 - n, \{n, n-1\}, 1)$ design with $(n-1)$ blocks of size n (the blocks of $P_{n+1} \setminus B_{n+1,n}$) and n^2 blocks of size $(n-1)$ (the blocks of $\mathcal{A}$). We shall use the $(n-1)$ blocks of size n to construct $2n$ blocks of size $(n-1)$ covering all the pairs contained in $P_{n+1} \setminus B_{n+1,n}$. Let

$$B_{n+1,1} = a_{11}\ a_{12}\ \dots\ a_{1n}$$
$$B_{n+1,2} = a_{21}\ a_{22}\ \dots\ a_{2n}$$
$$\cdots$$
$$B_{n+1,n-1} = a_{n-1,1}\ a_{n-1,2}\ \dots\ a_{n-1,n}$$

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be the lines of $P_{n+1} \setminus B_{n+1,n}$. Now it is easy to check that the $2n$ blocks

$$\begin{array}{cccc} a_{12} & a_{13} & \dots & a_{1n} \\ a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,n} \\ a_{11} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n-1,1} & a_{n-1,3} & \dots & a_{n-1,n} \end{array}$$

$$a_{11}\ a_{12}\ a_{21}\ a_{22}\ \dots\ a_{\frac{n-1}{2},1}\ a_{\frac{n-1}{2},2}$$

$$a_{\frac{n+1}{2},1}\ a_{\frac{n+1}{2},2}\ a_{\frac{n+3}{2},1}\ a_{\frac{n+3}{2},2}\ \dots\ a_{n-1,1}\ a_{n-1,2}$$

cover all the pairs contained in the blocks of $P_{n+1} \setminus B_{n+1,n}$. Let us denote the $2n$ blocks listed above by $\mathcal{C}$. It is clear now that we can substitute the blocks of $P_{n+1} \setminus B_{n+1,n}$ with the blocks of $\mathcal{C}$ in the 2-$(n^2-n, \{n, n-1\}, 1)$ design to get an $(n^2-n, n-1, 1)$ covering of size n^2+2n.

What remains to be shown is that the covering number $C(n^2-n, n-1, 2)$ is exactly n^2+2n. Schönheim's theorem applied to an $(n^2-n, n-1, 2)$ covering yields

$$C(n^2-n, n-1, 2) \geq \left\lceil \frac{n^2-n}{n-1} \left\lceil \frac{n^2-n-1}{n-2} \right\rceil \right\rceil = n \left\lceil \frac{n^2-n-1}{n-2} \right\rceil .$$

Since $\lceil \frac{n^2-n-1}{n-2} \rceil = n+2$, $C(n^2-n, n-1, 2) \geq n^2+2n$, which completes the proof. ■

Theorem 2.2 *If $n = 2^k$, $k \geq 2$, then*

$$n^2 + 2n \leq C(n^2-n, n-1, 2) \leq n^2 + 2n + 1.$$

Proof We use the proof of the preceding theorem except for the last two blocks of $\mathcal{C}$. Since n is even, three blocks are required to cover the $(n-1)$ pairs $a_{i1}a_{i2}$, $i = 1, 2, \dots, n-1$. ■

Note that it is not known whether this construction produces any covering numbers; it does not for $n = 4$ where a covering with $n^2 + 2n$ blocks meeting the Schönheim bound is known; that is, $C(12, 3, 2) = 24$ [5].

We now generalize the above construction. What we essentially used was a transversal design $\mathrm{TD}(n-1, n)$, where n is a prime power. We took the n^2 blocks of the $\mathrm{TD}(n-1, n)$, then covered the pairs contained in the groups of the transversal design by forming two blocks from the points of each group, and covering the remaining pairs by two (three) additional blocks. So, we shall proceed by using the same $\mathrm{TD}(n-1, n)$, noting that we can extend each group by a common set of new points and use the same type of construction to cover the pairs contained in each of the extended groups. We introduce new points $X = \{x_1, x_2, \dots, x_r\}$. Using the notation from Theorem 2.1, the expanded groups are

