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NEIGHBOURHOODS OF INDEPENDENT SETS FOR (a, b, k)-CRITICAL GRAPHS

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Abstract

Let G be a graph of order n. Let a, b and k be nonnegative integers such that $1 \leq a \leq b$. A graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. We provide a sufficient condition for a graph to be (a, b, k)-critical that extends a well-known sufficient condition for the existence of a k-factor.

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1. Introduction

In this paper we consider only finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$. The minimum degree of G is denoted by $\delta(G)$. The neighbourhood $N_G(x)$ of x is the set of all vertices in $V(G)$ adjacent to x, and for $X \subseteq V(G)$ we write $N_G(X) = \bigcup_{x \in X} N_G(x)$. For disjoint subsets S and T of $V(G)$, we denote by $e_G(S, T)$ the number of edges from S to T, by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph obtained from G by deleting all the vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges.

Let $1 \leq a \leq b$ and $k \geq 0$ be integers. A spanning subgraph F of G is called an $[a, b]$-factor if $a \leq d_F(x) \leq b$ for each $x \in V(G)$ (where of course d_F denotes the degree in F). And if $a = b = r$, then an $[a, b]$-factor of G is called an r-factor of G. A graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. If G is an (a, b, k)-critical graph, then we also say that G is (a, b, k)-critical. If $a = b = r$, then an (a, b, k)-critical graph is simply called an (r, k)-critical graph. In particular, a $(1, k)$-critical graph is simply

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called a k-critical graph. Terminology and notation not given in this paper can be found in [1].

Favaron [2] studied the properties of k-critical graphs. Liu and Yu [6] gave the characterization of (r, k)-critical graphs. Li [3] gave two sufficient conditions for graphs to be (a, b, k)-critical. Li [4] showed a degree condition for graphs to be (a, b, k)-critical. Zhou [8–10] investigated (a, b, k)-critical graphs and obtained some sufficient conditions for graphs to be (a, b, k)-critical. Liu and Wang [5] gave a necessary and sufficient condition for graphs to be (a, b, k)-critical. In this paper, we obtain a new sufficient condition for graphs to be (a, b, k)-critical. The main result will be given in the following section.

The following result on k-factors is known.

THEOREM 1 [7]. *Let $k \geq 2$ be an integer and G a graph of order n with $n \geq 4k - 6$. If k is odd, then n is even and G is connected. Let G satisfy*

$$|N_G(X)| \geq \frac{|X| + (k-1)n - 1}{2k-1},$$

for every nonempty independent subset X of $V(G)$, and

$$\delta(G) \geq \frac{k-1}{2k-1}(n+2).$$

Then G has a k-factor.

2. Main results

In this section, we prove the following theorem on (a, b, k)-critical graphs, which is an extension of Theorem 1.

THEOREM 2. *Let a, b and k be nonnegative integers with $1 \leq a < b$, and let G be a graph of order n with $n \geq (((a+b)\,(a+b-2))/b) + k$. Suppose that*

$$|N_G(X)| > \frac{(a-1)n + |X| + bk - 1}{a+b-1}, \tag{1}$$

for every nonempty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(a-1)n + a + b + bk - 2}{a+b-1}. \tag{2}$$

Then G is an (a, b, k)-critical graph.

In Theorem 2, if $k = 0$, then we obtain the following corollary.

COROLLARY 3. *Let a and b be integers such that $1 \leq a < b$, and let G be a graph of order n with $n \geq (((a+b)\,(a+b-2))/b)$. Let G satisfy*

$$|N_G(X)| > \frac{(a-1)n + |X| - 1}{a+b-1},$$

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for every nonempty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(a-1)n+a+b-2}{a+b-1}.$$

Then G has an $[a, b]$-factor.

3. Proof of Theorem 2

In order to prove our main theorem, we depend heavily on the following lemma.

LEMMA 4 [5]. *Let a, b and k be nonnegative integers with $a < b$, and let G be a graph of order $n \geq a+k+1$. Then G is an (a, b, k)-critical graph if and only if, for any $S \subseteq V(G)$ with $|S| \geq k$,*

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \geq bk,$$

where $T = \{x \mid x \in V(G) \setminus S,\ d_{G-S}(x) \leq a-1\}$.

