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HIGHER-ORDER OPTIMALITY CONDITIONS FOR A MINIMAX

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Higher-order necessary and sufficient optimality conditions for a nonsmooth minimax problem with infinitely many constraints of inequality type are established under suitable basic assumptions and regularity conditions.

1. INTRODUCTION

Let C be a nonempty subset of a normed space X, and let Q and B be nonempty sets. Let f_α $(\alpha \in Q)$ and g_β $(\beta \in B)$ be real-valued functions on X. We consider the following minimax problem:

$$\text{(P)} \qquad \min\{F(x) \mid x \in C,\ G(x) \leqslant 0\},$$

where $F(x) := \sup\limits_{\alpha\in Q} f_\alpha(x)$ and $G(x) := \sup\limits_{\beta\in B} g_\beta(x)$.

Our aim here is to develop higher-order optimality conditions for (P) by using suitable approximations to the functions involved. Thus our results will be formulated in terms of approximating functions $\phi_\alpha^{(k)}(.)$ and $\psi_\beta^{(k)}(.)$. These may be thought of as substitutes for the k-th order directional derivatives of f_α and g_β, but are considerably more general. For instance, $\phi_\alpha^{(k)}(.)$ and $\psi_\beta^{(k)}(.)$ do not need to be positively homogeneous of degree k. The lack of homogeneity forces us to use a particular kind of regularity condition (namely condition 2.3 below). Our approach extends the technique we have used in [6] to derive first-order conditions. Some optimality conditions from [8] are included as special cases in our results.

2. HIGHER-ORDER NECESSARY OPTIMALITY CONDITIONS

In the following we fix a reference point $x_0 \in C$ which is feasible for problem (P). We assume that $F(x_0)$ is finite. Let

$$Q_0 := \{\alpha \in Q \mid f_\alpha(x_0) = F(x_0)\}, \qquad B_0 := \{\beta \in B \mid g_\beta(x_0) = G(x_0)\}.$$

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We assume that Q, B are compact topological spaces, and that the mappings $\alpha \mapsto f_\alpha(x_0)$ and $\beta \mapsto g_\beta(x_0)$ are upper semicontinuous. Then Q_0 and B_0 are nonempty and compact. We recall [**2**, p.55] that the contingent cone to C at x_0 is the set

$$K_C(x_0) := \{d \in X \mid \exists \{d_n\} \subseteq X,\ \{t_n\} \subseteq \mathbb{R} : d_n \to d,\ t_n \downarrow 0,\ x_0 + t_n d_n \in C\}.$$

To derive necessary optimality conditions for (P), we introduce functions which play the roles of higher-order generalised directional derivatives of f_α and g_β. So, for each $\alpha \in Q$, $\beta \in B$, let $\varphi_\alpha^{(i)}$, $i \in I := \{1, \dots, k\}$, and $\psi_\beta^{(j)}$, $j \in J := \{1, \dots, p\}$, be real-valued functions on X satisfying the following:

Assumption 2.1.

(a) *$\varphi_\alpha^{(i)}(0) = 0$, $\psi_\beta^{(j)}(0) = 0$ for all $\alpha \in Q$, $\beta \in B$, $i \in I$, $j \in J$.*

(b) *The mappings $\alpha \mapsto \varphi_\alpha^{(i)}(d)$ and $\beta \mapsto \psi_\beta^{(j)}(d)$ are continuous for all $d \in K_C(x_0)$, $i \in I$, $j \in J$.*

(c) *If $d_n \to d$ as $n \to \infty$, then, for each $i \in I$,*

$$\liminf_{n\to\infty} \left[\varphi_\alpha^{(i)}(d_n) - \varphi_\alpha^{(i)}(d)\right] \leqslant 0 \quad \text{uniformly in } \alpha,$$

and, for each $j \in J$,

$$\liminf_{n\to\infty} \left[\psi_\beta^{(j)}(d_n) - \psi_\beta^{(j)}(d)\right] \leqslant 0 \quad \text{uniformly in } \beta.$$

(d) *The mapping $d \mapsto \max_{\alpha \in Q_0} \varphi_\alpha^{(k)}(d)$ is upper semicontinuous.*

Let us introduce relations between f_α and $\varphi_\alpha^{(k)}$, g_β and $\psi_\beta^{(p)}$.

