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STABILITY ANALYSIS FOR STOCHASTIC MCKEAN–VLASOV EQUATION

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Abstract

The pth ($p \geq 1$) moment exponential stability, almost surely exponential stability and stability in distribution for stochastic McKean–Vlasov equation are derived based on some distribution-dependent Lyapunov function techniques.

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1. Introduction

McKean–Vlasov stochastic differential equations (SDEs), originating from the seminal works [15, 18], are also known as mean-field SDEs or distribution-dependent SDEs which are used to study the interacting particle system and mean-field games. There are numerous works on the well-posedness, ergodicity and large deviations [10, 13, 17, 19]. Moreover, there are also several works on the stability of the McKean–Vlasov SDEs. Recently, Ding and Qiao [5] considered the stability for the McKean–Vlasov SDEs with non-Lipschitz coefficients

$$
\begin{aligned}
dX(t) &= b(X(t), \mathcal{L}(X(t)))\,dt + \sigma(X(t), \mathcal{L}(X(t)))\,dW(t),\\
X(0) &= x_0, \qquad (1.1)
\end{aligned}
$$

where $\mathcal{L}(X(t))$ is the distribution of $X(t)$, $W_\cdot = (W_\cdot^1, W_\cdot^2, \ldots, W_\cdot^l)$ is a $\mathcal{F}_t$-adapted standard Brownian motion and the coefficients $b : \mathbb{R}^d \times \mathcal{M}_{\lambda^2}(\mathbb{R}^d) \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times \mathcal{M}_{\lambda^2}(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^l$ are Borel measurable functions. The definition of $\mathcal{M}_{\lambda^2}(\mathbb{R}^d)$ is defined in the next section. Sufficient conditions are given for the exponential stability of the second moments for their solutions in terms of a Lyapunov

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function [12]. Furthermore, the almost surely (a.s.) asymptotic stability of their solutions is also discussed. Lv and Shan [14] considered the long time behaviour of stochastic McKean–Vlasov equations, and the exponential and logarithmic decay are discussed. Bahlali et al. [1] discussed the existence and uniqueness of solutions under a non-Lipschitz condition and derived various stability properties with respect to initial data, coefficients and driving processes. Wu et al. [20] studied the stability of solutions of McKean–Vlasov SDEs via feedback control based on discrete-time state observation and derived the H_∞ stability, asymptotic stability and exponential stability in mean square for the controlled systems.

In this paper, we first provide a sufficient condition for the pth moment exponential stability and a.s. exponential stability for (1.1) (Theorem 3.2) by using the classical Lyapunov function method. Furthermore, asymptotic stability in distribution is derived by introducing a distribution-dependent operator, together with a similar discussion as that for SDE with Markovian switching [21].

There are many recent works on the stability in distribution for stochastic differential equations with distribution-independent coefficients. Yuan et al. [22] discussed the stochastic differential equations with Markovian switching and investigated the stability in distribution of the equations. Further, Du et al. [6] improved the result of Yuan et al. [22] by giving a new sufficient condition for stability in distribution. Bao et al. [2] considered a neutral stochastic differential delay equation with Markovian switching and obtained sufficient conditions for stability in distribution. Fei et al. [8] considered the stability in distribution for a highly nonlinear stochastic differential equation driven by G-Brownian motion [16].

The rest of the paper is organized as follows. In Section 2, we recall some preliminary knowledge. The pth moment exponential stability and a.s. exponential stability is presented in Section 3.1. The stability in distribution is established in Section 3.2.

2. Preliminary and main result

Let $C(\mathbb{R}^d)$ be the collection of continuous functions on $\mathbb{R}^d$. For convenience, we denote the norm of vectors and matrices by $|\cdot|$ and $\|\cdot\|$, respectively. Furthermore, let $\langle\cdot,\cdot\rangle$ denote the scalar product in $\mathbb{R}^d$. Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel σ-algebra on $\mathbb{R}^d$ and $\mathcal{P}(\mathbb{R}^d)$ denote the space of all probability measures defined on $\mathcal{B}(\mathbb{R}^d)$ with the topology of weak convergence.

For $\lambda(x) = 1 + |x|$, $x \in \mathbb{R}^d$, define the Banach space

$$C_\lambda(\mathbb{R}^d) = \left\{\phi \in C(\mathbb{R}^d) \middle| \|\phi\|_{C_\lambda(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \frac{|\phi(x)|}{\lambda^2(x)} + \sup_{x\neq y} \frac{|\phi(x)-\phi(y)|}{|x-y|} < \infty\right\}.$$

Let $\mathcal{M}^s_{\lambda^p}(\mathbb{R}^d)$ be the space of signed measures m on $\mathcal{B}(\mathbb{R}^d)$ satisfying

$$\|m\|^p_{\lambda^p} = \int_{\mathbb{R}^d} \lambda^p(x)|m|(dx) < \infty, \quad p \geq 2,$$

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where $|m| = m^+ + m^-$, and $m = m^+ - m^-$ is the Jordan decomposition of m. Let $\mathcal{M}_{\lambda^p}(\mathbb{R}^d) = \mathcal{M}^s_{\lambda^p}(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$ be the set of probability measures on $\mathcal{B}(\mathbb{R}^d)$ with finite pth moments equipped with the metric,

$$\rho(\mu, \nu) \triangleq \sup_{\|\phi\|_{C_\lambda(\mathbb{R}^d)} \leq 1} \left| \int_{\mathbb{R}^d} \phi(x)\mu(dx) - \int_{\mathbb{R}^d} \phi(x)\nu(dx) \right|.$$

Then, $(\mathcal{M}_{\lambda^p}(\mathbb{R}^d), \rho)$ is a complete metric space.

