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SUFFICIENCY AND THE JACOBI CONDITION IN THE CALCULUS OF VARIATIONS

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1. Introduction. Besides stating the problem and the results, we shall give in this section a brief overview of the classical necessary and sufficient conditions in the calculus of variations, in order to clearly situate the contribution of this article.

1.1 *The problem.* We are given an interval $[a, b]$, two points x_a, x_b in $\mathbf{R}^n$, and a function L (the Lagrangian) mapping $[a, b] \times \mathbf{R}^n \times \mathbf{R}^n$ to $\mathbf{R}$. The basic problem in the calculus of variations, labeled (P), is that of minimizing the functional

$$J(x) := \int_a^b L(t, x(t), \dot{x}(t))dt$$

over some class X of functions x and subject to the constraints $x(a) = x_a$, $x(b) = x_b$. Let us take for now the class X of functions to be the continuously differentiable mappings from $[a, b]$ to $\mathbf{R}^n$; we call such functions *smooth arcs*.

A *tube* $T(x; \epsilon)$ about the smooth arc x, where ϵ is a positive number, consists of all points (t, y) in $[a, b] \times \mathbf{R}^n$ satisfying

$$|x(t) - y| < \epsilon.$$

A restricted tube $RT(x; \epsilon)$ consists of all (t, y, v) in $[a, b] \times \mathbf{R}^n \times \mathbf{R}^n$ satisfying

$$|y - x(t)| < \epsilon \quad \text{and} \quad |v - \dot{x}(t)| < \epsilon.$$

An arc y is said to lie in $T(x; \epsilon)$ (for example) if for each t in $[a, b]$, the point $(t, y(t))$ lies in $T(x; \epsilon)$. The feasible arc x (i.e., element of X satisfying the boundary conditions) is said to be a *weak local minimum* if for some restricted tube $RT(x; \epsilon)$, for all feasible y in $RT(x; \epsilon)$, one has $J(y) \geqq J(x)$. A *strong local minimum* corresponds to replacing $RT(x; \epsilon)$ by $T(x; \epsilon)$ in this definition.

1.2 *Necessary and sufficient conditions for a weak local minimum.* Let us suppose that L is C^2 and that x is a weak local minimum. The best-known necessary condition satisfied by x is due to Euler (1744):

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(E) $\dfrac{d}{dt} L_v(t, x, \dot{x}) = L_x(t, x, \dot{x})$ on $[a, b]$.

The Legendre condition (1786) states that in addition

(L) $L_{vv}(t, x, \dot{x}) \geqq 0$ on $[a, b]$ (positive semidefinite).

Unlike the above, the Jacobi necessary condition (J) (1837) has a global (i.e., not pointwise) nature. It states that (if strict inequality holds in (L)) there is no point c in (a, b) corresponding to which there is a nontrivial solution h on $[a, c]$ of the homogeneous boundary-value problem

$$\frac{d}{dt}[L_{vv}(t)\dot{h}(t) + L_{vx}(t)h(t)] - L_{xv}(t)\dot{h}(t) - L_{xx}(t)h(t) = 0$$

$$h(a) = h(c) = 0,$$

where (for example) $L_{vv}(t)$ is an abbreviation of $L_{vv}(t, x(t), \dot{x}(t))$. (The differential equation is called the Jacobi equation.) A point c for which a nontrivial solution does exist is called a *conjugate* point, so that the Jacobi condition may be rephrased as:

(J) there are no conjugate points in (a, b).

Jacobi was the first to observe that a slight strengthening of the necessary conditions (E) (L) (J) was adequate to render them sufficient. We define:

(L)′ $L_{vv}(t, x, \dot{x}) > 0$ on $[a, b]$.

(J)′ there are no conjugate points in $(a, b]$.

THEOREM 1. *A smooth arc x satisfying* (E), (L)′ *and* (J)′ *is a weak local minimum for the problem* (P).

1.3 *Necessary and sufficient conditions for a strong local minimum.* Of course (E), (L) and (J) continue to be necessary conditions if x is a strong local minimum for the problem (P). Weierstrass (c. 1879) provided another for this case, in terms of the "excess function"

$$E(t, x, v, w) := L(t, x, w) - L(t, x, v) - (w - v) \cdot L_v(t, x, v).$$

The Weierstrass necessary condition asserts

(W) $E(t, x(t), \dot{x}(t), w) \geqq 0$ for t in $[a, b]$ and for all w in $\mathbf{R}^n$.

