# ON THE FUNDAMENTAL GROUP OF A LIE SEMIGROUP

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The simplest type of Lie semigroups are closed convex cones in finite dimensional vector spaces. In general one defines a *Lie semigroup* to be a closed subsemigroup of a Lie group which is generated by one-parameter semigroups. If $W$ is a closed convex cone in a vector space $V$, then $W$ is convex and therefore simply connected. A similar statement for Lie semigroups is false in general. There exist generating Lie semigroups in simply connected Lie groups which are not simply connected (Example 1.15). To find criteria for cases when this is true, one has to consider the homomorphism

$$i_*:\pi_1(S)\rightarrow \pi_1(G)$$

induced by the inclusion mapping $i:S\rightarrow G$, where $S$ is a generating Lie semigroup in the Lie group $G$. Our main results concern the description of the image and the kernel of this mapping. We show that the image is the fundamental group of the largest covering group of $G$, into which $S$ lifts, and that the kernel is the fundamental group of the inverse image of $S$ in the universal covering group $\tilde{G}$. To get these results we construct a universal covering semigroup $\tilde{S}$ of $S$. If $j:H(S):=S\cap S^{-1}\rightarrow S$ is the inclusion mapping of the unit group of $S$ into $S$, then it turns out that the kernel of the induced mapping

$$j_*:\pi_1(H(S))\rightarrow \pi_1(S)$$

may be identfied with the fundamental group of the unit group $H(\tilde{S})$ of $\tilde{S}$ and that its image corresponds to the intersection $H(\tilde{S})_0\cap\pi_1(S)$, where $\pi_1(s)$ is identified with a central subgroup of $\tilde{S}$.

**1. The universal covering semigroup $\tilde{S}$.** Let $X$ be a path connected space and $x_0\in X$. In the following we write $\Omega(X,x_0)$ for the set of all continuous loops $\gamma:[0,1]\rightarrow X$ with $\gamma(0)=\gamma(1)=x_0$ and $\pi_1(X,x_0)$ for the quotient of $\Omega(X,x_0)$ modulo the homotopy relation with fixed endpoints. This is the *fundamental group of $X$ with respect to $x_0$*. If $\gamma:[0,1]\rightarrow X$ is a continuous path, which is not necessarily a loop, we write $[\gamma]$ for the homotopy class of $\gamma$ with fixed endpoints. For paths $\alpha,\beta:[0,1]\rightarrow X$, we set $\hat{\alpha}(t):=\alpha(1-t)$, and

$$\alpha\diamondsuit\beta(t)=\begin{cases}\alpha(2t) & \text{for } t\in[0,\frac{1}{2}]\\ \beta(2t-1) & \text{for } t\in[\frac{1}{2},1].\end{cases}$$

Note that this implies that $[\alpha\diamondsuit\beta]=[\alpha][\beta]$ if $\alpha$ and $\beta$ are loops. If $X$ is a topological monoid we usually use the unit element as base point. Since the isomorphy class of the group $\pi_1(X,x_0)$ is independent of $x_0$, we also write $\pi_1(X)$ for the fundamental group of $X$ without reference to a base point.

We recall the basic definitions and properties concerning Lie semigroups.

Definition 1.1. For a closed subsemigroup $S$ of a Lie group $G$ we define

$$\mathbf{L}(S):=\{X\in\mathbf{L}(G):\exp(\mathbb{R}^+X)\subseteq S\}$$

to be the *tangent wedge* of $S$. In fact, it turns out that $\mathbf{L}(S)$ is a *wedge*, i.e. a closed convex cone in the Lie algebra $\mathbf{L}(G)$ [**5**, V.1.6]. A closed subsemigroup $S$ of $G$ is called a *Lie semigroup* if it is reconstructable from its tangent wedge in the sense that

$$S = \overline{\langle \exp \mathbf{L}(S) \rangle}.$$

If $W$ is a wedge in a Lie algebra $\mathbf{g}$, we say that $W$ is *generating* if $\mathbf{g} = W - W$, and *Lie generating* if $\mathbf{g}$ is the smallest subalgebra containing $W$. The subspace $H(W) := W \cap -W$ is called *the edge of* $H(W)$ and $W$ is said to be *pointed* if $H(W) = \{0\}$.

We say that a Lie semigroup $S \subseteq G$ is *generating* if the wedge $\mathbf{L}(S)$ is Lie generating in $\mathbf{L}(G)$.

From now on $G$ denotes a connected Lie group and $S$ is a generating Lie semigroup in $G$.

PROPOSITION 1.2. †*For a generating Lie semigroup $S$ in $G$ the following assertions hold*

1) *There exists an analytic path $\alpha : [0,1] \to G$ such that*

$$\alpha(0) = \mathbf{1} \quad \textit{and} \quad \alpha(]0,1]) \subseteq \operatorname{int}(S).$$

2) *The interior* $\operatorname{int}(S)$ *is a dense semigroup ideal.*
3) *$S$ and* $\operatorname{int}(S)$ *are path connected.*
4) *$S$ is locally path connected.*
5) *$S$ is semi-locally simply connected.*

*Proof.* 1) follows from [**6**, Theorem 2.1].

2) That $\operatorname{int}(S)$ is a semigroup ideal is a consequence of the fact that a product of an arbitrary set and an open subset of a topological group is open. Let $\alpha$ be as in 1) and $s \in S$. Then $s\alpha\left(\frac{1}{n}\right)$ is a sequence in $\operatorname{int}(S)$ which converges to $s$. Hence $S \subseteq \overline{\operatorname{int}(S)}$.

3) It is immediate from the definition that a Lie semigroup is connected. Therefore the path connectedness of $S$ follows from Corollary 2.7 in [**6**]. If $a, b \in \operatorname{int}(S)$, then $U := aS^{-1} \cap bS^{-1}$ is a neighbourhood of $\mathbf{1}$ in $G$. Therefore there exists $s_0 \in \operatorname{int}(S) \cap U$. Hence $a, b \in s_0 S$ and $s_0 S$ is path connected. Thus $a$ and $b$ are connected by a path lying in $\operatorname{int}(S)$.