Given a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$, we recall the definition of the derivative for a function with respect to a probability measure [4]. A function $f : \mathcal{M}^s_{\lambda^p}(\mathbb{R}^d) \to \mathbb{R}$ is differential at $\mu \in \mathcal{M}^s_{\lambda^p}(\mathbb{R}^d)$ if for $\tilde{f}(\xi) \triangleq f(\mathbb{P}_\xi)$, $\xi \in L^p(\Omega; \mathbb{R}^d)$, there exists some $\zeta \in L^p(\Omega; \mathbb{R}^d)$ with $\mathbb{P}_\zeta = \mu$ such that $\tilde{f}$ is the Fréchet differential at ζ, that is, there exists a linear continuous mapping $D\tilde{f}(\zeta) : L^p(\Omega; \mathbb{R}^d) \to \mathbb{R}$ such that for all $\eta \in L^p(\Omega; \mathbb{R}^d)$,

$$\tilde{f}(\zeta + \eta) - \tilde{f}(\zeta) = D\tilde{f}(\zeta)(\eta) + o(|\eta|_{L^p}), \quad |\eta|_{L^p} \to 0.$$

Since $D\tilde{f}(\zeta) \in L(L^p(\Omega; \mathbb{R}^d), \mathbb{R})$, by the Riesz representation theorem [3], there exists a $\mathbb{P}$-a.s. unique random variable $\theta \in L^p(\Omega; \mathbb{R}^d)$ such that for $\eta \in L^p(\Omega; \mathbb{R}^d)$,

$$D\tilde{f}(\zeta)(\eta) = (\theta, \eta)_{L^p} = \mathbb{E}[\theta \cdot \eta].$$

Thus, there exists a Borel measurable function $h : \mathbb{R}^d \to \mathbb{R}^d$ which depends on the distribution $\mathbb{P}_\zeta$ rather than ζ itself such that $\theta = h(\zeta)$, and for $\xi \in L^2(\Omega; \mathbb{R}^d)$,

$$f(\mathbb{P}_\xi) - f(\mathbb{P}_\zeta) = \mathbb{E}[h(\zeta)(\xi - \zeta)] + o(|\xi - \zeta|_{L^2}).$$

We call $\partial_\mu f(\mathbb{P}_\zeta)(y) \triangleq h(y), y \in \mathbb{R}^d$ as the derivative of $f : \mathcal{M}_{\lambda^p}(\mathbb{R}^d) \to \mathbb{R}$ at $\mathbb{P}_\zeta$, $\zeta \in L^p(\Omega; \mathbb{R}^d)$.

Definition 2.1. Function f is said to be in $C^1(\mathcal{M}_{\lambda^p}(\mathbb{R}^d))$ if for each $\xi \in L^2(\Omega; \mathbb{R}^d)$, there exists a $\mathbb{P}_\xi$-modification of $\partial_\mu f(\mathbb{P}_\zeta)(\cdot)$ which is denoted by $\partial_\mu f(\mathbb{P}_\zeta)(\cdot)$ again, such that $\partial_\mu f : \mathcal{M}_{\lambda^p}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and we identify the function $\partial_\mu f$ with the derivative of f.

Definition 2.2. A function f belongs to $C_b^{1,1}(\mathcal{M}_{\lambda^p}(\mathbb{R}^d))$ if $f \in C^1(\mathcal{M}_{\lambda^p}(\mathbb{R}^d))$, and $\partial_\mu f$ is bounded and Lipschitz continuous, that is there exists a real number $C > 0$ such that:

(i) $\partial_\mu f(\mu)(x)| \leq C,\ \mu \in \mathcal{M}_{\lambda^p}(\mathbb{R}^d)$;
(ii) $\partial_\mu f(\mu)(x) - \partial_\mu f(\nu)(y)| \leq C(\rho(\mu, \nu) + |x - y|),\ \mu, \nu \in \mathcal{M}_{\lambda^p}(\mathbb{R}^d)$

for $x, y \in \mathbb{R}^d$.

Definition 2.3. The function f is said to be in $C^2(\mathcal{M}_{\lambda^p}(\mathbb{R}^d))$ if for every $\mu \in \mathcal{M}_{\lambda^p}(\mathbb{R}^d)$, $f \in C^1(\mathcal{M}_{\lambda^p}(\mathbb{R}^d))$ and $\partial_\mu f(\mathbb{P}_\xi)(\cdot)$ is differentiable and its derivative $\partial_y \partial_\mu f : \mathcal{M}_{\lambda^p}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is continuous.

Definition 2.4. The function f is said to be in $C_b^{2,1}(\mathcal{M}_{\lambda^p}(\mathbb{R}^d))$ if $f \in C^2(\mathcal{M}_{\lambda^p}(\mathbb{R}^d)) \cap C_b^{1,1}(\mathcal{M}_{\lambda^p}(\mathbb{R}^d))$ and its derivative $\partial_y \partial_\mu f$ is bounded and continuous.