Basic Assumption 2.2. *For all $d \in K_C(x_0)$ and sequences $d_n \to d$, $t_n \downarrow 0$ satisfying $x_0 + t_n d_n \in C$,*

$$\varphi_\alpha^{(k)}(d) \geqslant \liminf_{n\to\infty} \frac{1}{t_n^k}\left[f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0) - \sum_{i=1}^{k-1} t_n^i \varphi_\alpha^{(i)}(d_n)\right] \quad \text{uniformly in } \alpha,$$

and

$$\psi_\beta^{(p)}(d) \geqslant \liminf_{n\to\infty} \frac{1}{t_n^p}\left[g_\beta(x_0 + t_n d_n) - g_\beta(x_0) - \sum_{j=1}^{p-1} t_n^j \psi_\beta^{(j)}(d_n)\right] \quad \text{uniformly in } \beta.$$

To proceed further let us introduce the sets

$$M(x_0) := \left\{d \in K_C(x_0) \;\middle|\; \max_{\alpha\in Q_0} \varphi_\alpha^{(i)}(d) \leqslant 0\ \forall i \in I\setminus\{k\},\ \max_{\beta\in B_0} \psi_\beta^{(j)}(d) \leqslant 0\ \forall j \in J\right\},$$

$$\widetilde{M}(x_0) := \left\{d \in K_C(x_0) \;\middle|\; \max_{\alpha\in Q_0} \varphi_\alpha^{(i)}(d) < 0\ \forall i \in I\setminus\{k\},\ \max_{\beta\in B_0} \psi_\beta^{(j)}(d) < 0\ \forall j \in J\right\}.$$

Moreover, for $V \subseteq Q$ and $W \subseteq B$ let us define

$$\mathcal{C}(V,W) := \left\{ d \in K_C(x_0) \,\middle|\, \varphi_\alpha^{(i)}(d) < 0 \,\forall\, \alpha \in V,\, i \in I;\ \psi_\beta^{(j)}(d) < 0 \,\forall\, \beta \in W,\, j \in J \right\}.$$

Let us introduce a regularity condition of the type used in [**3**].

Regularity Condition 2.3.

(i) *For any closed sets V and W satisfying $Q_0 \subseteq V \subseteq Q$ and $B_0 \subseteq W \subseteq B$ it holds that $\mathcal{C}(V,W) \neq \emptyset$ implies $0 \in \operatorname{cl} \mathcal{C}(V,W)$.*

(ii) $M(x_0) \subseteq \operatorname{cl} \widetilde{M}(x_0)$.

Note that 2.3(i) holds, if the functions $\varphi_\alpha^{(i)}$, $\psi_\beta^{(j)}$ are positively homogeneous.

We are now in a position to formulate a general necessary optimality condition of order k for (P), which is the main result of the paper.

Theorem 2.4. *Let x_0 be a local minimiser for* (P). *Assume that assumption 2.1, the basic assumption 2.2, and the regularity condition 2.3 hold. Then*

$$\max_{\alpha \in Q_0} \varphi_\alpha^{(k)}(d) \geqslant 0 \quad \forall\, d \in M(x_0). \tag{1}$$

Proof: Suppose that (1) is not true. By 2.3.(ii), there exists $\overline{d} \in \operatorname{cl} \widetilde{M}(x_0)$ such that

$$\max_{\alpha \in Q_0} \varphi_\alpha^{(k)}\big(\overline{d}\big) < 0.$$

By 2.1.(d), $d \mapsto \max_{\alpha \in Q_0} \varphi_\alpha^{(k)}(d)$ is upper semicontinuous. So we can assume that $\overline{d} \in \widetilde{M}(x_0)$, that is, $\overline{d} \in K_C(x_0)$ and

$$\max_{\alpha \in Q_0} \varphi_\alpha^{(i)}\big(\overline{d}\big) < 0 \quad \forall\, i \in I, \quad \max_{\beta \in B_0} \psi_\beta^{(j)}\big(\overline{d}\big) < 0 \quad \forall\, j \in J.$$