4) Let $s \in S$ and $U$ be an open subset of $G$ containing $s$. We have to show that $U \cap S$ contains a path connected neighbourhood of $s$ with respect to $S$. Let $\alpha : [0,1] \to S$ be as in 1) with the additional condition that $s\alpha([0,1]) \subseteq S \cap U$ (reparametrization). Then $s\alpha(1) \in \operatorname{int}(S) \cap U$. Hence, there exists a contractible $\mathbf{1}$-neighbourhood $W$ in $G$ such that

$$Ws\alpha(1) \subseteq \operatorname{int}(S) \cap U \quad \text{and} \quad (Ws \cap S)\alpha([0,1]) \subseteq S \cap U.$$

Let $x, y \in V := (Ws \cap S)\alpha([0,1])$. Then $x = x'\alpha(t_x)$ and $y = y'\alpha(t_y)$, where $t_x, t_y \in [0,1]$ and $x', y' \in (Ws \cap S)$. To show that $V$ is path-connected we have to show the existence of a continuous path in $V$ from $x$ to $y$. First we observe that

$$\alpha_x : [t_x, 1] \to S, t \mapsto x'\alpha(t) \quad \text{and} \quad \alpha_y : [t_y, 1] \to S, t \mapsto y'\alpha(t)$$

are paths in $V$ connecting $x$ and $y$ with $x'\alpha(1)$ and $y'\alpha(1)$ respectively. But $x'\alpha(1)$, $y'\alpha(1) \in Ws\alpha(1)$, which is a contractible subset of $V$. Therefore $x'\alpha(1)$ and $y'\alpha(1)$ may be connected by a path in $V$. Consequently $V$ is a path-connected neighbourhood of $s$ in $S$ which is contained in $S \cap U$.

† For the proof of 4) the author thanks Prof. Dr. K. H. Hofmann.

5) We keep the notation from 4). We show that every loop $\beta:[0,1]\to Ws\cap S$ is in $S$ homotopic to the constant loop. Let

$$F(s,t):=\begin{cases}\beta(0)\alpha(3t) & \text{for } t\in\left[0,\frac{s}{3}\right]\\ \beta\left(\frac{3t-s}{3-2s}\right)\alpha(s) & \text{for } t\in\left[\frac{s}{3},1-\frac{s}{3}\right]\\ \beta(0)\alpha(3-3t) & \text{for } t\in\left[1-\frac{s}{3},1\right].\end{cases}$$

Then $F:[0,1]\times[0,1]\to S$ is continuous and satisfies $F(0,t)=\beta(t)$ and $F(s,0)=F(s,1)=\beta(0)$. Moreover, $\gamma:t\mapsto F(1,t)$ is homotopic to $\alpha\diamondsuit\beta\alpha(1)\diamondsuit\hat{\alpha}$ and $\beta\alpha(1)$ lies in the contractible subset $Ws\alpha(1)$ of $S$. Consequently

$$[\beta]=[\gamma]=[\alpha\diamondsuit\beta\alpha(1)\diamondsuit\hat{\alpha}]=[\alpha\diamondsuit\hat{\alpha}]=1.$$

THEOREM 1.3. *For every generating Lie semigroup $S\subseteq G$ there exists a locally compact topological monoid $\tilde{S}$ and a mapping $p:\tilde{S}\to S$ with the following properties.*

1) *$\tilde{S}$ is path connected, locally path connected, and $\pi_1(\tilde{S})=\{\mathbf{1}\}$.*
2) *$p:\tilde{S}\to S$ is a covering and a semigroup homomorphism.*
3) *$\operatorname{int}(\tilde{S}):=p^{-1}(\operatorname{int}(S))$ is a dense semigroup ideal in $\tilde{S}$.*
4) *If $q:T\to S$ is a covering homomorphism of path connected topological monoids, then there exists a unique covering homomorphism $\tilde{p}:\tilde{S}\to T$ such that $\tilde{p}(\mathbf{1}_{\tilde{S}})=\mathbf{1}_T$ and $q\circ\tilde{p}=p$.*

*Proof.* 1) The existence of a universal covering $p:\tilde{S}\to S$ follows from [**13**, p. 229] because $S$ is path connected, locally path connected, and semi-locally simply connected.

2) To define the structure of a monoid on $\tilde{S}$, we choose $\tilde{\mathbf{1}}\in p^{-1}(\mathbf{1})$. Let $m_S:S\times S\to S$ denote the multiplication of $S$. Then $m_S\circ(p\times p):\tilde{S}\times\tilde{S}\to S$ lifts uniquely to a continuous mapping $m_{\tilde{S}}:\tilde{S}\times\tilde{S}\to\tilde{S}$ such that $m_{\tilde{S}}(\tilde{\mathbf{1}},\tilde{\mathbf{1}})=\tilde{\mathbf{1}}$ and $p\circ m_{\tilde{S}}=m_S\circ(p\times p)$. This follows from [**13**, p. 221] because $\tilde{S}\times\tilde{S}$ is path connected, locally path connected, and simply connected [**13**, p. 203]. We show that $\tilde{S}$ is a monoid with respect to this multiplication. The mapping $\alpha:\tilde{S}\to\tilde{S}$, $s\mapsto\tilde{\mathbf{1}}s$ satisfies $p\circ\alpha=p=p\circ id_{\tilde{S}}$ and $\alpha(\tilde{\mathbf{1}})=\tilde{\mathbf{1}}$. Now the uniqueness of the lift [**13**, p. 221] implies that $\alpha=id_{\tilde{S}}$. Thus $\tilde{\mathbf{1}}s=s$ for all $s\in S$. That $\tilde{\mathbf{1}}$ is a right unit follows similarly. Since

$$p\circ(m_{\tilde{S}}\times id_{\tilde{S}})\circ m_{\tilde{S}}=p\circ(id_{\tilde{S}}\times m_{\tilde{S}})\circ m_{\tilde{S}}$$

and $(\tilde{\mathbf{1}}\tilde{\mathbf{1}})\tilde{\mathbf{1}}=\tilde{\mathbf{1}}=\tilde{\mathbf{1}}(\tilde{\mathbf{1}}\tilde{\mathbf{1}})$, the fact that $\tilde{S}\times\tilde{S}\times\tilde{S}$ is path connected, locally path connected, and simply connected [**13**, p. 203] implies that