DEFINITION 2.5. The function $\Phi \in C_b^{2,2,1}(\mathcal{M}_{\lambda^p}(\mathbb{R}^d) \times \mathbb{R}^d)$ if:

(i) Φ is bi-continuous with respect to (x, μ);
(ii) for any x, $\Phi(x, \cdot) \in C_b^{2,1}(\mathcal{M}_{\lambda^p}(\mathbb{R}^d))$ and for any $\mu \in \mathcal{M}_{\lambda^p}(\mathbb{R}^d)$, $\Phi(\cdot, \mu) \in C^2(\mathbb{R}^d)$.

If $\Phi \in C_b^{2,2,1}(\mathcal{M}_{\lambda^p}(\mathbb{R}^d \times \mathbb{R}^d))$ and $\Phi \geq 0$, then we say $\Phi \in C_{b+}^{2,2,1}(\mathbb{R}^d \times \mathcal{M}_{\lambda^p}(\mathbb{R}^d))$.

DEFINITION 2.6. The function $\Phi \in C(\mathbb{R}^d \times \mathcal{M}_{\lambda^p}(\mathbb{R}^d))$, if $\Phi \in C^{2,2}(\mathbb{R}^d \times \mathcal{M}_{\lambda^p}(\mathbb{R}^d))$ and for every compact set $K \subseteq \mathbb{R}^d \times \mathcal{M}_{\lambda^p}(\mathbb{R}^d)$,

$$\sup_{(x,\mu)\in K} \int_{\mathbb{R}^d} (\|\partial_y \partial_\mu \Phi(x, \mu)(y)\|^2 + |\partial_\mu \Phi(x, \mu)(y)|^2)\mu(dy) < \infty.$$

If $\Phi \in C(\mathbb{R}^d \times \mathcal{M}_{\lambda^p}(\mathbb{R}^d))$ and $\Phi \geq 0$, then we say that $\Phi \in C_+(\mathbb{R}^d \times \mathcal{M}_{\lambda^p}(\mathbb{R}^d))$.

For (1.1), we make the following assumptions.

ASSUMPTION 2.7. Functions b, σ are continuous with respect to $(x, \mu) \in \mathbb{R}^d \times \mathcal{M}_{\lambda^2}(\mathbb{R}^d)$, and there is a constant $L_1 > 0$ such that

$$|b(x, \mu)|^2 + \|\sigma(x, \mu)\|^2 \leq L_1(1 + |x|^2 + \|\mu\|_{\lambda^2}^2).$$

ASSUMPTION 2.8. There is a constant $L_2 > 0$ such that

$$2\langle x - y, b(x, \mu) - b(y, \mu)\rangle + \|\sigma(x, \mu) - \sigma(y, \nu)\|^2 \leq L_2(|x - y|^2 + \rho(\mu, \nu)^2).$$

ASSUMPTION 2.9. There exists a function $v(\cdot, \cdot) : \mathbb{R}^d \times \mathcal{M}_{\lambda^p}(\mathbb{R}^d) \to \mathbb{R}$ such that:

(i) $v \in C_+(\mathbb{R}^d \times \mathcal{M}_{\lambda^2}(\mathbb{R}^d))$;
(ii) $\int_{\mathbb{R}^d} (L^\mu v(x, \mu) + \gamma v(x, \mu))\mu(dx) \leq 0$;
(iii) for some $p \geq 1$, $a_1 \int_{\mathbb{R}^d} |x|^p \mu(dx) \leq \int_{\mathbb{R}^d} v(x, \mu)\mu(dx) \leq a_2 \int_{\mathbb{R}^d} |x|^p \mu(dx)$.

By the classical result of Wang [19], under Assumptions 2.7–2.9, there exists a unique strong solution $X_t^{x_0}$, with initial value x_0, to (1.1), and for $p \geq 2$, $\mathbb{E}\sup_{0\leq t\leq T} |X_t^{x_0}|^p < \infty$. Let $C^2(\mathbb{R}^d \times \mathcal{M}_{\lambda^2}(\mathbb{R}^d); \mathbb{R}^+)$ denote the space of nonnegative functions which are continuous and twice differentiable. For $V \in C^2(\mathbb{R}^d \times \mathcal{M}_{\lambda^2}(\mathbb{R}^d); \mathbb{R}^+)$, we have the following generator of (1.1):

$$L^\mu V(x, \mu) = (b^i \partial_{x_i})(x, \mu) + \frac{1}{2}((\sigma\sigma^*)^{ij} \partial^2_{x_i x_j})(x, \mu) + \int_{\mathbb{R}^d} b^i(y, \mu)(\partial_\mu V)_i(x, \mu)(y)\mu(dy) + \frac{1}{2} \int_{\mathbb{R}^d} (\sigma\sigma^*)^{ij}(y, \mu)\partial_{y_i}(\partial_\mu V)_j(y)\mu(dy)\,.$$

REMARK 2.10. In fact, the strong solution to (1.1) defines a Markov process [19]. Let $p(t, x_0, dz)$ be the transition probability distribution to process $X_t^{x_0}$ and $p(t, x_0, \Gamma)$ be the probability for the event $\{X_t^{x_0} \in \Gamma\}$ with the initial value x_0, that is,

$$p(t, x_0, \Gamma) = \int_\Gamma p(t, x_0, dz), \quad \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

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DEFINITION 2.11. The process $X_t^{x_0}$ with initial value x_0 is called stable in distribution if there exists a probability measure $\Pi(\cdot)$ such that for any initial value x_0, its transition probability $p(t, x_0, dz)$ weakly converges to $\Pi(\cdot)$, as $t \to \infty$. Equation (1.1) is said to be stable in distribution if $X_t^{x_0}$ is stable in distribution.