$X \cup B_{n+1,i}$, $i = 1, 2, \ldots, n-1$. For each of the expanded groups we form two blocks of size $(n-1)$ and a third block of size $(2r+2)$:

$$B^1_{n+1,i} = x_1\ x_2\ \ldots\ x_r\ a_{i,r+2}\ a_{i,r+3}\ \ldots\ a_{i,n}$$
$$B^2_{n+1,i} = a_{i,1}\ a_{i,2}\ \ldots\ a_{i,r}\ a_{i,r+1}\ a_{i,r+3}\ \ldots\ a_{i,n}$$
$$B^3_{n+1,i} = x_1\ x_2\ \ldots\ x_r\ a_{i,1}\ a_{i,2}\ \ldots\ a_{i,r+2}.$$

We certainly want $2r + 2 \le n - 1$; that is, $r \le \frac{n-3}{2}$. It is clear that these three blocks cover all pairs contained in $X \cup B_{n+1,i}$. Now, if $(r+2)$ is small compared to $(n-1)$ we will be able to unite sets of $s = \lfloor \frac{n-1-r}{r+2} \rfloor$ "third" blocks into single blocks of size at most $(n-1)$ containing all pairs from the corresponding third blocks. So, we form the blocks:

$$x_1\ x_2\ \ldots\ x_r\ \ a_{1,1}\ a_{1,2}\ \ldots\ a_{1,r+2}\ \ a_{2,1}\ a_{2,2}\ \ldots\ a_{2,r+2}\ \ldots\ a_{s,1}\ a_{s,2}\ \ldots\ a_{s,r+2}\ \ *$$
$$x_1\ x_2\ \ldots\ x_r\ \ a_{s+1,1}\ a_{s+1,2}\ \ldots\ a_{s+1,r+2}\ \ a_{s+2,1}\ a_{s+2,2}\ \ldots\ a_{s+2,r+2}\ \ldots\ a_{2s,1}\ a_{2s,2}\ \ldots\ a_{2s,r+2}\ \ *$$
$$\ldots$$
$$x_1\ x_2\ \ldots\ x_r\ \ a_{qs+1,1}\ a_{qs+1,2}\ \ldots\ a_{qs+1,r+2}\ \ldots\ a_{n-1,1}\ a_{n-1,2}\ \ldots\ a_{n-1,r+2}\ \ *$$

The asterisk at the end of each block denotes the remaining $(n-1) - \big((r+2)s + r\big)$ positions for the first q blocks and $(n-1) - \big(r + (r+2)(n-1-s\lfloor \frac{n-1}{s} \rfloor)\big)$ for the last block. Clearly, $\lceil \frac{n-1}{s} \rceil = \lceil \frac{n-1}{\lfloor \frac{n-1-r}{r+2} \rfloor} \rceil$ is the number of blocks needed to cover all pairs contained in the third blocks of the extended groups. Thus we have proved the following:

Theorem 2.3 *If $0 \le r \le \frac{n-3}{2}$, where $n \ge 3$ is a prime power, then*

$$C(n^2 - n + r, n-1, 2) \le n^2 + 2(n-1) + \left\lceil \frac{n-1}{\lfloor \frac{n-1-r}{r+2} \rfloor} \right\rceil.$$

The interesting question now is whether this construction produces new covering numbers. In what follows we answer this question affirmatively. First we calculate the Schönheim bound for an $(n^2 - n + r, n-1, 2)$ covering, where $0 \le r \le \frac{n-3}{2}$.

$$\begin{aligned} C(n^2 - n + r, n-1, 2) &\ge \left\lceil \frac{n^2 - n + r}{n-1} \left\lceil \frac{n^2 - n + r - 1}{n-2} \right\rceil \right\rceil \\ &= \left\lceil \frac{n^2 - n + r}{n-1} \left(n + 1 + \left\lceil \frac{r+1}{n-2} \right\rceil \right) \right\rceil \\ &= \left\lceil \frac{(n^2 - n + r)(n+2)}{n-1} \right\rceil \end{aligned}$$