$$(m_{\tilde{S}}\times id_{\tilde{S}})\circ m_{\tilde{S}}=(id_{\tilde{S}}\times m_{\tilde{S}})\circ m_{\tilde{S}},$$

i.e. multiplication on $\tilde{S}$ is associative [**13**, p. 221]. That $p:\tilde{S}\to S$ is a homomorphism is a consequence of

$$m_S\circ(p\times p)=p\circ m_{\tilde{S}}.$$

3) As the inverse image of an ideal, the subset $\operatorname{int}(\tilde{S}):=p^{-1}(\operatorname{int}(S))$ is a semigroup ideal. Since $p$ is a local homeomorphism and $\mathbf{1}\in\overline{\operatorname{int}(S)}$, it follows that $\tilde{\mathbf{1}}\in\overline{\operatorname{int}(\tilde{S})}$. Therefore $s\in s\,\overline{\operatorname{int}(\tilde{S})}\subseteq\overline{\operatorname{int}(\tilde{S})}$ for all $s\in\tilde{S}$.

4) That there exists a continuous mapping $\tilde{p}:\tilde{S}\to T$ such that $q\circ\tilde{p}=p$ and $\tilde{p}(\tilde{\mathbf{1}})=\mathbf{1}_T$ follows again from [**13**, p. 221] and Proposition 1.2. If $m_T: T\times T\to T$ denotes multiplication in $T$, then

$$\tilde{p}\circ m_{\tilde{S}}(\tilde{\mathbf{1}},\tilde{\mathbf{1}})=m_T\circ(\tilde{p}\times\tilde{p})(\tilde{\mathbf{1}},\tilde{\mathbf{1}})$$

and

$$\begin{aligned} q\circ m_T\circ(\tilde{p}\times\tilde{p}) &= m_S\circ(q\times q)\circ(\tilde{p}\times\tilde{p}) \\ &= m_S\circ(p\times p)=p\circ m_{\tilde{S}} \\ &= q\circ\tilde{p}\circ m_{\tilde{S}}. \end{aligned}$$

Now the uniqueness assertion of [**13**, p. 221] for the lift of this mapping shows that

$$m_T\circ(\tilde{p}\times\tilde{p})=\tilde{p}\circ m_{\tilde{S}},$$

i.e. $\tilde{p}$ is a morphism of topological monoids.

We claim that $\tilde{p}$ is surjective. Let $t\in T$ and $\beta:[0,1]\to T$ be a path with $\beta(0)=\mathbf{1}_T$ and $\beta(1)=t$. Then there exists a path $\alpha:[0,1]\to\tilde{S}$ such that $\alpha(0)=\mathbf{1}$, $p\circ\alpha(1)=q(t)$, and $p\circ\alpha=q\circ\beta$. Hence $\beta(1)=t=\tilde{p}(\alpha(1))$ [**13**, p. 221] and $\tilde{p}$ is surjective. Now it follows immediately from the definition that $\tilde{p}$ is a covering.

Remark 1.4. In [**7**] Kahn defines the notion of a covering semigroup $(\bar{S},\varphi)$ of a topological semigroup $S$ as a pair of a topological semigroup $\bar{S}$ and a covering $\varphi:\bar{S}\to S$. He calls a semigroup $S$ simply connected if for every covering semigroup $(\bar{S},\varphi)$ of $S$ the mapping $\varphi$ is a homeomorphism. Now Theorem 1.3.1 and [**14**, p. 84] show that our $\tilde{S}$ is simply connected in this sense. Therefore $(\tilde{S},p)$ is simply connected covering semigroup of $S$ in the sense of Kahn [**7**, p. 430]. Some of the results of Section 1, in particular Corollary 1.8, can already be found in Kahn's paper. Since the proofs are rather short we include them for the sake of completeness.

Proposition 1.5. *Let $I$ be a dense path connected semigroup ideal in the path connected topological monoid S. Suppose that there exists a path $\beta:[0,1]\to S$ such that*

$$\beta(0)=\mathbf{1}\quad\text{and}\quad\beta(]0,1])\subseteq I.$$

*Then the inclusion $i:I\to S$ induces an isomorphism*

$$i_*:\pi_1(I)\to\pi_1(S).$$

*Proof.* Let $x_0\in I$ be a fixed point which serves as base point for $I$ and $S$ simultaneously

1) $i_*$ is injective. Let $\gamma\in\Omega(I,x_0)$ such that $i_*[\gamma]=[i\circ\gamma]=\mathbf{1}$ in $\pi_1(S)$. Then there exists a continuous mapping $F:[0,1]\times[0,1]\to S$ such that

$$F(0,t)=\gamma(t), F(1,t)=x_0,\quad\text{and}\quad F(s,0)=F(s,1)=x_0$$

for $s,t\in[0,1]$. We define $G:[0,1]\times[0,1]\to S$ by $G(s,t):=F(s,t)\beta(s(1-s)t(1-t))$, where $\beta$ is a continuous curve $[0,1]\to$ S such that $\beta(0)=1$ and $\beta(]0,1])\subseteq I$. Then $G$ is a deformation of $\gamma$ to the constant path in $x_0$ and im $G\subseteq I$ because $F(0,t), F(1,t)\in I$ for all $t\in[0,1]$. Hence $[\gamma]=[x_0]$ and $i_*$ is injective.

2) $i_*$ is surjective. Let $[\gamma]\in\pi_1(S)$ and $\gamma:[0,1]\to S$ with $\gamma(0)=\gamma(1)=x_0$. Then

$$F:[0,1]\times[0,1]\to S,\qquad(s,t)\mapsto\beta(t(1-t)s)\gamma(t)$$

deforms the path $\gamma$ into a path which lies entirely in $I$. Hence $[\gamma]\in\operatorname{im}(i_*)$.