To study the stability in distribution, we need the following assumption. First, for a given function $U \in C^2(\mathbb{R}^d; \mathbb{R})$, we define the operator

$$\begin{aligned} L(x, y, \mu, \nu)U(x-y) &= [b(x,\mu) - b(y,\nu)]U_x(x-y) \\ &\quad + \tfrac{1}{2} Tr[(\sigma(x,\mu) - \sigma(y,\nu))^* U_{xx}(x-y)(\sigma(x,\mu) - \sigma(y,\nu))]. \end{aligned}$$

ASSUMPTION 2.12. There exist a function $U \in C^2(\mathbb{R}^d; \mathbb{R}_+)$ and a constant $K > 0$, such that for any two solutions $(X_t^{x_0})_{t\geq 0}$ and $(X_t^{y_0})_{t\geq 0}$ with its distributions $\mathcal{L}(X_t^{x_0}) = \mu_t$ and $\mathcal{L}(X_t^{y_0}) = \nu_t$, and for all couplings $\pi \in \Pi(\mu_t, \nu_t)$,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} L(x, y, \mu_t, \nu_t)U(x-y)\pi(dx, dy) \leq -K \int_{\mathbb{R}^d\times\mathbb{R}^d} U(x-y)\pi(dx, dy).$$

3. Stability analysis

3.1. Exponential stability This section gives the exponential stability in the pth moment and in a.s. sense.

DEFINITION 3.1. Let $p \geq 1$. The solution $X_t^{x_0}$ of (1.1) is said to be pth moment exponentially stable if there is a pair of constants $\gamma > 0$ and $C > 0$ such that

$$\mathbb{E}|X_t^{x_0}|^p \leq C|x_0|^p e^{-\gamma t}, \quad t \geq 0.$$

Further, it is said to be a.s. exponentially stable if

$$\limsup_{t\to\infty} \frac{\log|X_t^{x_0}|}{t} \leq -\gamma, \quad \text{a.s.}$$

For this, we further assume that for some constant $N > 0$ and $p \geq 1$,

$$|b(x,\mu)|^p + \|\sigma(x,\mu)\|^p \leq N\Big(|x|^p + \int_{\mathbb{R}^d} |x|^p \mu(dx)\Big). \tag{3.1}$$

THEOREM 3.2. *Assume that Assumptions 2.7–2.9 hold. For every $x_0 \in \mathbb{R}^d$, $X_t^{x_0}$ is pth moment exponentially stable and a.s. exponentially stable. Furthermore, for every $\epsilon > 0$, there exists an $R > 0$ such that for all $t \geq 0$, $\mathbb{P}\{|X_t^{x_0}| \geq R\} < \epsilon$.*

PROOF. For $x_0 \in \mathbb{R}^d$, let $X_t^{x_0}$ be the solution of (1.1), for positive integer k, define the stopping times

$$\rho_k = \inf\{t > 0 \mid |X_t^{x_0}| \geq k\},$$

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then obviously, $\rho_k \to \infty$, a.s. as $k \to \infty$. Then for the stopped processes $(X^{x_0}_{t\wedge\rho_k})_{t\geq 0}$, function v and distribution processes $\mathcal{L}(X^{x_0}_t)_{t\geq 0}$, by Itô's formula [9],

$$
\begin{aligned}
&e^{\gamma(t\wedge\rho_k)}v(X^{x_0}_{t\wedge\rho_k}, \mathcal{L}(X^{x_0}_t)) - v(x_0, \delta_{x_0}) \\
&\qquad = \int_0^t \gamma e^{\gamma(s\wedge\rho_k)} v(X^{x_0}_{s\wedge\rho_k}, \mathcal{L}(X^{x_0}_s))\, ds \\
&\qquad\quad + \int_0^t e^{\gamma(s\wedge\rho_k)} (b^i \partial_{x_i} v)(X^{x_0}_{s\wedge\rho_k}, \mathcal{L}(X^{x_0}_s))\, ds \\
&\qquad\quad + \frac{1}{2}\int_0^t e^{\gamma(s\wedge\rho_k)} ((\sigma\sigma^*)^{ij} \partial^2_{x_i x_j} v)(X^{x_0}_{s\wedge\rho_k}, \mathcal{L}(X^{x_0}_s))\, ds \\
&\qquad\quad + \int_0^t e^{\gamma(s\wedge\rho_k)} (\partial_{x_i} v \sigma^{ij})(X^{x_0}_{s\wedge\rho_k}, \mathcal{L}(X^{x_0}_s))\, dW^j_s \\
&\qquad\quad + \int_0^t \int_{\mathbb{R}^d} e^{\gamma(s\wedge\rho_k)} ((b^i \partial_{x_i} v)(X^{x_0}_{s\wedge\rho_k}, \mathcal{L}(X^{x_0}_s))(y)\mathcal{L}(X^{x_0}_s)(dy)\, ds \\
&\qquad\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} e^{\gamma(s\wedge\rho_k)} (\sigma\sigma^*)^{ij} (X^{x_0}_{s\wedge\rho_k}, \mathcal{L}(X^{x_0}_s))(y)\mathcal{L}(X^{x_0}_s)(dy)\, ds.
\end{aligned}
$$