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$$= n^2 + 2n + r + \left\lceil \frac{3r}{n-1} \right\rceil$$
$$= n^2 + 2(n-1) + r + 2 + \epsilon,$$

where

$$\epsilon = \left\lceil \frac{3r}{n-1} \right\rceil = \begin{cases} 0 & \text{if } r = 0, \\ 1 & \text{if } 0 < r \leq \frac{n-1}{3} \text{ and } n \geq 7, \\ & \text{or } 0 < r \leq \frac{n-3}{2} \text{ and } n < 7, \\ 2 & \text{if } \frac{n-1}{3} < r \leq \frac{n-3}{2}. \end{cases}$$

The question is: When does $r + 2 + \epsilon = \left\lceil \frac{n-1}{\lfloor \frac{n-1-r}{r+2} \rfloor} \right\rceil$?

Case 1. $r = 0$**.** Then $\epsilon = 0$, so we need to solve $2 = \left\lceil \frac{n-1}{\lfloor \frac{n-1}{2} \rfloor} \right\rceil$, which is true only for odd n. Thus we get $C(n^2 - n, n-1, 2) = n^2 + 2n$, where n is a power of an odd prime; that is, the result from Theorem 2.1.

Case 2. $0 < r \leq \frac{n-1}{3}$**,** $n \geq 7$**.** Now, $\epsilon = 1$, so we have to find the solutions to

$$r + 3 = \left\lceil \frac{n-1}{\lfloor \frac{n-1-r}{r+2} \rfloor} \right\rceil, \tag{1}$$

which is equivalent to

$$r + 2 < \frac{n-1}{\lfloor \frac{n-1-r}{r+2} \rfloor} \leq r + 3,$$

or,

$$\frac{n-1}{r+3} \leq \left\lfloor \frac{n-1-r}{r+2} \right\rfloor < \frac{n-1}{r+2}. \tag{2}$$

The right hand inequality of (2) is always true, so (1) is equivalent to

$$\frac{n-1}{r+3} \leq \left\lfloor \frac{n-1-r}{r+2} \right\rfloor. \tag{3}$$

Finally, condition (3) is satisfied if and only if $\left[\frac{n-1}{r+3}, \frac{n-1-r}{r+2}\right]$ is an interval and contains an integer, so we get the following.

Theorem 2.4 *If $n \geq 7$ is a prime power and $1 \leq r \leq \frac{n-1}{3}$ is an integer such that $\left[\frac{n-1}{r+3}, \frac{n-1-r}{r+2}\right]$ is an interval containing an integer, then*

$$C(n^2 - n + r, n-1, 2) = (n+1)^2 + r.$$

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n	Range of r	n	Range of r	n	Range of r
7	-	29	1–4	64	1–6
8	1	31	1–3	67	1–4
9	1	32	1–2	71	1–7
11	1–2	37	1–3	73	1–6
13	1	41	1–5	79	1–6
16	1–2	43	1–4	81	1–7
17	1	47	1–4	83	1–5
19	1–3	49	1–5	89	1–8
23	1–2	53	1–4	97	1–6
25	1–3	59	1–4		
27	1–2	61	1–4		

Table 1: New covering numbers for $n < 100$.

Note that we have to test only pairs (n, r) such that $\left[\frac{n-1}{r+3}, \frac{n-1-r}{r+2}\right]$ is indeed an interval, that is, $\frac{n-1}{r+3} \le \frac{n-1-r}{r+2}$. The last inequality is equivalent to $r^2 + 3r + (1 - n) \le 0$, which gives the following range for r:

$$1 \le r \le \left\lfloor \frac{\sqrt{5+4n} - 3}{2} \right\rfloor. \tag{4}$$

Since $\frac{\sqrt{5+4n}-3}{2} \le \frac{n-1}{3}$ for every n, (4) is a necessary (but not sufficient) condition for the existence of a solution to equation (1).