We choose $\delta > 0$ such that $\varphi_\alpha^{(i)}\big(\overline{d}\big) \leqslant -2\delta$ for all $\alpha \in Q_0$, $i \in I$, and $\psi_\beta^{(j)}\big(\overline{d}\big) \leqslant -2\delta$ for all $\beta \in B_0$, $j \in J$. We define

$$U_1 := \{\alpha \in Q \mid \varphi_\alpha^{(i)}\big(\overline{d}\big) < -\delta \,\forall\, i \in I\}, \quad U_2 := \{\beta \in B \mid \psi_\beta^{(j)}\big(\overline{d}\big) < -\delta \,\forall\, j \in J\}.$$

Then $Q_0 \subseteq U_1$, $B_0 \subseteq U_2$. By 2.1.(b), U_1 and U_2 are open, and

$$\varphi_\alpha^{(i)}\big(\overline{d}\big) \leqslant -\delta \quad \forall\, \alpha \in \operatorname{cl} U_1,\, i \in I, \quad \psi_\beta^{(j)}\big(\overline{d}\big) \leqslant -\delta \quad \forall\, \beta \in \operatorname{cl} U_2,\, j \in J.$$

So $\overline{d} \in \mathcal{C}(\operatorname{cl} U_1, \operatorname{cl} U_2)$ and, by 2.3.(i), $0 \in \operatorname{cl} \mathcal{C}(\operatorname{cl} U_1, \operatorname{cl} U_2)$. Hence there exists a sequence $\{h_n\} \subseteq \mathcal{C}(\operatorname{cl} U_1, \operatorname{cl} U_2)$ converging to 0.

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We assert that there exist $d \in K_C(x_0)$ and $\varepsilon > 0$ satisfying

$$\begin{aligned} &\varphi_\alpha^{(i)}(d) + \varepsilon \leqslant F(x_0) - f_\alpha(x_0) \quad \forall i \in I, \\ &\psi_\beta^{(j)}(d) + \varepsilon \leqslant G(x_0) - g_\beta(x_0) \quad \forall j \in J \end{aligned} \tag{2}$$

for every $\alpha \in Q$, $\beta \in B$. To prove this, observe that $f_\alpha(x_0) < F(x_0)$ for all $\alpha \in Q \setminus U_1$ and $g_\beta(x_0) < G(x_0)$ for all $\beta \in B \setminus U_2$. Since $Q \setminus U_1$, $B \setminus U_2$ are compact and $\alpha \mapsto f_\alpha(x_0)$, $\beta \mapsto g_\beta(x_0)$ are upper semicontinuous, there exists $\varepsilon_1 > 0$ such that

$$f_\alpha(x_0) \leqslant F(x_0) - 2\varepsilon_1 \quad \forall \alpha \in Q \setminus U_1, \quad g_\beta(x_0) \leqslant G(x_0) - 2\varepsilon_1 \quad \forall \beta \in B \setminus U_2.$$

Since $h_n \to 0$, by 2.1.(a) and (c) there exists $m \in \mathbb{N}$ with

$$\varphi_\alpha^{(i)}(h_m) \leqslant \varepsilon_1 \quad \forall \alpha \in Q, i \in I, \quad \psi_\beta^{(j)}(h_m) \leqslant \varepsilon_1 \quad \forall \beta \in B, j \in J.$$

Let $d := h_m$. Then (2) holds for all $\alpha \in Q \setminus U_1$, $\beta \in B \setminus U_2$, and every $\varepsilon \leqslant \varepsilon_1$. On the other hand, $d \in C(\operatorname{cl} U_1, \operatorname{cl} U_2)$. Hence by 2.1.(b) and the compactness of $\operatorname{cl} U_1$, $\operatorname{cl} U_2$ there exists $\varepsilon_2 > 0$ such that

$$\varphi_\alpha^{(i)}(d) + \varepsilon_2 \leqslant 0 \quad \forall \alpha \in \operatorname{cl} U_1, i \in I, \quad \psi_\beta^{(j)}(d) + \varepsilon_2 \leqslant 0 \quad \forall \beta \in \operatorname{cl} U_2, j \in J.$$

This implies (2) for all $\alpha \in \operatorname{cl} U_1$, $\beta \in \operatorname{cl} U_2$, and every $\varepsilon \leqslant \varepsilon_2$.