Then taking expectation, by Assumption 2.9,

$$\mathbb{E}[e^{\gamma(t\wedge\rho_k)} v(X^{x_0}_{t\wedge\rho_k}, \mathcal{L}(X^{x_0}_t))] - v(x_0, \delta_{x_0}) \leq 0.$$

Let $k \to \infty$ and together with the Fatou lemma [7],

$$\mathbb{E}[e^{\gamma t} v(X_t, \mathcal{L}(X^{x_0}_t))] \leq v(x_0, \delta_{x_0}) \leq 0.$$

Furthermore, by Assumption 2.7,

$$a_1 \mathbb{E}|X^{x_0}_t|^p \leq \mathbb{E} v(X_t, \mathcal{L}(X^{x_0}_t)) \leq e^{-\gamma t} v(x_0, \delta_{x_0}) \leq a_2 e^{-\gamma t}|x_0|^p.$$

Thus,

$$\mathbb{E}|X^{x_0}_t|^p \leq \frac{a_2}{a_1} e^{-\gamma t}|x_0|^p.$$

Then by Chebyshev's inequality [7], for $R > 0$,

$$\mathbb{P}\{|X^{x_0}_t| \geq R\} \leq \frac{\mathbb{E}|X^{x_0}_t|^p}{R^p}.$$

Noticing that from (1.1),

$$X^{x_0}_{t+s} = X^{x_0}_t + \int_t^{t+s} b(X^{x_0}_u, \mathcal{L}(X^{x_0}_u))\, du + \int_t^{t+s} \sigma(X^{x_0}_u, \mathcal{L}(X^{x_0}_u))\, dW_u,$$

and for $p \geq 1$, $\tau > 0$, by (3.1),

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$$
\begin{aligned}
\mathbb{E}\sup_{0\le s\le\tau}|X_{t+s}^{x_0}|^p &\le C_p\mathbb{E}|X_t^{x_0}|^p + C_p\mathbb{E}\sup_{0\le s\le\tau}\Big|\int_t^{t+s} b(X_u^{x_0},\mathcal{L}(X_u^{x_0}))\,du\Big|^p\\
&\quad + C_p\mathbb{E}\sup_{0\le s\le\tau}\Big|\int_t^{t+s}\sigma(X_u^{x_0},\mathcal{L}(X_u^{x_0}))\,dW_u\Big|^p\\
&\le C_p\mathbb{E}|X_t^{x_0}|^p + C_{p,\tau}\mathbb{E}\int_t^{t+\tau}|b(X_u^{x_0},\mathcal{L}(X_u^{x_0}))|^p\,du\\
&\quad + C_p\mathbb{E}\int_t^{t+\tau}|\sigma(X_u^{x_0},\mathcal{L}(X_u^{x_0}))|^p\,du\\
&\le C_p\mathbb{E}|X_t^{x_0}|^p + C_{p,\tau,N}\mathbb{E}\int_t^{t+\tau}\Big(|X_u^{x_0}|^p + \int_{\mathbb{R}^d}|x|^p\mathcal{L}(X_u^{x_0})(dx)\Big)\,du\\
&\le C_p\mathbb{E}|X_t^{x_0}|^p + C_{p,\tau,N}\int_t^{t+\tau}\mathbb{E}|X_u^{x_0}|^p\,du\\
&\le C_{p,x_0}e^{-\gamma t} + C_{p,\tau,x_0,N}\int_t^{t+\tau}e^{-\gamma u}\,du \le C_{p,\tau,x_0,\gamma,N}e^{-\gamma t}.
\end{aligned}
$$

Then for $n = 1, 2, \ldots,$

$$
\mathbb{E}\sup_{n\tau\le t\le(n+1)\tau}|X_t^{x_0}|^p \le C_{p,\tau,x_0,\gamma,N}e^{-\gamma n\tau},
$$

thus, for $\epsilon \in (0,\gamma)$ and $n \in \mathbb{N}$, by Chebychev's inequality,

$$
\mathbb{P}\Big(\omega : \sup_{n\tau\le t\le(n+1)\tau}|X_t^{x_0}|^p > e^{-(\gamma-\epsilon)n\tau}\Big) \le C_{p,\tau,x_0,\gamma,N}e^{-\epsilon n\tau}.
$$

By the Borel–Cantelli lemma [11], there exists a random constant $n_0(\omega)$ such that for almost all $\omega \in \Omega$, for $n > n_0(\omega)$,

$$
\sup_{n\tau\le t\le(n+1)\tau}|X_t^{x_0}|^p \le C_{p,\tau,x_0,\gamma,N}e^{-(\gamma-\epsilon)n\tau}, \quad \text{a.s.}
$$

Thus, for any $n\tau \le t \le (n+1)\tau$,

$$
\frac{\log|X_t^{x_0}|}{t} = \frac{\log|X_t^{x_0}|^p}{pt} \le \frac{\log\sup_{n\tau\le t\le(n+1)\tau}|X_t^{x_0}|^p}{pn\tau} \le -\frac{\gamma-\epsilon}{p}, \quad \text{a.s.,}
$$

and

$$
\limsup_{t\to\infty}\frac{\log|X_t^{x_0}|}{t} \le -\frac{\gamma-\epsilon}{p}, \quad \text{a.s.}
$$