Corollary 1.6. 1) *The inclusions* $i:\mathrm{int}(S)\to S$, $\tilde{\imath}:\mathrm{int}(\tilde{S})\to\tilde{S}$ *induce isomorphisms*

$$i_*:\pi_1(\mathrm{int}(S))\to\pi_1(S) \quad \textit{and} \quad \tilde{\imath}_*:\pi_1(\mathrm{int}(\tilde{S}))\to\pi_1(\tilde{S}).$$

2) $\pi_1(\mathrm{int}(\tilde{S}))=\{\mathbf{1}\}$.

*Proof.* The first statement follows from Propositions 1.2 and 1.5, and the second statement is a consequence of 1) and Theorem 1.3 because $\pi_1(\tilde{S})=\{\mathbf{1}\}$.

Lemma 1.7. (Hilton's Lemma for monoids) *Let $S$ be a topological monoid, $\gamma:[0,1]\to S$ a continuous path with $\gamma(0)=\mathbf{1}$, and $\gamma'\in\Omega(S,\mathbf{1})$. Then*

$$[\gamma\gamma']=[\gamma'\gamma]=[\gamma'\diamond\gamma],$$

*where* $\gamma\gamma'(t)=\gamma(t)\gamma'(t)$.

*Proof.* We set

$$\eta'(x)=\begin{cases}\gamma'(2t) & \text{if } t\in[0,\frac{1}{2}]\\ \gamma'(1) & \text{if } t\in[\frac{1}{2},1]\end{cases}$$

and

$$\eta(t)=\begin{cases}\mathbf{1} & \text{if } t\in[0,\frac{1}{2}]\\ \gamma(2t-1) & \text{if } t\in[\frac{1}{2},1].\end{cases}$$

Then

$$\eta\eta'=\eta'\eta=\eta'\diamond\eta.$$

Clearly $[\eta]=[\gamma]$ and $[\eta']=[\gamma']$. Therefore

$$[\gamma\gamma']=[\eta\eta']=[\eta'\eta]=[\gamma'\gamma]=[\eta'\diamond\eta]=[\gamma'\diamond\gamma].$$

Corollary 1.8. *Let $p:\tilde{S}\to S$ be as above, then the following assertions hold:*

1) *Let $\tilde{\gamma}$ denote the lift of $\gamma$ with $\tilde{\gamma}(0)=\tilde{\mathbf{1}}$. Then the mapping $[\gamma]\mapsto\tilde{\gamma}(1)$, $\pi_1(S)\to\ker p$ is an isomorphism of groups.*
2) $\ker p\subseteq Z(\tilde{S}):=\{s\in\tilde{S}:(\forall t\in\tilde{S})st=ts\}$.
3) *$\pi_1(S)$ is abelian.*
4) *$\pi_1(S)$ acts freely on $\tilde{S}$, i.e. $d\in\pi_1(S)$, $s\in\tilde{S}$ and $ds=s$ implies that $d=\tilde{\mathbf{1}}$.*
5) *$\tilde{S}$ is cancellative, i.e. $ab=ac$ or $ba=ca$ implies that $b=c$.*

*Proof.* (cf. [**7**, pp. 432, 433]) 1) Let $\gamma,\gamma'\in\Omega(S,\mathbf{1})$. Using Lemma 1.7 we find that

$$[\gamma][\gamma']=[\gamma\gamma']\mapsto\tilde{\gamma}(1)\tilde{\gamma}'(1).$$

Hence $[\gamma]\mapsto\tilde{\gamma}(1)$ defines a monoid homomorphism. It follows from the construction of $\tilde{S}$ that it is bijective and therefore $p^{-1}(\mathbf{1})$ is a group isomorphic to $\pi_1(S)$.

2) Let $d\in p^{-1}(\mathbf{1})$ and $s\in\tilde{S}$. Then there exists $\gamma'\in\Omega(S,\mathbf{1})$ with $\tilde{\gamma}'(1)=d$ and a path $\gamma:[0,1]\to S$ such that $\tilde{\gamma}(1)=s$. Then Lemma 1.7 shows that

$$sd=(\gamma\gamma')^{\sim}(1)=(\gamma'\gamma)^{\sim}(1)=ds.$$

Hence $d$ is central in $\tilde{S}$.

3) This is a consequence of 2).

4) Suppose that $ds = s$, that $s = \tilde{\gamma}(1)$, and that $d = \tilde{\alpha}(1)$ with $\alpha \in \Omega(S, \mathbf{1})$. Then

$$(\alpha\gamma)^{\sim}(1) = ds = s = \tilde{\gamma}(1)$$

and therefore Hilton's Lemma implies that

$$[\alpha \diamond \gamma] = [\alpha\gamma] = [\gamma].$$

This shows that

$$[\alpha] = [\alpha \diamond (\gamma \diamond \hat{\gamma})] = [(\alpha \diamond \gamma) \diamond \hat{\gamma}] = [\gamma \diamond \hat{\gamma}] = [\mathbf{1}],$$

i.e. $d = \tilde{\mathbf{1}}$.

5) Suppose that $ab = ac$ with $a, b, c \in \tilde{S}$. Then

$$p(a)p(b) = p(ab) = p(ac) = p(a)p(c)$$

implies that $p(b) = p(c)$. Therefore we find $d \in \pi_1(S)$ such that $c = bd$. Hence $ab = a(bd) = (ab)d = d(ab)$ and 4) shows that $d = \tilde{\mathbf{1}}$, i.e. $b = c$. The other implication follows similarly.

PROPOSITION 1.9. *The set $p^{-1}(H(S))$ agrees with the unit group $H(\tilde{S})$ of $\tilde{S}$. The mapping $p|_{H(\tilde{S})}$ defines a covering of Lie groups.*

*Proof.* It is clear that $p^{-1}(H(S))$, as the inverse image of a subsemigroup, is a closed subsemigroup of $\tilde{S}$ and that it contains $H(\tilde{S})$. Let $x \in p^{-1}(H(S))$. Then there exists $s \in S$ such that $p(x)s = \mathbf{1}$. Let $s = p(y)$. Then $p(xy) = p(x)p(y) = \mathbf{1}$ and therefore $xy \in p^{-1}(\mathbf{1})$. Since $p^{-1}(\mathbf{1})$ is a subgroup of $\tilde{S}$ (Corollary 1.8), it is contained in $H(\tilde{S})$. Thus $x \in H(\tilde{S})$ because $\tilde{S} \backslash H(\tilde{S})$ is a semigroup ideal.