The strengthened form of this leading to a set of sufficient conditions is

(W)′ $E(t, y, v, w) \geqq 0$ for (t, y, v) in $RT(x; \epsilon)$ and for all w in $\mathbf{R}^n$.

THEOREM 2. (E) (L)′ (J)′ (W)′ *together imply that* x *is a strong local minimum for* (P).

We remark that (W)′ is satisfied whenever L is (globally) convex in v (i.e., $L_{vv} \geqq 0$).

1.4 *The Hamilton-Jacobi inequality.* The purpose of this article is to provide a new, simple and unified proof of Theorems 1 and 2. It utilizes an observation concerning an inequality for a C^1 function $W(t, y)$:

$$(1.1) \quad W_t(t, y) + W_y(t, y) \cdot v - L(t, y, v) \leqq W_t(t, x(t)) + W_y(t, x(t)) \cdot \dot{x}(t) - L(t, x(t), \dot{x}(t)).$$

PROPOSITION 3. *Suppose that there is a function* W *satisfying* (1.1) *for all* (t, y, v) *in a restricted tube* $RT(x; \epsilon)$. *Then* x *is a weak local minimum. If* (1.1) *is satisfied by all* (t, y) *in a tube* $T(x; \epsilon)$ *and for all* v, *then* x *is a strong local minimum.*

The proof consists of substituting $(t, y(t), \dot{y}(t))$ for (t, y, v) in (1.1) and integrating, where y is any arc feasible for (P).

The connection between certain equalities related to (1.1) on the one hand and fields (geodesic coverings) on the other is well known [**4**][**11**][**14**, Chapter 15][**16**][**17**]. Perhaps less appreciated is the utility of this simple (one-sided) verification technique in the absence of fields of extremals, especially when extended to a nonsmooth setting (see for example [**1**][**6**][**7**, Section 3.7][**8**]).

Our proof of Theorems 1 and 2 will hinge upon exhibiting a verification function W for x, thereby forging a pedagogically useful link to a different approach. Besides the fact that both theorems can be proven at once this way, we achieve a reduction in the smoothness required of L (i.e., from C^3 to C^2), and we gain a lot in simplicity. In particular, we eliminate the need to invoke general imbedding or field theorems of the theory of differential equations, an integral component of the usual proofs (see for example [**2**][**3**][**4**][**10**][**12**][**13**][**16**]). Our approach actually yields the optimality of x relative to all absolutely continuous feasible arcs, since Proposition 3 is robust for this modification. (Surprisingly, this fact does not follow from simple approximation, as shown by the possibility of the "Lavrentiev phenomenon" [**5**][**9**]). Finally, we remark that the proof adapts very easily to the situation in which the arc x is piecewise-smooth (see Section 3) which is not a feature of the classical approach (see for example the discussion, in [**10**, Section 3.8]).

None of the components of our approach is new per se; it is a matter of how they are used. In a related vein, we remark that the relationships between the Jacobi condition and the Hamilton-Jacobi inequality, and between both these concepts and that of convex integrands and canonical transformations, are studied in depth in **[18][19][20]** in a more general setting.

2. An associated matrix equation. There exists a complete theory relating the Jacobi equation to a certain Riccati equation and to a certain quadratic variational problem; see for example **[15]**. In order to make this article self-contained, we shall give an elementary proof of just those results of differential equations needed later. The hypotheses are those of Theorem 1. We define the following continuous $n \times n$-matrix-valued functions on $[a, b]$.

$$A(t) = -L_{vv}^{-1}(t)L_{vx}(t)$$

$$B(t) = L_{vv}^{-1}(t)$$

$$C(t) = L_{xx}(t) - L_{xv}(t)L_{vv}^{-1}(t)L_{vx}(t).$$

Note that B is symmetric and positive definite. We shall consider the linear matrix system

$$\text{(2.1)} \qquad \begin{cases} \dot{U}(t) = A(t)U(t) + B(t)V(t) \\ \dot{V}(t) = C(t)U(t) - A^*(t)V(t). \end{cases}$$