Now letting $\epsilon \to 0$, the proof is complete. □

REMARK 3.3. From Theorem 3.2, the transition probability family $\{p(t,x_0,dz) \mid t \ge 0\}$ is tight, that is, for $\epsilon > 0$, there exists a compact set $\mathcal{K} = \mathcal{K}(x_0,\epsilon)$ such that

$$
\mathbb{P}(t,x_0,\mathcal{K}) \ge 1-\epsilon.
$$

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3.2. Stability in distribution Next, we consider the stability in distribution. For this, we need to consider the difference between two solutions with different initial values, that is,

$$X_t^{x_0} - X_t^{y_0} = x_0 - y_0 + \int_0^t [b(X_s^{x_0}, \mathcal{L}(X_s^{x_0})) - b(X_s^{y_0}, \mathcal{L}(X_s^{y_0}))]\, ds$$
$$+ \int_0^t [\sigma(X_s^{x_0}, \mathcal{L}(X_s^{x_0})) - \sigma(X_s^{y_0}, \mathcal{L}(X_s^{y_0}))]\, dW_s.$$

We need two more notation. Let $\mathcal{H}$ be the set consisting of nondecreasing functions $K : \mathbb{R}_+ \to \mathbb{R}_+$ such that $K(0) = 0$, and $\mathcal{H}_\infty$ be the set of functions $K \in \mathcal{H}$ such that $K(x) \to \infty$ as $x \to \infty$.

LEMMA 3.4. *If there exists a function $U \in C^2(\mathbb{R}^d; \mathbb{R}_+)$ satisfying $U(0) = 0$ and a function $\alpha_1 \in \mathcal{H}_\infty$ such that*

$$\alpha_1(|x|) \le U(x) \quad \textit{for} \ \ x \in \mathbb{R}^d,$$

then for every $\epsilon > 0$ and compact set $\mathcal{K}$ on $\mathbb{R}^d$, there exists $T = T(\mathcal{K}, \epsilon) > 0$ such that

$$\mathbb{P}\{\|X_t^{x_0} - X_t^{y_0}\| > \epsilon\} < 1 - \epsilon, \quad t \ge T,\ x_0,\ y_0 \in \mathcal{K}.$$

For the convenience of presentation in the following, we rewrite (1.1) as

$$X_t^{x_0} = x_0 + \int_0^t \tilde{b}(u, X_u^{x_0})\, du + \int_0^t \tilde{\sigma}(u, X_u^{x_0})\, dW_u, \quad t \ge 0, \tag{3.2}$$

where $\tilde{b}(u, X_u^{x_0}) = b(X_u^{x_0}, \mathcal{L}(X_u^{x_0}))$ and $\tilde{\sigma}(u, X_u^{x_0}) = \sigma(X_u^{x_0}, \mathcal{L}(X_u^{x_0}))$. Since (1.1) has a unique strong solution $(X_t^{x_0})_{t\ge 0}$ with the initial distribution δ_{x_0}, the distribution of process $(X_t^{x_0})_{t\ge 0}$ is known, and (3.2) is a classic SDE. Then, the solution of (3.2) is a strong Markov process [5, Lemma 5.3].

PROOF OF LEMMA 3.4. For $\epsilon > 0$, by the continuity of function U with $U(0) = 0$, we choose $\alpha \in (0, \epsilon)$ small enough such that

$$\frac{\sup_{|x|\le\alpha} U(x)}{\mu_1(\epsilon)} < \frac{\epsilon}{2}.$$

Let $\mathcal{K}$ be a compact set on $\mathbb{R}^d$ and for fixed x_0, $y_0 \in \mathcal{K}$ and $\beta > \alpha$, we define two stopping times

$$\tau_\alpha = \inf\{t \ge 0 \mid |X_t^{x_0} - X_t^{y_0}| \le \alpha\},$$
$$\tau_\beta = \inf\{t \ge 0 \mid |X_t^{x_0} - X_t^{y_0}| \ge \beta\}.$$

By Itô's formula for the stopped process $U(X_{\tau_\beta\wedge t}^{x_0} - X_{\tau_\beta\wedge t}^{y_0})$ and Assumption 2.12,

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$$
\begin{aligned}
\mathbb{E}U(X^{x_0}_{\tau_\beta\wedge t}-X^{y_0}_{\tau_\beta\wedge t}) &\le U(x_0-y_0)-K\int_0^{\tau_\beta\wedge t}U(X^{x_0}_s-X^{y_0}_s)\,ds\\
&\quad+\mathbb{E}\int_0^{\tau_\beta\wedge t}U_x(X^{x_0}_s-X^{y_0}_s)(\tilde{\sigma}(s,X^{x_0}_s)-\tilde{\sigma}(u,X^{y_0}_s))\,dW_s\\
&=U(x_0-y_0)-K\int_0^{\tau_\beta\wedge t}U(X^{x_0}_s-X^{y_0}_s)\,ds.
\end{aligned}
$$