Now we use (2) to find a sequence $\{x_n\}$ of feasible points for problem (P) converging to x_0 such that $F(x_n) < F(x_0)$ for all n, which contradicts the hypothesis that x_0 is a local minimiser.

Since $d \in K_C(x_0)$, there exist sequences $\{d_n\} \subseteq X$, $\{t_n\} \subseteq \mathbb{R}$ such that $d_n \to d$, $t_n \downarrow 0$, and $x_n := x_0 + t_n d_n \in C$ for all n. Replacing these by appropriate subsequences we can assume that, for every n,

$$\begin{aligned} &\frac{1}{t_n^k}\Big[f_\alpha(x_n) - f_\alpha(x_0) - \sum_{i=1}^{k-1} t_n^i \varphi_\alpha^{(i)}(d_n)\Big] \leqslant \varphi_\alpha^{(k)}(d) + \frac{\varepsilon}{2} \quad \forall \alpha \in Q, \\ &\frac{1}{t_n^p}\Big[g_\beta(x_n) - g_\beta(x_0) - \sum_{j=1}^{p-1} t_n^j \psi_\beta^{(j)}(d_n)\Big] \leqslant \psi_\beta^{(p)}(d) + \frac{\varepsilon}{2} \quad \forall \beta \in B \end{aligned}$$

by 2.2,

$$\begin{aligned} &\varphi_\alpha^{(i)}(d_n) \leqslant \varphi_\alpha^{(i)}(d) + \varepsilon \quad \forall \alpha \in Q, i \in I, \\ &\psi_\beta^{(j)}(d_n) \leqslant \psi_\beta^{(j)}(d) + \varepsilon \quad \forall \beta \in B, j \in J \end{aligned}$$

by 2.1.(c), and $\sum_{i=1}^{k} t_n^i \leqslant 1$, $\sum_{j=1}^{p} t_n^j \leqslant 1$. Then, for all $\alpha \in Q$,

$$\begin{aligned} f_\alpha(x_n) &\leqslant t_n^k\Big(\varphi_\alpha^{(k)}(d) + \frac{\varepsilon}{2}\Big) + \sum_{i=1}^{k-1} t_n^i \varphi_\alpha^{(i)}(d_n) + f_\alpha(x_0) \\ &\leqslant t_n^k\Big(\varphi_\alpha^{(k)}(d) + \varepsilon\Big) + \sum_{i=1}^{k-1} t_n^i\Big(\varphi_\alpha^{(i)}(d) + \varepsilon\Big) + f_\alpha(x_0) - t_n^k\frac{\varepsilon}{2} \\ &\leqslant \sum_{i=1}^{k} t_n^i(F(x_0) - f_\alpha(x_0)) + f_\alpha(x_0) - t_n^k\frac{\varepsilon}{2} \\ &\leqslant F(x_0) - t_n^k\frac{\varepsilon}{2}, \end{aligned}$$

and similarly $g_\beta(x_n) \leqslant G(x_0) - t_n^k\varepsilon/2$ for all $\beta \in B$. Hence $F(x_n) < F(x_0)$ and $G(x_n) < G(x_0) \leqslant 0$. □

In what follows, we give an application of Theorem 2.4.

EXAMPLE 2.5: Recall that $\overline{f}_\alpha^{(k)}(x_0; d)$, the upper Dini directional derivative of order k of f_α at x_0 in the direction d, is defined recursively as follows (see for example, [**4, 5, 8**]):

$$\overline{f}_\alpha^{(k)}(x_0; d) := k! \limsup_{h\to d,\, t\downarrow 0} \frac{1}{t^k}\left[f_\alpha(x_0 + th) - f_\alpha(x_0) - \sum_{i=1}^{k-1} \frac{t^i \overline{f}_\alpha^{(i)}(x_0; h)}{i!}\right],$$

provided that each $\overline{f}_\alpha^{(i)}$ is real-valued.