The Table 2 gives new covering numbers produced by Theorem 2.4 in the range $n < 100$ for n a prime power.

Case 3. **$0 < r \le \frac{n-3}{2}$, $n < 7$.** In this case, $n = 3, 4$, or 5. If $n = 3$ or 4 we do not get any values for r, so the only possibility is $n = 5$. Then $r = 1$, and we obtain $C(21, 4, 2) = 25 + 10 + 1 + 1 = 37$ (as given in [5]).

Case 4. **$\frac{n-1}{3} < r \le \frac{n-3}{2}$, $n > 7$.** Now, $\epsilon = 2$, so we want to solve

$$r + 4 = \left\lceil \frac{n-1}{\left\lfloor \frac{n-1-r}{r+2} \right\rfloor} \right\rceil.$$

The restrictions on r imply $\lfloor \frac{n-1-r}{r+2} \rfloor = 1$, so the equation becomes $r + 4 = n - 1$. Thus $r = n - 5$, which is larger than $\frac{n-3}{2}$, so we do not get any solutions in this case.

We proceed with another generalization of the construction from Theorem 2.1; initially without introducing new points. Instead of starting with a $\mathrm{TD}(n-1, n)$, we start with a $\mathrm{TD}(n-m, n)$, where $m \ge 1$ and $n \ge 3m$. We basically follow the same construction as in Theorems 2.1 and 2.3: first take the n^2 blocks of size $n - m$ from the $\mathrm{TD}(n-m, n)$, then form two blocks of size $n-m$ from each of the $n-m$ groups of the $\mathrm{TD}(n-m, n)$, and finally form blocks of size $n-m$ to cover the pairs contained in the $(m+m)$-sets remaining in each

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group. We certainly want $n - m \geq 2m$, that is, $m \leq \frac{n}{3}$. Thus we get an $(n^2 - mn, n - m, 2)$ covering of size

$$n^2 + 2(n-m) + \left\lceil \frac{n-m}{\lfloor \frac{n-m}{2m} \rfloor} \right\rceil,$$

which proves the following.

Theorem 2.5 *If $n \geq 3$ is a prime power and $1 \leq m \leq \frac{n}{3}$, then*

$$C(n^2 - mn, n-m, 2) \leq n^2 + 2(n-m) + \left\lceil \frac{n-m}{\lfloor \frac{n-m}{2m} \rfloor} \right\rceil.$$

The Schönheim bound for an $(n^2 - mn, n - m, 2)$ covering, where $1 \leq m \leq \frac{n}{3}$, is

$$\begin{aligned} C(n^2 - mn, n-m, 2) &\geq \left\lceil \frac{n^2 - mn}{n-m} \left\lceil \frac{n^2 - mn - 1}{n-m-1} \right\rceil \right\rceil \\ &= \left\lceil n \left\lceil n + 1 + \frac{m}{n-m-1} \right\rceil \right\rceil \\ &= n^2 + 2n \\ &= n^2 + 2(n-m) + 2m. \end{aligned}$$

Now, if $2m = \left\lceil \frac{n-m}{\lfloor \frac{n-m}{2m} \rfloor} \right\rceil$, then we get a covering number. But $2m = \left\lceil \frac{n-m}{\lfloor \frac{n-m}{2m} \rfloor} \right\rceil$ if and only if $2m$ divides $n - m$, and we apply the following.

Claim 2.6 *If $n = p^a$, where p is a prime and $n > m$, then $2m$ divides $(n - m)$ if and only if $m = p^b$, $0 \leq b \leq a - 1$, and p is odd.*

Proof ($\Leftarrow$) $n - m = p^a - p^b = p^b(p^{a-b} - 1)$ is divisible by $2p^b = 2m$ for p odd.
($\Rightarrow$) Now, let $n = p^a$, and let $m = k \cdot p^b$, where $(p, k) = 1$. Then

$$\frac{p^a - kp^b}{2kp^b} = \frac{(p^{a-b} - k)}{2k}$$

is an integer, so k divides p^{a-b}. However, $(k, p) = 1$; consequently, $k = 1$, that is, $m = p^b$. Furthermore, $2k$ divides $p^{a-b} - k$, and since $k = 1$, p must be odd. ■

Thus we prove that the construction preceding the claim produces covering numbers if and only if n and m are powers of the same odd prime. More precisely, we have the following.