The corresponding system in $\mathbf{R}^n$ is

$$\text{(2.2)} \qquad \begin{cases} \dot{u}(t) = A(t)u(t) + B(t)v(t) \\ \dot{v}(t) = C(t)u(t) - A^*(t)v(t), \end{cases}$$

which is easily seen to be the Jacobi equation rewritten as a first-order system (the correspondence being u to h and v to $L_{vv}\dot{h} + L_{vx}h$). (In fact (2.2) defines the Jacobi equation when A, B, C are not differentiable.) In the following, * denotes transpose.

PROPOSITION 4. *There is a solution* (U, V) *of* (2.1) *on* $[a, b]$ *with* $\det U(t) \neq 0$ *and* $U^*(t)V(t) = V^*(t)U(t)$.

Proof. For ease of notation, let us take $[a, b] = [0, 1]$. Define (U_0, V_0) to be the solution of (2.1) on $[0, 1]$ satisfying

$$(U_0, V_0)(0) = (0, I)$$

(where I is the $n \times n$ identity matrix), and (U_1, V_1) to be the one satisfying

$$(U_1, V_1)(1) = (0, I).$$

We note

(2.3) $\quad U_1(0)$ and $U_0(t)$ are invertible for $t > 0$.

To prove the first of these assertions, simply suppose that $U_1(0)\zeta = 0$ for some nonzero element ζ of $\mathbf{R}^n$. Then

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$$u(t):=U_1(t)\zeta \quad \text{and} \quad v(t):=V_1(t)\zeta$$

provides a solution of (2.2) with $u(\cdot)$ nontrivial (since v is nontrivial) but vanishing at 0 and 1. This contradicts the strengthened Jacobi condition (J)′. The proof for $U_0(t)$ is the same. We note further

(2.4) $$U_i^*(t)V_i(t) = V_i^*(t)U_i(t), \quad t \in [a, b], i = 0, 1.$$

This results from the fact that the quantity

$$U_i^*(t)V_i(t) - V_i^*(t)U_i(t)$$

has zero derivative (substitution of (2.1)) and value 0 at $t = 0$ (for $i = 0$) or $t = 1$ (for $i = 1$). One final property is useful:

(2.5) $$U_0^*(t)V_1(t) - V_0^*(t)U_1(t) = -M^{-1}$$

for some constant invertible M. That the left side of (2.5) is constant is easily seen by differentiating it and substituting (2.1); the constant is invertible since the value of the left side at $t = 0$ is nonsingular by (2.3).

We now define

$$(U, V) = (U_0, V_0) + (U_1M, V_1M).$$

The desired property $U^*V = V^*U$ is an immediate consequence of (2.4) and (2.5), and $U(0)$, $U(1)$ are invertible by (2.3). Thus we need only prove that $U(\gamma)$ is invertible when γ lies in (0, 1).

Suppose not. Then for some nonzero ζ in $\mathbf{R}^n$ we have

(2.6) $$U(\gamma)\zeta = U_0(\gamma)\zeta + U_1(\gamma)M\zeta = 0.$$

We proceed to define

$$\begin{aligned}(u(t), v(t)) &= (U_0(t)\zeta, V_0(t)\zeta) \quad \text{on } [0, \gamma]\\ &= (-U_1(t)M\zeta, -V_1(t)M\zeta) \quad \text{on } (\gamma, 1].\end{aligned}$$

Note that (u, v) satisfies (2.2) on each of the subintervals $[0, \gamma)$ and $(\gamma, 1]$, and that u (but not necessarily v) is continuous on [0, 1] and vanishes at 0 and at 1. The following fact follows from substitution of (2.2):

(2.7) $$\frac{d}{dt}\langle u(t), v(t)\rangle = \langle u(t), C(t)u(t)\rangle + \langle v(t), Bv(t)\rangle.$$