Since $p: \tilde{S} \to S$ is a covering, it is obvious that the restriction of $p$ to $H(\tilde{S})$ is a covering morphism of topological groups. Therefore $H(\tilde{S})$ is a Lie group and $p|_{H(\tilde{S})}$ a covering morphism of Lie groups.

EXAMPLE 1.10. We note that the semigroups $\tilde{S}$ need not be generated, not even topologically, by an arbitrary small neighbourhood of $\mathbf{1}$. To see this, let $G_1 := \mathbb{R}^2$ and $S_1 := \mathbb{R}^+(1, -1) \times \mathbb{R}^+(1, 1)$. Set $G := \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ and write $p: G_1 \to G$, $(x, y) \mapsto (x, y + \mathbb{Z})$ for the quotient homomorphism. Then the image $S = p(S_1)$ of $S_1$ in $G$ is a generating Lie semigroup with $\pi_1(S) \cong \mathbb{Z}$ (Theorem 3.4 below). The universal covering $\tilde{S}$ corresponds to the subsemigroup $\mathbb{Z} + S_1$ of $G_1$. The subsemigroup $S_1$ is a closed neighbourhood of $\mathbf{1}$ in $\tilde{S}$ and $H(\tilde{S}) \cong \mathbb{Z}$ is not connected.

PROPOSITION 1.11. *For $X \in \mathbf{L}(S)$ we set $\gamma_X: \mathbb{R}^+ \to S$, $t \mapsto \exp(tX)$. Then $X \mapsto \tilde{\gamma}_X$ is a bijection from* $\mathbf{L}(S) \to \operatorname{Hom}(\mathbb{R}^+, \tilde{S})$. *Define* $\operatorname{Exp}: \mathbf{L}(S) \to \tilde{S}$, $X \mapsto \tilde{\gamma}_X(1)$. *Then the semigroup*

$$\tilde{S}_L := \overline{\langle \operatorname{Exp}(\mathbf{L}(S)) \rangle}$$

*is a neighbourhood of $\mathbf{1}$ in $\tilde{S}$. It is the smallest subsemigroup topologically generated by every neighbourhood of $\mathbf{1}$ in $\tilde{S}$. Moreover*

$$\tilde{S} = \overline{\pi_1(S)\tilde{S}_L}.$$

*Proof.* The first statement follows from the fact that $p: \tilde{S} \to S$ induces a local isomorphism from a neighbourhood of $\tilde{\mathbf{1}}$ in $\tilde{S}$ to a $\mathbf{1}$-neighbourhood in $S$. Now the second statement follows from the assumption that $S$ is a Lie semigroup, and the last assertion is clear because $\overline{\pi_1(S)\tilde{S}_L}$ is a $\pi_1(S)$-saturated subset of $\tilde{S}$ which is mapped surjectively onto $S$.

Lemma 1.12. *Let $q:\tilde{G}\to G$ denote the universal covering group of $G$, identify $\pi_1(G)$ with* $\ker q$ *and $\pi_1(S)$ with $p^{-1}(\mathbf{1})$. Then there exists a continuous homomorphism $\tilde{\imath}:\tilde{S}\to\tilde{G}$ such that $q\circ\tilde{\imath}=i\circ p$, $\tilde{\imath}\,|_{\pi_1(S)}=i_*$, and the image of $\tilde{\imath}$ is the path-component of $\mathbf{1}$ in $q^{-1}(S)$.*

*Proof.* The only thing we have to prove is the existence of $\tilde{\imath}$. The rest follows from the identification of $\pi_1(S)$ and $\pi_1(G)$ with subgroups of $\tilde{S}$ and $\tilde{G}$ respectively. Let $S_1$ be the path-component of $\mathbf{1}$ in $q^{-1}(S)$. It follows from Proposition 1.2 that $q^{-1}(S)$ is locally path connected because $q$ is a local homeomorphism. Therefore $S_1$ is an open closed connected component of $q^{-1}(S)$. Now the universal property of $\tilde{S}$ (Theorem 1.3.4) implies the existence of a surjective semigroup covering $\tilde{\imath}:\tilde{S}\to S_1$ such that $q\circ\tilde{\imath}=p$.

Theorem 1.13. *Let $j:H(S)\to S$ be the inclusion mapping and*

$$j_*:\pi_1(H(S))\to\pi_1(S)$$

*the induced homomorphism of the fundamental groups. Then*

$$\ker j_*=\pi_1(H(\tilde{S})) \quad \textit{and} \quad \operatorname{im} j_*=H(\tilde{S})_0\cap\pi_1(S).$$

*Proof.* Let $\widetilde{H(S)}$ be the universal covering group of $H(S)$. Then there exists a Lie group homomorphism $q:\widetilde{H(S)}\to H(\tilde{S})_0$ such that $p\circ q:\widetilde{H(S)}\to H(S)$ is the universal covering morphism of $H(S)$. The homomorphism $j_*$ corresponds to the homomorphism

$$q\,|_{\pi_1(H(S))}:\pi_1(H(S))\to\pi_1(S),$$

where $\pi_1(H(S))$ is identified with the corresponding subgroup of $\widetilde{H(S)}$. Thus $H(\tilde{S})_0\cong\widetilde{H(S)}/\ker j_*$ implies that $\ker j_*=\pi_1(H(\tilde{S})_0)$.

The image is clearly contained in $D':=H(\tilde{S})_0\cap\pi_1(S)$. But $H(S)\cong H(\tilde{S})_0/D'$ and therefore $D'=\operatorname{im} j_*$.