Then,

$$\alpha_1(\beta)\mathbb{P}\{\tau_\beta\le t\}\le U(x_0-y_0),$$

that is,

$$\mathbb{P}\{\tau_\beta\le t\}\le\frac{U(x_0-y_0)}{\alpha_1(\beta)}.$$

Notice that for all $x_0, y_0 \in \mathcal{K}$ and $U(x_0-y_0)$ bounded, there exists $\beta=\beta(\mathcal{K},\epsilon)>0$ such that

$$\mathbb{P}\{\tau_\beta<\infty\}\le\frac{\epsilon}{4}.$$

Fix the β and let $t_\alpha=\tau_\alpha\wedge\tau_\beta\wedge t$, then similar discussion yields

$$
\begin{aligned}
\mathbb{E}U(X^{x_0}_{t_\alpha}-X^{y_0}_{t_\alpha}) &\le U(x_0-y_0)-K\mathbb{E}\int_0^{t_\alpha}U(X^{x_0}_s-X^{y_0}_s)\,ds\\
&\le U(x_0-y_0)-K\mathbb{E}\int_0^{t_\alpha}\alpha_1(|X^{x_0}_s-X^{y_0}_s|)\,ds\\
&\le U(x_0-y_0)-K\alpha_1(\alpha)\mathbb{E}(\tau_\alpha\wedge\tau_\beta\wedge t),
\end{aligned}
$$

which implies that

$$\mathbb{P}\{\tau_\alpha\wedge\tau_\beta\ge t\}\le\frac{U(x_0-y_0)}{K\alpha_1(\alpha)t}.$$

Moreover, this implies that for a given $\epsilon\in(0,1)$, there exists $T=T(\mathcal{K},\epsilon)>0$ such that

$$\mathbb{P}\{\tau_\alpha\wedge\tau_\beta\le T\}>1-\frac{\epsilon}{4}.$$

Thus,

$$
\begin{aligned}
1-\frac{\epsilon}{4}<\mathbb{P}\{\tau_\alpha\wedge\tau_\beta\le T\} &\le \mathbb{P}\{\tau_\alpha\le T\}+\mathbb{P}(\tau_\beta\le T)\\
&\le \mathbb{P}\{\tau_\alpha\le T\}+\mathbb{P}(\tau_\beta\le\infty)\\
&\le \mathbb{P}\{\tau_\alpha\le T\}+\frac{\epsilon}{4}
\end{aligned}
$$

and

$$\mathbb{P}\{\tau_\alpha\le T\}\ge 1-\frac{\epsilon}{2}.$$

Now we define the stopping time

$$\sigma = \inf\{t \geq \tau_\alpha \wedge T \mid |X_t^{x_0} - X_t^{y_0}| \geq \epsilon\}.$$

Let $t > T$, then

$$\begin{aligned}\mathbb{P}(\tau_\alpha \leq T \cap \sigma \leq t)\mu_1(\epsilon) &\leq \mathbb{E}(I_{\tau_\alpha \leq T, \sigma \leq t} U(X_{t\wedge\sigma}^{x_0} - X_{t\wedge\sigma}^{y_0})) \\ &\leq \mathbb{E}(I_{\tau_\alpha \leq T} U(X_{t\wedge\tau_\alpha}^{x_0} - X_{t\wedge\tau_\alpha}^{y_0})) \\ &\leq \mathbb{E}(I_{\tau_\alpha \leq T} U(X_{\tau_\alpha}^{x_0} - X_{\tau_\alpha}^{y_0})) \leq \mathbb{P}(\tau_\alpha \leq T) \sup_{|x|\leq\alpha} U(x).\end{aligned}$$

Thus,

$$\mathbb{P}(\{\tau_\alpha \leq T\} \cap \{\sigma \leq t\}) \leq \frac{\epsilon}{2}$$

and

$$\mathbb{P}\{\sigma \leq t\} \leq \mathbb{P}(\{\tau_\alpha \leq T\} \cap \{\sigma \leq t\}) + \mathbb{P}\{\tau_{\alpha > T}\} < \epsilon.$$

Let $t \to \infty$, then

$$\mathbb{P}\{\sigma < \infty\} \leq \epsilon.$$

This indicates that for all $x_0, y_0 \in \mathcal{K}$, $t \geq T$,

$$\mathbb{P}\{|X_t^{x_0} - X_t^{y_0}| < \epsilon\} \geq 1 - \epsilon.$$

This completes the proof. □

LEMMA 3.5. *For every compact set $\mathcal{K}$,*

$$\lim_{t\to\infty} \rho(p(t, x_0, \cdot), p(t, y_0, \cdot)) = 0, \quad x_0, \ y_0 \in \mathcal{K}.$$

PROOF. We only need to show that there exists $T > 0$, such that for all $\epsilon > 0$ and x_0, $y_0 \in \mathcal{K}$,

$$\rho(p(t, x_0, \cdot), p(t, y_0, \cdot)) \leq \epsilon, \quad t \geq T.$$

It is equivalent to show that

$$\sup_{\phi \in C_\lambda(\mathbb{R}^d)} |\mathbb{E}\phi(X_t^{x_0}) - \mathbb{E}\phi(X_t^{y_0})| \leq \epsilon, \quad t \geq T.$$

However, notice that for every $\phi \in C_\lambda(\mathbb{R}^d)$,

$$|\mathbb{E}\phi(X_t^{x_0}) - \mathbb{E}\phi(X_t^{y_0})| \leq \mathbb{E}[2 \wedge |X_t^{x_0} - X_t^{y_0}|].$$

By Lemma 3.4, there exists a $T_1 > 0$ such that

$$\mathbb{E}[2 \wedge |X_t^{x_0} - X_t^{y_0}|] < \epsilon, \quad t \geq T_1.$$

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Due to the arbitrariness of $\phi \in C_\lambda(\mathbb{R}^d)$,

$$\sup_{\phi \in C_\lambda(\mathbb{R}^d)} |\mathbb{E}\phi(X_t^{x_0}) - \mathbb{E}\phi(X_t^{y_0})| \le \epsilon, \quad t \ge T_1.$$