Let I denote the value obtained when the right side of (2.7) is integrated over [0, 1]. In view of (2.7) we have

$$\begin{aligned}I &= \langle u(\gamma), v(\gamma-)\rangle - \langle u(\gamma), v(\gamma+)\rangle\\ &= \langle U_0(\gamma)\zeta, V_0(\gamma)\zeta\rangle - \langle U_1(\gamma)M\zeta, V_1(\gamma)M\zeta\rangle\\ &= \langle -U_1(\gamma)M\zeta, V_0(\gamma)\zeta\rangle - \langle -U_0(\gamma)\zeta, V_1(\gamma)M\zeta\rangle \quad \text{(by (2.6))}\end{aligned}$$

$$= \langle (U_0^*(\gamma)V_1(\gamma)M - V_0^*(\gamma)U_1(\gamma)M)\zeta, \zeta\rangle$$
$$= \langle -\zeta, \zeta\rangle = -|\zeta|^2 < 0 \quad \text{(by (2.5))}.$$

We shall now calculate I a different way to deduce that it is nonnegative, which gives the desired contradiction. We set

$$h(t) = \zeta \qquad \text{on } [0, \gamma]$$
$$= U_0^{-1}(t)u(t) \quad \text{on } [\gamma, 1]$$

(note that $U_0^{-1}(t)$ exists by (2.3), and that h is continuous), and we define

$$w(t) = V_0(t)h(t).$$

Note that $u(t) = U_0(t)h(t)$. It follows from this and (2.1) (2.4) that we have

$$\text{(2.8)} \quad \frac{d}{dt}\langle u(t), w(t)\rangle = \langle w(t), B(t)w(t)\rangle + 2\langle V_0(t)h(t), U_0(t)\dot{h}(t)\rangle + \langle u(t), C(t)u(t)\rangle.$$

We calculate

$$I = \int_0^1 \{ \langle u, Cu\rangle + \langle B^{-1}(\dot{u} - Au), \dot{u} - Au\rangle \}dt \quad \text{(by (2.2))}$$
$$= \int_0^1 \{ \langle u, Cu\rangle + \langle B^{-1}(\dot{U}_0 h + U_0\dot{h} - Au), \dot{U}_0 h + U_0\dot{h} - Au\rangle \}dt$$

(putting $u = Ah$)

$$= \int_0^1 \{ \langle u, Cu\rangle + \langle w, Bw\rangle + 2\langle V_0 h, U_0\dot{h}\rangle + \langle U_0\dot{h}, B^{-1}U_0\dot{h}\rangle \}dt$$

(after substituting from (2.1) for $\dot{U}_0$)

$$= \int_0^1 \left\{ \frac{d}{dt}\langle u, w\rangle + \langle U_0\dot{h}, B^{-1}U_0\dot{h}\rangle \right\} dt$$

(by (2.8))

$$= \int_0^1 \langle U_0\dot{h}, B^{-1}U_0\dot{h}\rangle dt \geqq 0$$

(since $B > 0$).

This completes the proof. The following corollary results immediately from the proposition by taking

$$Q(t) = V(t)U^{-1}(t)$$

and expanding the equation

$$\frac{d}{dt}(QU) = \dot{V}:$$

COROLLARY. *There exists a symmetric solution Q on $[a, b]$ of the matrix Riccati equation*

$$\dot{Q}(t) + Q(t)A(t) + A^*(t)Q(t) + Q(t)B(t)Q(t) - C(t) = 0.$$

The following technical result asserts that the Jacobi condition actually holds in a somewhat "stricter" form.

PROPOSITION 5. *Consider the system* (2.2) *with C replaced by $C - \delta I$. Then for $\delta > 0$ sufficiently small, there is no nontrivial solution (u, v) for which u vanishes both at a and at some point c in $(a, b]$.*

If the result is false, then there exist a sequence δ_i decreasing to 0 and corresponding (u_i, v_i) and c_i in $(a, b]$ such that on $[a, b]$ we have

$$(2.9) \quad \begin{cases} \dot{u}_i = Au_i + Bv_i \\ \dot{v}_i = (C - \delta_i I)u_i - A^* v_i \end{cases}$$

$$u_i(a) = u_i(c_i) = 0,$$

and without loss of generality

$$2.10) \quad \max_{[a, b]} |(u_i(t), v_i(t))| = 1.$$

The relation (2.9) implies

$$(2.11) \quad |(\dot{u}_i, \dot{v}_i)| \leqq k|(u_i, v_i)| \leqq k \quad \text{on } [a, b]$$

for some constant k not depending on i, whence

$$(2.12) \quad |(u_i(t), v_i(t))| \leqq K|(u_i(a), v_i(a))| = K|v_i(a)|$$

by Gronwall's lemma. We may assume $c_i \to c_0$, $v_i(a) \to v_0$. In view of (2.12) and (2.10), we have $v_0 \neq 0$. Since B is invertible, we have $Bv_0 \neq 0$; let w be a vector such that $\langle w, Bv_0 \rangle = 1$.