The situation of Theorem 1.13 is illustrated in the following diagram.

$$\begin{array}{ccccc}
\pi_1(H(S)) & \xrightarrow{j_*} & \pi_1(S)\cap H(\tilde{S})_0 & \longrightarrow & \mathbf{1} \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{H(S)} & \longrightarrow & H(\tilde{S})_0 & \xrightarrow{p} & H(S) \\
 & & \downarrow & & \downarrow \\
 & & \tilde{S} & \xrightarrow{p} & S
\end{array}$$

Corollary 1.14. *The mapping $j_*:\pi_1(H(S))\to\pi_1(S)$ is*
1) *injective iff $H(\tilde{S})$ is simply connected.*
2) *surjective iff $H(\tilde{S})$ is connected.*

*Proof.* In view of Theorem 1.13 we only have to show that the connectedness of $H(\tilde{S})$ follows from the surjectivity of $j_*$. If $\pi_1(S)=\pi_1(S)\cap H(\tilde{S})_0$ then $\pi_1(S)\subseteq H(\tilde{S})_0$ and therefore, according to Proposition 1.9,

$$H(\tilde{S})_0=p^{-1}(H(S))=H(\tilde{S}).$$

Example 1.15. 1) Let $G := \mathrm{SU}(2) \times \mathbb{R}$. Then $\mathbf{L}(G) = \mathrm{su}(2) \oplus \mathbb{R} \cong \mathrm{u}(2)$. Then there exists a pointed invariant wedge $C \subseteq \mathbf{L}(G)$ with non-empty interior (take the matrices with spectrum on the positive imaginary axis in u(2)). Take $X \in \mathrm{su}(2)$. Then $\exp \mathbb{R}X$ is a circle group in $G$ because $\exp \mathbb{R}X$ is a torus and the maximal tori in SU(2) are of dimension 1. We set $W := \mathbb{R}X + C$ and $S := \overline{\langle \exp W \rangle}$. Then $S$ is a generating Lie semigroup in $G$. That $\mathbf{L}(S) = W$ follows from [**9**, Proposition 3.11] because $\exp H(W)$ is a closed subgroup of $G$, SU(2) is the unique maximal compact subgroup of $G$, and $\mathrm{su}(2) \cap W = H(W)$.

According to Theorem 3.11 below, we know that $G = SS^{-1} = S^{-1}S$. Therefore $S$ is simply connected (Theorem 3.4). Hence $\tilde{S}$ agrees with $S$ and we have an example, where $H(\tilde{S})$ is not simply connected. In view of Corollary 1.12 this is related to the fact that the circle $\exp \mathbb{R}X$ cannot be deformed to a constant loop in $H(S)$, but if one pushes it far enough into the interior of $S$, for example into a coset of SU(2), the contraction becomes possible.

2) That $H(\tilde{S})$ need not be connected follows from Example 1.10.

3) Let $\mathbf{g}$ be a Lie algebra which contains a pointed generating invariant cone $C$, $G_{\mathbb{C}}$ the simply connected Lie group with $\mathbf{L}(G_{\mathbb{C}}) = \mathbf{g}_{\mathbb{C}}$, and $G := \langle \exp_{G_{\mathbb{C}}} \mathbf{g} \rangle \subseteq G_{\mathbb{C}}$. Then the set $S := G \exp(iC)$ is a generating Lie semigroup in $G_{\mathbb{C}}$ ([**8**, Cor. 3.6] or [**2**, Theorem 3.12]). We claim that $H(\tilde{S}) = \tilde{G}$ and that $\tilde{S} = \tilde{G}\,\mathrm{Exp}(iC)$.

Clearly $\tilde{G} \times C$ is a simply connected, locally path connected space. Therefore the mapping $\varphi : \tilde{G} \times C \to S$, $(g, c) \mapsto g \exp(ic)$, which is a covering, lifts to a covering $\tilde{\varphi} : \tilde{G} \times C \to \tilde{S}$ with $p \circ \tilde{\varphi} = \varphi$. Since $\tilde{S}$ is simply connected and locally path connected (Proposition 1.2), the mapping $\tilde{\varphi}$ is a homeomorphism. This proves that $H(\tilde{S}) \cong \tilde{G}$ and that $\tilde{S} = \tilde{G}\,\mathrm{Exp}(iC)$.

We will return to these examples later (Example 2.11), where we will see that the semigroups $\tilde{S}$ are not realizable as subsemigroups of groups.

**2. The relation to the free group over $S$.** Let us return to the problem from the beginning. Given a generating Lie semigroup $S \subseteq G$, we consider the inclusion mapping $i : S \to G$ and the associated homomorphism $i_* : \pi_1(S) \to \pi_1(G)$ with respect to the base point $\mathbf{1}$. The main achievement of Section 1 is the realization of $\pi_1(S)$ as a concrete subgroup of the centre of the locally compact semigroup $\tilde{S}$. In the following we identify $\pi_1(S)$ with this subgroup of $\tilde{S}$, and similarly $\pi_1(G)$ with the corresponding subgroup of $\tilde{G}$, the universal covering group of $G$.

We start with the determination of the image of $i_*$. To state the first main theorem, we recall the following result from [**5**, VII.3.28]:

Theorem 2.1. *Let $S \subseteq G$ be a generating Lie semigroup. Then there exists a covering group $p : G(S) \to G$ and a continuous homomorphism $\gamma_S : S \to G(S)$ which has the universal property of the free topological group on $S$, i.e. for every continuous homomorphism $\varphi : S \to K$, where $K$ is a topological group, there exists a continuous homomorphism $\tilde{\varphi} : G(S) \to K$ such that $\varphi = \tilde{\varphi} \circ \gamma_S$. The group $G(S)$ is the largest covering of $G$ in which $S$ lifts.*

Our first main result will be the identification of $\pi_1(G(S))$, as a subgroup of $\pi_1(G)$, as the image of $i_*$.

First we need more detailed information about the situation of this theorem. We start with a general lemma about subsemigroups of metrizable topological groups.

LEMMA 2.2. *Let $G$ be a metrizable topological group, $S \subseteq G$ a closed subsemigroup with non-empty interior, and $D \subseteq Z(G)$ a discrete central subgroup. Then the following assertions hold:*

1) $SS^{-1} = \mathrm{int}(S)\mathrm{int}(S)^{-1}$.
2) $D_1 := D \cap SS^{-1} = D \cap S^{-1}S$ *is a subgroup of* $D$.
3) *The semigroup* $S_1 := D_1 S$ *is relatively open and closed in the semigroup* $S_2 := DS$.
4) $\overline{S_2} = D\overline{S_1}$.
5) $dS_1 = d'S_1$ *iff* $d \in d'D_1$.