The proof is now complete. □

LEMMA 3.6. *Under the assumptions of Theorem 3.2 and Lemma 3.4, for $x_0 \in \mathbb{R}^d$, $\{p(t, x_0, \cdot), t \ge 0\}$ is a Cauchy sequence.*

PROOF. We need to show that for every $x_0 \in \mathbb{R}^d$ and $\epsilon > 0$, there exists $T > 0$ such that for $t \ge T$ and $s > 0$,

$$\rho(p(t+s, x_0, \cdot), p(t, x_0, \cdot)) \le \epsilon,$$

which is equivalent that for every $\phi \in C_\lambda(\mathbb{R}^d)$,

$$\sup_{\phi \in C_\lambda(\mathbb{R}^d)} |\mathbb{E}\phi(X_{t+s}^{x_0}) - \mathbb{E}\phi(X_t^{x_0})| \le \epsilon, \quad t \ge T, \; s > 0.$$

By Lemma 3.2, there exists a compact set $\mathcal{K}$ on $\mathbb{R}^d$ such that for $\epsilon > 0$,

$$p(t, x_0, \mathcal{K}) > 1 - \frac{\epsilon}{8}.$$

Furthermore, by the strong Markov property of $X_t^{x_0}$, for $\phi \in C_\lambda(\mathbb{R}^d)$ and $t, s > 0$,

$$\begin{aligned}
|\mathbb{E}\phi(X_{t+s}^{x_0}) - \mathbb{E}\phi(X_t^{x_0})| &= |\mathbb{E}[\mathbb{E}(\phi(X_{t+s}^{x_0})|\mathcal{F}_s)] - \mathbb{E}\phi(X_t^{x_0})| \\
&= |\mathbb{E}[\mathbb{E}\phi(X_t^{s, X_s^{x_0}})] - \mathbb{E}\phi(X_t^{x_0})| \\
&= \Big| \int_{\mathbb{R}^d} \mathbb{E}\phi(X_t^z) p(s, x_0, dz) - \mathbb{E}\phi(X_t^{x_0}) \Big| \\
&\le \int_{\mathbb{R}^d} \mathbb{E}|\phi(X_t^z) - \phi(X_t^{x_0})| p(s, x_0, dz) \\
&= \int_{\mathcal{K}} \mathbb{E}|\phi(X_t^z) - \phi(X_t^{x_0})| p(s, x_0, dz) \\
&\quad + \int_{\mathbb{R}^d - \mathcal{K}} \mathbb{E}|\phi(X_t^z) - \phi(X_t^{x_0})| p(s, x_0, dz) \\
&\le \int_{\mathcal{K}} \mathbb{E}|\phi(X_t^z) - \phi(X_t^{x_0})| p(s, x_0, dz) + \frac{\epsilon}{4}.
\end{aligned}$$

By Lemma 3.4, there exists $T > 0$ such that for every $\epsilon > 0$,

$$\mathbb{E}|\phi(X_t^z) - \phi(X_t^{x_0})| \le \mathbb{E}[2 \wedge |X_t^z - X_t^{x_0}|] \le \frac{3\epsilon}{4}, \quad t \ge T.$$

Thus,

$$|\mathbb{E}\phi(X_{t+s}^{x_0}) - \mathbb{E}\phi(X_t^{x_0})| \le \epsilon, \quad t \ge T, \; s > 0,$$

which completes the proof. □

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THEOREM 3.7. *Under assumptions of Theorem 3.2 and Lemma 3.4, (1.1) is stable in distribution.*

PROOF. By Definition 2.11, we need to show that there exists a probability measure $\pi(\cdot)$ such that for every $x_0 \in \mathbb{R}^d$, the transition probability family $\{p(t, x_0, \cdot) : t \geq 0\}$ weakly converges to $\pi(\cdot)$. In fact, we show that for every $x \in \mathbb{R}^d$,

$$\lim_{t\to\infty} \rho(p(t, x_0, \cdot), \pi(\cdot)) = 0.$$

By Lemma 3.6, $\{p(t, 0, \cdot) : t \geq 0\}$ is a Cauchy sequence in $\mathcal{P}(\mathbb{R}^d)$ with the metric ρ. Since $\mathcal{P}(\mathbb{R}^d)$ is a complete metric space, there exists a probability measure $\pi(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ such that

$$\lim_{t\to\infty} \rho(p(t, x, \cdot), \pi(\cdot)) = 0.$$

By a triangle inequality,

$$\begin{aligned}\lim_{t\to\infty} \rho(p(t, x, \cdot), \pi(\cdot)) &\leq \lim_{t\to\infty} \rho(p(t, x, \cdot), p(t, 0, \cdot)) + \lim_{t\to\infty} \rho(p(t, 0, \cdot), \pi(\cdot)) \\ &= 0.\end{aligned}$$

The proof is complete. □

4. Conclusions

In the study, we first prove the pth moment exponential stability and a.s. exponential stability of the solution of (1.1) by using the distribution-dependent Itô's formula, and then obtain the tightness of the transition probability family corresponding to the solution of (1.1). Based on this, we introduce a distribution-dependent operator, that is, Assumption 2.12, and combined with the method of Yuan and Mao [21], we get that when the time is long enough, the transition probability family tends to a unique probability measure, that is, the solution of (1.1) is asymptotically stable in distribution. It would be valuable to use a similar method to analyse the long time behaviour of (1.1) with jump noise.

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