Theorem 2.7 *If p is an odd prime and $0 \leq b \leq a - 1$, then*

$$C(p^{2a} - p^{a+b}, p^a - p^b, 2) = p^{2a} + 2p^a.$$

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Note that any particular value of b produces a new infinite class of covering numbers.

We can now generalize Theorem 2.3 by starting with a $\mathrm{TD}(n-m,n)$, $m \geq 1$, and extending each group by r new points. The result is the following.

Theorem 2.8 *If n is a prime power, m is an integer such that $n \geq 3m$, and r is an integer such that $0 \leq r \leq \frac{n-3m}{2}$, then*

$$C(n^2-mn+r,n-m,2) \leq n^2+2(n-m)+\left\lceil \frac{n-m}{\lfloor \frac{n-m-r}{2m+r} \rfloor} \right\rceil.$$

The Schönheim bound for an $(n^2-mn+r,n-m,2)$ covering, where $0 \leq r \leq \frac{n-3m}{2}$, is (after routine calculations):

$$C(n^2-mn+r,n-m,2) \geq n^2+2(n-m)+2m+r+\left\lceil \frac{r(m+2)}{n-m} \right\rceil.$$

Consequently, if

$$2m+r+\left\lceil \frac{r(m+2)}{n-m} \right\rceil = \left\lceil \frac{n-m}{\lfloor \frac{n-m-r}{2m+r} \rfloor} \right\rceil,$$

then we get a covering number. This allow us to formulate the following general result.

Theorem 2.9 *If $n \geq 3$ is a prime power, m is an integer such that $1 \leq m \leq \frac{n}{3}$, r is an integer such that $0 \leq r \leq \frac{n-3m}{2}$, and (n,m,r) is a solution to the equation*

$$2m+r+\left\lceil \frac{r(m+2)}{n-m} \right\rceil = \left\lceil \frac{n-m}{\lfloor \frac{n-m-r}{2m+r} \rfloor} \right\rceil, \tag{5}$$

then

$$C(n^2-mn+r,n-m,2) = n^2+2n+r+\left\lceil \frac{r(m+2)}{n-m} \right\rceil.$$

By studying the solutions of equation (5) we are able to extract infinite classes of covering numbers. As much as possible, we restrict ourselves to following the discussion after Theorem 2.3. Difficulties arise because of the many possible values of $\epsilon = \lceil \frac{r(m+2)}{n-m} \rceil$ in the general case. Nevertheless, by adapting Cases 2 and 3 ($\epsilon = 1$) we obtain some explicit results.

Let $\epsilon = 1$. Then $\frac{r(m+2)}{n-m} \leq 1$, so $1 \leq r \leq \frac{n-m}{m+2}$. Equation (5) becomes

$$2m+r+1 = \left\lceil \frac{n-m}{\lfloor \frac{n-m-r}{2m+r} \rfloor} \right\rceil. \tag{6}$$

Comparing the upper bounds $\frac{n-m}{m+2}$ and $\frac{n-3m}{2}$ on r, we see that $\frac{n-m}{m+2} \leq \frac{n-3m}{2}$ if $n \geq 3m+4$, and $\frac{n-m}{m+2} \geq \frac{n-3m}{2}$ if $3m \leq n \leq 3m+3$. Thus $\epsilon = 1$ if a) $n \geq 3m+4$ and $1 \leq r \leq \frac{n-m}{m+2}$, or b) $3m \leq n \leq 3m+3$ and $1 \leq r \leq \frac{n-3m}{2}$.