Note that the mapping $d \mapsto \overline{f}_\alpha^{(k)}(x_0; d)$ is upper semicontinuous. By applying Theorem 2.4 to the upper Dini directional derivatives we obtain:

COROLLARY 2.6. *Let x_0 be a local minimiser for* (P). *Assume that for*

$$\varphi_\alpha^{(i)}(d) := \frac{1}{i!}\overline{f}_\alpha^{(i)}(x_0; d) \quad \forall i \in I, \quad \psi_\beta^{(j)}(d) := \frac{1}{j!}\overline{g}_\beta^{(j)}(x_0; d) \quad \forall j \in J,$$

assumptions 2.1.(b)–(d) *and* 2.3.(ii) *hold. Suppose, in addition, that the limits in the definitions of $\overline{f}_\alpha$ and $\overline{g}_\beta$ at x_0 are uniformly in α and β, respectively. Then*

$$\max_{\alpha \in Q_0} \overline{f}_\alpha^{(k)}(x_0; d) \geqslant 0$$

holds for every $d \in K_C(x_0)$ with

$$\max_{\alpha \in Q_0} \overline{f}_\alpha^{(i)}(x_0; d) \leqslant 0 \quad \forall i \in I \setminus \{k\}, \quad \max_{\beta \in B_0} \overline{g}_\beta^{(j)}(x_0; d) \leqslant 0 \quad \forall j \in J.$$

PROOF: It is easy to see that assumption 2.1.(a) and the basic assumption 2.2 are satisfied. Moreover, 2.3.(i) holds since $\overline{f}_\alpha^{(i)}$, $\overline{g}_\beta^{(j)}$ are positively homogeneous. So the conclusion follows from Theorem 2.4. ∎

EXAMPLE 2.7: If f_α is $(k-1)$ times Fréchet differentiable on X $(k > 1)$ and the Fréchet derivative of order k of f_α at x_0, $f_\alpha^{(k)}(x_0)$, exists, then (see for example, [7])

$$f_\alpha^{(k)}(x_0)d^k = \overline{f}_\alpha^{(k)}(x_0; d) \quad \forall\, d \in X,$$

where $d^k := (d, \ldots, d) \in X^k$. Similarly as in Example 2.5, we get the necessary condition for a local minimiser in this case:

$$\max_{\alpha \in Q_0} f_\alpha^{(k)}(x_0)d^k \geqslant 0$$

for every $d \in K_C(x_0)$ with

$$\max_{\alpha \in Q_0} f_\alpha^{(i)}(x_0)d^i \leqslant 0 \quad \forall\, i \in I \setminus \{k\}, \quad \max_{\beta \in B_0} g_\beta^{(j)}(x_0)d^j \leqslant 0 \quad \forall\, j \in J.$$

3. Higher-Order Sufficient Optimality Conditions

In this section we assume that $X = \mathbb{R}^m$.

DEFINITION 3.1: [10] The point $x_0 \in D$ is said to be a strict local minimiser of order k for the mathematical program $\min\{F(x) \mid x \in D\}$ if there exist $\sigma > 0$ and a neighbourhood U of x_0 such that

$$F(x) \geqslant F(x_0) + \sigma \|x - x_0\|^k \quad \forall\, x \in U \cap D.$$

Let x_0 be a feasible point for (P). As in the previous section, we consider real-valued functions $\varphi_\alpha^{(i)}$, $\alpha \in Q$, $i \in I$, and $\psi_\beta^{(j)}$, $\beta \in B$, $j \in J$, on $\mathbb{R}^m$, and we define

$$M(x_0) := \Big\{ d \in K_C(x_0) \;\Big|\; \sup_{\alpha \in Q_0} \varphi_\alpha^{(i)}(d) \leqslant 0 \;\forall\, i \in I \setminus \{k\},\; \sup_{\beta \in B_0} \psi_\beta^{(j)}(d) \leqslant 0 \;\forall\, j \in J \Big\}.$$

Let us introduce relations between f_α and $\varphi_\alpha^{(i)}$, g_β and $\psi_\beta^{(j)}$.