PROOF OF THEOREM 2. In order to prove the theorem by contradiction, we assume that G is not an (a, b, k)-critical graph. Then, by Lemma 4, there exists a subset S of $V(G)$ with $|S| \geq k$ such that

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \leq bk - 1, \tag{3}$$

where $T = \{x \mid x \in V(G) \setminus S,\ d_{G-S}(x) \leq a-1\}$. We choose such subsets S and T so that $|T|$ is as small as possible.

If $T = \emptyset$, then by (3), $bk - 1 \geq \delta_G(S, T) = b|S| \geq bk$, a contradiction. Hence, $T \neq \emptyset$. Let

$$h = \min\{d_{G-S}(x) \mid x \in T\}.$$

Obviously,

$$\delta(G) \leq h + |S|. \tag{4}$$

According to the definition of T,

$$0 \leq h \leq a-1.$$

We shall consider two cases according to the value of h and derive a contradiction in each case.

CASE 1. $h = 0$. Let $Y = \{x \in T \mid d_{G-S}(x) = 0\}$. Clearly, $Y \neq \emptyset$. Since Y is independent we obtain, by (1),

$$\frac{(a-1)n + |Y| + bk - 1}{a+b-1} < |N_G(Y)| \leq |S|. \tag{5}$$

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On the other hand, from (3) and $|S| + |T| \le n$, we obtain

$$\begin{aligned} bk - 1 \ge \delta_G(S, T) &= b|S| + d_{G-S}(T) - a|T| \\ &\ge b|S| + |T| - |Y| - a|T| \\ &= b|S| - (a-1)\,|T| - |Y| \\ &\ge b|S| - (a-1)\,(n - |S|) - |Y| \\ &= (a+b-1)\,|S| - |Y| - (a-1)n, \end{aligned}$$

which implies that

$$|S| \le \frac{(a-1)n + |Y| + bk - 1}{a+b-1}.$$

This contradicts (5).

CASE 2. $1 \le h \le a-1$. According to (3) and $|S| + |T| \le n$ and $a - h \ge 1$, we obtain

$$\begin{aligned} bk - 1 \ge \delta_G(S, T) &= b|S| + d_{G-S}(T) - a|T| \\ &\ge b|S| - (a-h)|T| \\ &\ge b|S| - (a-h)\,(n - |S|) \\ &= (a+b-h)\,|S| - (a-h)n, \end{aligned}$$

which implies that

$$|S| \le \frac{(a-h)n + bk - 1}{a+b-h}. \tag{6}$$

On the other hand, by (2), (4) and (6),

$$\frac{(a-1)n + a + b + bk - 2}{a+b-1} < \delta(G) \le |S| + h \le \frac{(a-h)n + bk - 1}{a+b-h} + h,$$

that is,

$$(a+b-h)\left(\frac{(a-1)n + a + b + bk - 2}{a+b-1} - h\right) - (a-h)n - bk + 1 < 0. \tag{7}$$

Let

$$\begin{aligned} f(h) = (a+b-h)\,((((a-1)n + a + b + bk - 2)/(a+b-1)) - h) \\ - (a-h)n - bk + 1. \end{aligned}$$

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Then, by $1 \le h \le a-1$ and $n \ge (((a+b)\,(a+b-2))/b)+k$,

$$\begin{aligned}
f'(h) &= -\frac{(a-1)n+a+b+bk-2}{a+b-1} + h - a - b + h + n \\
&= 2h + \frac{bn-a-b-bk+2}{a+b-1} - a - b \\
&\ge 2 + \frac{bn-a-b-bk+2}{a+b-1} - a - b \\
&= \frac{bn-(a+b)\,(a+b-2)-bk}{a+b-1} \\
&\ge \frac{b((((a+b)\,(a+b-2))/b)+k)-(a+b)\,(a+b-2)-bk}{a+b-1} \\
&= 0.
\end{aligned}$$

Thus we obtain, using $1 \le h \le a-1$,

$$f(h) \ge f(1). \tag{8}$$

In view of (7) and (8), we obtain

$$\begin{aligned}
0 &> f(h) \ge f(1) \\
&= (a+b-1)\left(\frac{(a-1)n+a+b+bk-2}{a+b-1} - 1\right) - (a-1)n - bk + 1 \\
&= 0,
\end{aligned}$$

which is a contradiction.