Let us consider case a). By an argument similar to Case 2 (from the discussion after Theorem 2.3), we see that equation (6) is equivalent to the condition that $\left[\frac{n-m}{2m+r+1}, \frac{n-m-r}{2m+r}\right]$ is an interval containing an integer. This proves the following.

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Theorem 2.10 *If n is a prime power, m an integer such that $n \geq 3m + 4$, and $r \geq 1$ is an integer such that $\left[\frac{n-m}{2m+r+1}, \frac{n-m-r}{2m+r}\right]$ is an interval containing an integer, then*

$$C(n^2 - mn + r, n - m, 2) = (n + 1)^2 + r.$$

The necessary condition (4), which ensures that $\left[\frac{n-m}{2m+r+1}, \frac{n-m-r}{2m+r}\right]$ is an interval, is replaced by:

$$1 \leq r \leq \left\lfloor \frac{\sqrt{4m^2 + 4n + 1} - (2m + 1)}{2} \right\rfloor. \tag{7}$$

Note that the condition $r \leq \frac{n-3m}{2}$ is weaker than the necessary condition (7). Thus Theorem 2.10 produces covering numbers even if m is not a power of the same prime as n (or power of a prime at all). It also produces covering numbers when n is a power of two. For example,

$n = 13, m = 2, r = 1$ gives $C(144, 11, 2) = 197$;
$n = 16, m = 2, r = 2$ gives $C(226, 14, 2) = 291$;
$n = 59, m = 6, r = 1$ gives $C(3128, 53, 2) = 3601$.

Now, let us consider case b) (an extension of Case 3). If $n = 3m$ or $n = 3m + 1$ we get no values of r. If $n = 3m + 2$, then $r = 1$, and the equation (6) holds for every m, so we obtain the following.

Theorem 2.11 *If $m = \frac{n-2}{3}$, where $n \equiv 2 \pmod 3$ is a prime power, then*

$$C(6m^2 + 10m + 5, 2m + 2, 2) = 9m^2 + 18m + 10.$$

Note that letting $m = 2$ in the above result produces the "lottery" covering number $C(49, 6, 2) = 82$.

If $n = 3m + 3$, then, again, $r = 1$. Equation (6) is not satisfied for any m.

Rather than trying to extract further infinite classes of covering numbers from Theorem 2.9, we restrict ourselves to listing all the triples (n, m, r) producing covering numbers from Theorem 2.9 for $n \leq 40$ (see Table 2).

The idea of taking a group divisible design (such as a transversal design), and then covering the groups in some way, is a natural approach to producing a covering design. Todorov [8, Theorem 4] found the construction we have given in Theorem 2.5; however he did not investigate when this construction met the Schönheim bound (nor did he add extra points to the basic design).

As a further example of the use of group divisible designs we give the following.

Theorem 2.12 *If there exists a group divisible design with n groups of size m, blocks of size $k \geq m$, and $\lambda = 1$, then*

$$C(mn, k, 2) \leq \frac{m^2 n(n-1)}{k(k-1)} + \left\lceil \frac{n}{\lfloor \frac{k}{m} \rfloor} \right\rceil. \tag{8}$$