In view of (2.11), we have for some constant k_1:

$$|Au_i(t)| \leqq k_1(c_i - a),$$

$$|Bv_i(t) - Bv_i(a)| \leqq k_1(c_i - a) \quad \text{for } t \in [a, c_i].$$

If $c_0 = a$ (i.e., $c_i \to a$) it follows that for i sufficiently large we have

$$|\langle w, Au_i(t) \rangle| < \frac{1}{2}, \ \langle w, Bv_i(t) \rangle > \frac{1}{2}, \ t \in [a, c_i].$$

But this together with (2.9) implies

$$\langle \dot{u}_i(t), w \rangle > 0 \quad \text{on } [a, c_i],$$

a contradiction since

$$u_i(a) = u_i(c_i) = 0.$$

Thus $c_0 > a$.

Now $\{ (u_i, v_i) \}$ is uniformly equicontinuous on $[a, b]$ by (2.11) and bounded, so by the theorem of Arzela-Ascoli, there is a subsequence (we eschew relabeling) converging uniformly to (u_0, v_0). It follows from (2.9) that

$$\{ (\dot{u}_i, \dot{v}_i + \delta_i u_i) \} = \{ (Au_i + Bv_i, Cu_i - A^* v_i) \}$$

is also equicontinuous (and bounded by (2.11)), so that a further subsequence is such that $\dot{u}_i$ and $\dot{v}_i + \delta_i u_i$ converge uniformly, to $\dot{u}_0$ and $\dot{v}_0$ necessarily, in view of, for example,

$$u_i(t) = \int_a^t \dot{u}_i.$$

But then (u_0, v_0) is a nontrivial solution of the original system (2.2) admitting a conjugate point c_0 in $(a, b]$. This contradiction completes the proof.

The following is obtained by applying the Corollary to Proposition 4 with C replaced by $C - \delta I$, which is permissible in light of Proposition 5.

COROLLARY. *There exists a symmetric solution Q on $[a, b]$ of the matrix Riccati inequality*

$$\dot{Q}(t) + Q(t)A(t) + A^*(t)Q(t) + Q(t)B(t)Q(t) - C(t) < 0.$$

We remark that the results of this section remain valid if A, B, C are only assumed piecewise-continuous, with minor and evident changes in the proof.

3. The proof of theorems 1 and 2. We set

$$p(t) := L_v(t, x(t), \dot{x}(t))$$

$$W(t, y) := \langle p(t), y \rangle + \frac{1}{2} \langle y - x(t), Q(t)(y - x(t)) \rangle,$$

where Q is provided by the Corollary to Proposition 5, Section 2. The proof consists of showing that W has the properties of a verification function described in Proposition 3, Section 1.

The Euler equation (E) asserts that $\dot{p}(t)$ is equal to $L_x(t, x, \dot{x})$; it follows that p and hence W are C^1. Consider the equation

(3.1) $\quad p(t) + Q(t)(y - x(t))(= W_y(t, y)) = L_v(t, y, \zeta).$

A solution is $\zeta = \dot{x}(\gamma)$ when $t = \gamma$, $y = x(\gamma)$, and we have

$$L_{vv}(\gamma, x, \dot{x}) > 0$$

by (L)′. The implicit function theorem guarantees the existence of a C^1 function $\zeta(t, y)$ defining near $(\gamma, x(\gamma))$ the locally unique solution of (3.1). By an application of the Heine-Borel theorem, we may suppose that ζ is defined on a neighborhood of the graph of x; i.e., on a tube $T(x; \epsilon_1)$. By reducing ϵ_1 if necessary we may further suppose