*Proof.* 1) If $g = s_1 s_2^{-1}$ with $s_1, s_2 \in S$, then $g = s_1 s_0 s_0^{-1} s_2^{-1}$, where $s_0 \in \mathrm{int}(S)$ is arbitrary. Then $s_1 s_0, s_2 s_0 \in \mathrm{int}(S)$ and the assertion follows.

2) Set $D_1 := D \cap SS^{-1}$. It is clear that $D_1 = D_1^{-1}$. Let $d = s_1 s_2^{-1} \in D_1$, where $s_1, s_2 \in S$. Then $s_1$ and $s_2$ commute with $d$ and therefore with each other. Hence $d = s_2^{-1} s_1 \in S^{-1}S$. By symmetry we see that $S^{-1}S \cap D$ equals $D_1$, too. If $d' = s_1' s_2'^{-1}$ with $s_1', s_2' \in S$, then $dd' = s_1 s_2^{-1} d' = s_1 d' s_2^{-1} \in SS^{-1}$. Thus $D_1^2 \subseteq D_1$ and consequently $D_1$ is a group.

3) Let $g = \lim_{n\to\infty} d_n s_n$ with $d_n \in D$ and $s_n \in S$. Suppose first that $d_n \in D_1$ and $g = ds \in DS = S_2$. We choose an element $s_0 \in \mathrm{int}(S)$. Then $gs_0 = dss_0 = \lim_{n\to\infty} d_n s_n s_0$ and $ss_0 \in \mathrm{int}(S)$. Therefore there exists $n_0 \in \mathbb{N}$ such that $d^{-1} d_{n_0} s_{n_0} s_0 \in \mathrm{int}(S)$. Then $d^{-1} d_{n_0} \in SS^{-1} \cap D = D_1$. This shows that $d \in D_1$ and $g \in S_1 = D_1 S$. So we have proved that $S_1$ is relatively closed in $S_2$.

To show that $S_1$ is also relatively open in $S_2$ we assume that $g = ds \in D_1 S = S_1$. By the same argument as above we find $n_0 \in \mathbb{N}$ such that $d^{-1} d_n \in D_1$ for $n \geq n_0$. But this means that eventually $d_n \in D_1$ and $d_n s_n \in S_1$. Thus $S_1$ is also relatively open.

4) We only have to prove that $\overline{S_1}D$ is closed. So let $g = \lim_{n\to\infty} d_n s_n$ with $d_n \in D$ and $s_n \in \overline{S_1}$. Because $G$ was supposed to be metrizable we may replace $s_n$ by $d_n' s_n'$, where $d_n' \in D_1$ and $s_n' \in S$. Hence we may assume that $s_n \in S$. Then there exists $m \in \mathbb{N}$ such that $d_m s_m \in gSS^{-1}$ because $SS^{-1}$ is a $\mathbf{1}$-neighbourhood in $G$. Thus $g \in d_m s_m SS^{-1} \subseteq DSS^{-1} = D\,\mathrm{int}(S)\mathrm{int}(S)^{-1}$. Choose $d \in D$ and $a, b \in \mathrm{int}(S)$ such that $g = dab^{-1}$. Then $a = \lim_{n\to\infty} d^{-1} d_n s_n b \in \mathrm{int}(S)$ and there exists $n_0 \in \mathbb{N}$ such that $d^{-1} d_n s_n b \in S$ whenever $n \geq n_0$. In this case $d^{-1} d_n \in S(s_n b)^{-1} \subseteq SS^{-1}$ and so $d_n \in dD_1$. Now $g = \lim_{n\to\infty} d_n s_n \in d\overline{D_1 S} = d\overline{S_1}$.

5) If $dS_1 = dD_1 S = d'D_1 S = d'S_1$, then $d^{-1}d' \in SS^{-1}D_1 \subseteq SS^{-1}$. Therefore $d' \in dD_1$.

PROPOSITION 2.3. *Let $S \subseteq G$ be a generating Lie semigroup, $p: G_1 \to G$ a covering morphism with $\exp_G = p \circ \exp_{G_1}$, and $S_1 \subseteq G_1$ the Lie semigroup with $\mathbf{L}(S_1) = \mathbf{L}(S)$. Then the following are equivalent:*

1) *$S$ lifts into $G_1$.*
2) *The subsets $dS_1$, $d \in \ker p$ are the connected components of the closed semigroup $S_2 := \ker \varphi \,.\, S_1$.*
3) $S_1 S_1^{-1} \cap D = \{\mathbf{1}\}$.

*Proof.* 1)$\Rightarrow$2): Suppose that $\gamma: S \to G_1$ is a lift of $S$ into $G_1$. Then $p \circ \gamma = \mathrm{id}_S$ and $\gamma(S)$ is a locally compact subsemigroup of $G_1$ with

$$\exp_G X = p \circ \gamma(\exp_G X) = p(\exp_{G_1} X) \quad \text{for all} \quad X \in \mathbf{L}(S).$$

Thus $\gamma(\exp_G X) = \exp_{G_1} X$ for all $X \in \mathbf{L}(S)$. This proves that $\gamma(S) \subseteq S_1$. On the other hand it is clear that $p(S_1) \subseteq S$. The mapping $\gamma \circ p$ agrees with the identity on the dense subsemigroup $\langle \exp_{G_1} \mathbf{L}(S) \rangle$ of $S_1$ and therefore $\gamma \circ p\,|_{S_1} = \mathrm{id}_{S_1}$. In particular $\gamma(S) = S_1$.

This proves that $S_1^{-1} S_1 \cap D = \{\mathbf{1}\}$ because $s = s'd$ with $s, s' \in S_1$ implies that $p(s) = p(s')$ and therefore $s = s'$. It is clear that the subsets $dS_1 \subseteq \tilde{S}$ are connected. We prove that they are pairwise disjoint. If this is false, we find $d \in \ker p \backslash \{\mathbf{1}\}$ such that $dS_1 \cap S_1 \neq \varnothing$. Choose $s, s' \in S_1$ with $ds = s'$. Then $p(s) = p(s')$ which proves that $s = s'$ because $p\,|_{S_1}$ is injective. It follows from Lemma 2.2 that the sets $dS_1$ are open closed subsets of the closed semigroup $DS_1$.