From the contradictions we deduce that G is an (a, b, k)-critical graph. This completes the proof of Theorem 2.

REMARK. Let us show that the condition

$$|N_G(X)| > (((a-1)n + |X| + bk - 1)/(a+b-1))$$

in Theorem 2 cannot be replaced by

$$|N_G(X)| \ge (((a-1)n + |X| + bk - 1)/(a+b-1)).$$

Let $b > a \ge 2$, $k \ge 0$ be three integers such that $(((a+b-1)^2)/(a-1))$ is an integer, and let $n = (((a+b-1)^2)/(a-1)) + k$. Clearly, n is an integer. Let

$$H = K_{a+b+k} \bigvee ((a+b)K_1 \cup ((((a+b-1)^2)/(a-1)) - 2(a+b))K_2).$$

Let $X = V((a+b)K_1)$. Obviously,

$$|N_H(X)| = (((a-1)n + |X| + bk - 1)/(a+b-1))$$

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and

$$\delta(H) = a + b + k > (((a-1)n + a + b + bk - 2)/(a + b - 1)).$$

Let

$$S = V(K_{a+b+k}) \subseteq V(H)$$

and

$$T = V((a+b)K_1 \cup ((((a+b-1)^2)/(a-1)) - 2(a+b))K_2) \subseteq V(H);$$

then

$$|S| = a + b + k \geq k, \quad |T| = (((a+b-1)^2)/(a-1)) - (a+b).$$

Thus,

$$\begin{aligned}\delta_H(S, T) &= b|S| + d_{H-S}(T) - a|T| \\ &= b(a+b+k) + \frac{(a+b-1)^2}{a-1} - 2(a+b) \\ &\quad - a\left(\frac{(a+b-1)^2}{a-1} - (a+b)\right) \\ &= bk - 1 < bk.\end{aligned}$$

By Lemma 4, H is not an (a, b, k)-critical graph. In the above sense, the condition

$$|N_G(X)| > (((a-1)n + |X| + bk - 1)/(a + b - 1))$$

in Theorem 2 is the best possible.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph theory with applications* (Macmillan Press, London, 1976).
[2] O. Favaron, 'On k-factor-critical graphs', *Discuss. Math. Graph Theory* **16**(1) (1996), 41–51.
[3] J. Li, 'Sufficient conditions for graphs to be (a, b, n)-critical graphs', *Math. Appl.* **17**(3) (2004), 450–455 (in Chinese).
[4] J. Li, 'A new degree condition for graph to have $[a, b]$-factor', *Discrete Math.* **290** (2005), 99–103.
[5] G. Liu and J. Wang, '(a, b, k)-critical graphs', *Adv. Math.* **27**(6) (1998), 536–540 (in Chinese).
[6] G. Liu and Q. Yu, 'k-factors and extendability with prescribed components', *Congr. Numer.* **139**(1) (1999), 77–88.
[7] D. R. Woodall, 'k-factors and neighbourhoods of independent sets in graphs', *J. London Math. Soc.* **41**(2) (1990), 385–392.
[8] S. Zhou, 'Some sufficient conditions for graphs to have (g, f)-factors', *Bull. Austral. Math. Soc.* **75** (2007), 447–452.

https://doi.org/10.1017/S0004972708000270 Published online by Cambridge University Press

[9] S. Zhou and J. Jiang, 'Notes on the binding numbers for (a, b, k)-critical graphs', *Bull. Austral. Math. Soc.* **76** (2007), 307–314.

[10] S. Zhou and M. Zong, 'Some new sufficient conditions for graphs to be (a, b, k)-critical graphs', *Ars Combin.* to appear.

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