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n	m	r	Covering number	n	m	r	Covering number
3	1	0	$C(6,2,2)=15$	5	1	0	$C(20,4,2)=35$
5	1	1	$C(21,4,2)=37$	7	1	0	$C(42,6,2)=63$
8	1	1	$C(57,7,2)=82$	8	2	1	$C(49,6,2)=82$
9	1	0	$C(72,8,2)=99$	9	1	1	$C(73,8,2)=101$
9	3	0	$C(54,6,2)=99$	11	1	0	$C(110,10,2)=143$
11	1	1	$C(111,10,2)=145$	11	1	2	$C(112,10,2)=146$
11	3	1	$C(89,8,2)=145$	13	1	0	$C(156,12,2)=195$
13	1	1	$C(157,12,2)=197$	13	2	1	$C(144,11,2)=197$
16	1	1	$C(241,15,2)=290$	16	1	2	$C(242,15,2)=291$
16	2	2	$C(226,14,2)=291$	17	1	0	$C(272,16,2)=323$
17	1	1	$C(273,16,2)=325$	17	5	1	$C(205,12,2)=325$
19	1	0	$C(342,18,2)=399$	19	1	1	$C(343,18,2)=401$
19	1	2	$C(344,18,2)=402$	19	1	3	$C(345,18,2)=403$
19	2	1	$C(324,17,2)=401$	19	3	1	$C(305,16,2)=401$
23	1	0	$C(506,22,2)=575$	23	1	1	$C(507,22,2)=577$
23	1	2	$C(508,22,2)=578$	23	2	1	$C(484,21,2)=577$
23	2	2	$C(485,21,2)=578$	23	4	1	$C(438,19,2)=577$
23	7	1	$C(369,16,2)=577$	25	1	0	$C(600,24,2)=675$
25	1	1	$C(601,24,2)=677$	25	1	2	$C(602,24,2)=678$
25	1	3	$C(603,24,2)=679$	25	2	1	$C(576,23,2)=677$
25	3	1	$C(551,22,2)=677$	25	5	0	$C(500,20,2)=675$
27	1	0	$C(702,26,2)=783$	27	1	1	$C(703,26,2)=785$
27	1	2	$C(704,26,2)=786$	27	3	0	$C(648,24,2)=783$
27	3	1	$C(649,24,2)=785$	27	9	0	$C(486,18,2)=783$
29	1	0	$C(812,28,2)=899$	29	1	1	$C(813,28,2)=901$
29	1	2	$C(814,28,2)=902$	29	1	3	$C(815,28,2)=903$
29	1	4	$C(816,28,2)=904$	29	2	1	$C(784,27,2)=901$
29	2	2	$C(785,27,2)=902$	29	3	2	$C(756,26,2)=902$
29	5	1	$C(697,24,2)=901$	29	9	1	$C(581,20,2)=901$
31	1	0	$C(930,30,2)=1023$	31	1	1	$C(931,30,2)=1025$
31	1	2	$C(932,30,2)=1026$	31	1	3	$C(933,30,2)=1027$
31	2	1	$C(900,29,2)=1025$	31	5	2	$C(808,26,2)=1026$
32	1	1	$C(993,31,2)=1090$	32	1	2	$C(994,31,2)=1091$
32	2	1	$C(961,30,2)=1090$	32	3	1	$C(929,29,2)=1090$
32	4	1	$C(897,28,2)=1090$	32	10	1	$C(705,22,2)=1090$
37	1	0	$C(1332,36,2)=1443$	37	1	1	$C(1333,36,2)=1445$
37	1	2	$C(1334,36,2)=1446$	37	1	3	$C(1335,36,2)=1447$
37	2	1	$C(1296,35,2)=1445$	37	2	2	$C(1297,35,2)=1446$
37	3	2	$C(1260,34,2)=1446$	37	4	2	$C(1223,33,2)=1446$

Table 2: Covering numbers from Theorem 2.9 for $n \leq 40$.

Proof The number of blocks of the group divisible design is $\frac{m^2 n(n-1)}{k(k-1)}$. The pairs contained in the groups can be covered by $\left\lceil \frac{n}{\lfloor \frac{k}{m} \rfloor} \right\rceil$ additional blocks of size k. ■

Obviously, if the Schönheim bound $\lceil \frac{mn}{k} \lceil \frac{mn-1}{k-1} \rceil \rceil$ equals the right hand side of inequality (8), then so does the covering number $C(mn, k, 2)$. We know of three cases, where this observation can be applied to find new covering numbers. The first two are due to Mathon [4]: group divisible designs with 33 groups of size 3 and block-size 9, and 45 groups of size 3 and block-size 12. Another case is Baker's design [1] with 15 groups of size 3 and block-size 7. These lead to the following.

Corollary 2.13 $C(99, 9, 2) = 143$, $C(135, 12, 2) = 147$ *and* $C(46, 7, 2) = 53$.