BASIC ASSUMPTION 3.2. *For all $d \in K_C(x_0)$ and sequences $d_n \to d$, $t_n \downarrow 0$ satisfying $x_0 + t_n d_n \in C$ we have*

$$\varphi_\alpha^{(i)}(d) \leqslant \limsup_{n \to \infty} \frac{1}{t_n^i}\left[f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)\right] \quad \forall\, \alpha \in Q,\, i \in I,$$

$$\psi_\beta^{(j)}(d) \leqslant \limsup_{n \to \infty} \frac{1}{t_n^j}\left[g_\beta(x_0 + t_n d_n) - g_\beta(x_0)\right] \quad \forall\, \beta \in B,\, j \in J.$$

A higher-order sufficient optimality condition for (P) can be stated as follows.

Theorem 3.3. *Let $G(x_0) = 0$ and let the basic assumption 3.2 be satisfied. Assume that*

$$\sup_{\alpha \in Q_0} \varphi_\alpha^{(k)}(d) > 0 \quad \forall d \in M(x_0) \setminus \{0\}. \tag{3}$$

Then x_0 is a strict local minimiser of order k for (P).

Proof: Assume that x_0 is not a strict local minimiser of order k for (P). Then there exists a sequence $\{x_n\} \subseteq C \setminus \{x_0\}$ such that $G(x_n) \leqslant 0$, $\|x_n - x_0\| \leqslant 1/n$, and $F(x_n) < F(x_0) + \|x_n - x_0\|^k / n$ for every n. Since $f_\alpha(x_0) = F(x_0)$ for $\alpha \in Q_0$, we obtain

$$f_\alpha(x_n) \leqslant f_\alpha(x_0) + \|x_n - x_0\|^k / n \quad \forall \alpha \in Q_0,\, n \in \mathbb{N}. \tag{4}$$

Let $t_n := \|x_n - x_0\|$ and $d_n := (x_n - x_0)/t_n$. Then $\|d_n\| = 1$, so by the compactness of the unit sphere in $\mathbb{R}^m$ there exists a subsequence $\{d_{n_\nu}\}$ converging to d with $\|d\| = 1$. Since $t_{n_\nu} \downarrow 0$ and $x_0 + t_n d_n = x_n \in C$, it follows that $d \in K_C(x_0)$.

Using $G(x_0) = 0$ we have $g_\beta(x_n) - g_\beta(x_0) = g_\beta(x_n) \leqslant G(x_n) \leqslant 0$ for all $\beta \in B_0$, $n \in \mathbb{N}$. Thus

$$\psi_\beta^{(j)}(d) \leqslant \limsup_{\nu \to \infty} \frac{g_\beta(x_{n_\nu}) - g_\beta(x_0)}{t_{n_\nu}^k} \leqslant 0 \quad \forall \beta \in B_0,\, j \in J$$

by 3.2. By combining (4) and 3.2 we obtain

$$\begin{aligned} \varphi_\alpha^{(i)}(d) &\leqslant \limsup_{\nu \to \infty} \frac{f_\alpha(x_{n_\nu}) - f_\alpha(x_0)}{t_{n_\nu}^i} \\ &\leqslant \limsup_{\nu \to \infty} \frac{\|x_{n_\nu} - x_0\|^k}{n_\nu t_{n_\nu}^i} = \lim_{\nu \to \infty} \frac{t_{n_\nu}^{k-i}}{n_\nu} = 0 \quad \forall \alpha \in Q_0,\, i \in I. \end{aligned}$$

Hence $d \in M(x_0) \setminus \{0\}$ and $\sup_{\alpha \in Q_0} \varphi_\alpha^{(k)}(d) \leqslant 0$, which contradicts (3). ∎

Theorem 3.3 includes [**8**] Corollary 2.1 as a special case.

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