(3.2) $\quad L_{vv}(t, y, v) > 0 \quad \text{on } RT(x; \epsilon_2),$

where

$$|\zeta(t, y) - \dot{x}(t)| < \epsilon_2 \quad \text{for all } (t, y) \text{ in } T(x; \epsilon_1).$$

Thus for fixed (t, y) in $T(x; \epsilon_1)$, the function

$$v \rightarrow \langle W_y(t, y), v\rangle - L(t, y, v)$$

is concave by (3.2) on the set $|v - \dot{x}(t)| < \epsilon_2$ and has zero gradient at $v = \zeta(t, y)$ by (3.1). We deduce

(3.3)
$$\max_{|v - \dot{x}| < \epsilon_2} \{W_t(t, y) + \langle W_y(t, y), v\rangle - L(t, y, v)\}$$
$$=: F(t, y) = W_t(t, y) + \langle W_y(t, y), \zeta(t, y)\rangle - L(t, y, \zeta(t, y)).$$

The function F defined above is also given by

$$\begin{aligned} &\langle \dot{p}(t), y\rangle - \langle \dot{x}(t), Q(t)(y - x(t))\rangle \\ &\quad + \frac{1}{2}\langle y - x(t), \dot{Q}(t)(y - x(t))\rangle \\ &\quad + \langle p(t) + Q(t)(y - x(t)), \zeta(t, y)\rangle \\ &\quad - L(t, y, \zeta(t, y)) \end{aligned}$$

once the derivatives W_t and W_y are substituted. Routine calculation then confirms that F_y, F_{yy} are continuous in (t, y) and satisfy

(3.4)
$$\begin{cases} F_y(t, x(t)) = 0 \\ F_{yy}(t, x(t)) = \dot{Q} + QA + A^*Q + QBQ - C < 0. \end{cases}$$

It follows from Taylor's formula that for every (s, y) near $(t, x(t))$, one has

$$F(s, y) \leqq F(s, x(s)).$$

Another application of the Heine-Borel theorem yields a tube $T(x; \epsilon_3)$ with the property that for every (t, y) in $T(x; \epsilon_3)$ one has

(3.5) $$F(t, y) \leqq F(t, x(t)) = W_t(t, x) + \langle W_y(t, x), \dot{x} \rangle - L(t, x, \dot{x}).$$

The inequality combined with (3.3) immediately yields the Hamilton-Jacobi inequality (1.1) on $RT(x; \epsilon)$, where

$$\epsilon = \min[\epsilon_1, \epsilon_2, \epsilon_3].$$

The fact that x is a weak local minimum results (Theorem 1).

Suppose now that (W)′ also holds. Let (t, y) belong to $T(x; \epsilon)$ and let v be any point in $\mathbf{R}^n$. Then

$$W_t(t, y) + \langle W_y(t, y), v \rangle - L(t, y, v)$$

$$= W_t(t, y) + \langle L_v(t, y, \zeta(t, y)), v \rangle - L(t, y, v)$$

(by (3.1))

$$\leqq W_t(t, y) + \langle L_v(t, y, \zeta(t, y)), \zeta(t, y) \rangle - L(t, y, \zeta(t, y))$$

(by (W)′ with $v = \zeta(t, y)$, $w = v$)

$$= F(t, y)$$

(by (3.3) and (3.1))

$$\leqq F(t, x(t)) = W_t(t, x) + \langle W_y(t, x), \dot{x} \rangle - L(t, x, \dot{x})$$

(by (3.5)). That x is a strong local minimum (Theorem 2) follows, again by Proposition 3.

Remarks. Because the inequalities in (3.2) and (3.4) are strict, the inequality (3.5) holds strictly for $y \neq x(t)$. This leads to the additional assertion that x is the unique (weak or strong) local minimum relative to $RT(x; \epsilon)$ or $T(x; \epsilon)$.

If x is piecewise-smooth, then A, B, C are piecewise-continuous and the resulting Q is piecewise-smooth. We proceed to verify (1.1) (for the same W, by the same method), on subintervals of $[a, b]$ on which $\dot{x}$ is continuous. W is now itself piecewise-smooth (rather than C^1), but this is quite adequate for Proposition 3. Thus Theorems 1 and 2 follow as before.

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