3 Packings

Although the primary focus of this paper is on covering designs, it is worth mentioning packing designs. With the same notation as in the introduction, a (v, k, t) *packing design* or *packing* is a family of k-subsets (blocks) of $X(v)$ such that every t-subset of $X(v)$ is contained in *at most* one block of the design. The *packing number* $D(v, k, t)$ is the maximum size of a (v, k, t) packing. There is also a general upper bound on packings due to Schönheim.

Theorem 3.1

$$D(v, k, t) \geq \left\lfloor \frac{v}{k} \left\lfloor \frac{v-1}{k-1} \cdots \left\lfloor \frac{v-t+1}{k-t+1} \right\rfloor \cdots \right\rfloor \right\rfloor = U(v, k, t).$$

If we take the blocks of a $\mathrm{TD}(n-1, n)$, plus a block of $n-1$ points from each of the $n-1$ groups (thereby omitting one point per group), then we have a packing design of n^2+n-1 blocks. Using this and the Schönheim packing bound we obtain the following.

Theorem 3.2 *If $n \geq 3$ is a prime power, then*

$$n^2 + n - 1 \leq D(n^2 - n, n-1, 2) \leq n^2 + n = U(n^2, n-1, 2).$$

Theorem 3.2 can be extended to the following.

Theorem 3.3 *If there exists a group divisible design with $\lambda = 1$, v points, block size k and every group of size less than k, then that design is a packing with $U(v, k, 2)$ blocks.*

Proof Note that a point in a group of size g must meet each of the $v - g$ points outside its group exactly once, so $k-1$ divides $v-g$. Now suppose g_1 and g_2 are the sizes of two groups and $g_1 \geq g_2$. Since $k-1$ divides $v - g_1$ and $v - g_2$, it divides their difference. But $0 \leq (v - g_2) - (v - g_1) = g_1 - g_2 \leq (k-1) - 1$, so we see that all groups are of the same size. Next we note that

$$\frac{v-g}{k-1} \leq \left\lfloor \frac{v-1}{k-1} \right\rfloor < \frac{v-1+k-g}{k-1} = \frac{v-g}{k-1} + 1$$

so the first inequality is actually an equality, and the number of blocks in the group divisible design is $\bigl(v(v-g)\bigr)/\bigl(k(k-1)\bigr)$ which is $U(v, k, 2)$. ■

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Corollary 3.4 $D(45, 7, 2) = 45$, $D(99, 9, 2) = 132$, *and* $D(135, 12, 2) = 135$.

Proof Use Baker's design [1] with $k = 7$ and 15 groups of size 3, and the two Mathon designs [4] with $k = 9$ and 33 groups of size 3, and $k = 12$ and 45 groups of size 3 in Theorem 3.3. ■

References

[1] R. D. Baker, *An elliptic semiplane*. J. Combin. Theory Ser. A **25**(1978), 193–195.

[2] I. Bluskov, *Some t-Designs are Minimal $(t+1)$-Coverings*. Discrete Math. **188**(1998), 245–251.

[3] D. M. Gordon, G. Kuperberg and O. Patashnik, *New Constructions for Covering Designs*. J. Combin. Des. **3**(1995), 269–284.

[4] R. Mathon, *On a new divisible semiplane*. Unpublished manuscript.

[5] W. H. Mills and R. C. Mullin, *Coverings and Packings*. In: Contemporary Design Theory: A Collection of Surveys (eds. J. H. Dinitz and D. R. Stinson), Wiley, 1992, 371–399.

[6] J. Schönheim, *On Coverings*. Pacific J. Math. **14**(1964), 1405–1411.

[7] S. S. Shrikhande, *A note on mutually orthogonal Latin squares*. Shankhyā A **23**(1961), 115–116.

[8] D. T. Todorov, *Some coverings derived from finite planes*. In: Finite and Infinite Sets, Vol. 2 (eds. A. Hajnal, L. Lovász and V. T. Sós), Colloq. Math. Soc. Janos Bolyai **37**, Elsevier, New York, 1984, 697–710.

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