2)$\Rightarrow$3): Assume that the sets $dS_1$, $d \in \ker p$ are the connected components of the closed semigroup $S_2$. If $d \in S_1 S_1^{-1} \cap D$, then $dS_1 = S_1$ and therefore $d = \mathbf{1}$.

3)$\Rightarrow$1): Suppose that $D \cap S_1 S_1^{-1} = \{\mathbf{1}\}$. Lemma 2.2 implies that the subsets $dS_1$ of $DS_1$ are open and closed in the closed semigroup $S_2$.

Now $p(S_2) = p(S_1) = S$ because this is a closed subsemigroup of $G$ which contains $\exp_G \mathbf{L}(S)$. We claim that $p\,|_{S_1}$ is injective. To see this, let $s, s' \in S_1$ with $p(s) = p(s')$. Then there exists $d \in \ker p$ with $s' = ds \in S_1 \cap dS_1$, i.e. $d \in S_1 S_1^{-1}$. Thus $d = \mathbf{1}$ and $s = s'$. Therefore the restriction $p\,|_{S_1}: S_1 \to S$ is a continuous locally homeomorphic bijection, whence an isomorphism of topological semigroups. We conclude that $(p\,|_{S_1})^{-1}: S \to S_1$ is a lift of $S$ into $G_1$.

Remark 2.4. Note that the proof of Proposition 2.3 even shows that the condition $D \cap S_1 S_1^{-1} = \{\mathbf{1}\}$ implies the existence of a closed subsemigroup $S$ of $G = G_1/D$ with $\mathbf{L}(S) = \mathbf{L}(S_1)$ (cf. Theorem 1.13 in **[11]**).

So far this was not directly related to the fundamental group of $S$ but now the largest part of the work is done and we can put the pieces together.

Proposition 2.5. *Let* $S_1 := \overline{\langle \exp_{\tilde{G}} \mathbf{L}(S) \rangle}$. *Then*

$$\operatorname{im} i_* = S_1 S_1^{-1} \cap \pi_1(G) \quad \textit{and} \quad \ker i_* = \pi_1(\overline{(\operatorname{im} i_*) S_1}),$$

*where* $\overline{(\operatorname{im} i_*) S_1}$ *is the path-component of* $\mathbf{1}$ *in* $q^{-1}(S)$.

*Proof.* Let $D_1 := S_1 S_1^{-1} \cap \pi_1(G)$. According to Lemma 2.2 this is a subgroup of $\pi_1(G)$. Let $\tilde{\imath}: \tilde{S} \to \tilde{G}$ be the homomorphism from Lemma 1.12.

If $d = s_1 s_2^{-1} \in D_1$ with $s_1, s_2 \in \operatorname{int}(S)$, then there exist continuous mappings $\alpha, \beta: [0, 1] \to S$ such that $\alpha(0) = \mathbf{1}$, $\alpha(1) = s_1$, $\beta(0) = s_2$, and $\beta(1) = \mathbf{1}$. Thus $p(\alpha(1)) = p(s_1) = p(s_2) = p(\beta(0))$ and therefore $(p \circ \alpha) \diamondsuit (p \circ \beta)$ is a continuous path in $S$ whose homotopy class corresponds to $d$. Hence $d \in \tilde{\imath}(\pi_1(S))$.

If, conversely, $d = \tilde{\imath}(x)$, then there exists $\gamma \in \Omega(S, \mathbf{1})$ such that $d = [\gamma]$. According to Proposition 1.2 and Corollary 1.6 we may assume that $\gamma([0, 1]) \subseteq \operatorname{int}(S) \cup \{\mathbf{1}\} \subseteq \langle \exp_G \mathbf{L}(S) \rangle$. Therefore $\tilde{\gamma}([0, 1]) \subseteq DS_1$. Using Lemma 2.2 we find that $\tilde{\gamma}([0, 1]) \subseteq D_1 S_1$ because it is connected. Consequently $\tilde{\gamma}(1) \in D_1$.

With Proposition 1.11 we conclude that

$$S_1 D_1 \subseteq \tilde{\imath}(\tilde{S}) \subseteq \overline{\tilde{\imath}(\tilde{S}_L) \tilde{\imath}(\pi_1(S))} \subseteq \overline{S_1 D_1}.$$

According to Lemma 1.12 the semigroup $\tilde{\imath}(\tilde{S})$ agrees with the path-component of $\mathbf{1}$ in $q^{-1}(S)$ which is open and closed in $q^{-1}(S)$ because $q^{-1}(S)$ is locally path connected (Proposition 1.2). Therefore $\overline{S_1 D_1} = \tilde{\imath}(\tilde{S})$ and $\pi_1(\overline{S_1 D_1}) \cong \ker \tilde{\imath}$.

Theorem 2.6. $\operatorname{im} i_* = \pi_1(G(S))$.

*Proof.* Let $D' \subseteq D := \pi_1(G) \subseteq \tilde{G}$ be a subgroup, $G' := \tilde{G}/D'$, $q' : \tilde{G} \to G'$ the corresponding covering homomorphism, and $S' := \overline{\langle \exp_{G'} \mathbf{L}(S) \rangle}$, $S_1 := \overline{\langle \exp_{\tilde{G}} \mathbf{L}(S) \rangle}$ the Lie subsemigroups of $G'$ and $\tilde{G}$ generated by $\mathbf{L}(S)$.

Then

$$S'S'^{-1} = \operatorname{int}(S')\operatorname{int}(S')^{-1} \subseteq q'(S_1)q'(S_1)^{-1}.$$

Therefore

$$S'S'^{-1} \cap q'(D) = q'(S_1 S_1^{-1}) \cap q'(D) = q'(S_1 S_1^{-1} \cap D) = q'(\operatorname{im} i_*).$$

Now Proposition 2.3 shows that $S$ lifts to $G'$ if and only if $q'(\operatorname{im} i_*) = \{\mathbf{1}\}$, i.e. $\operatorname{im} i_* \subseteq D'$. So the largest covering group of $G$ into which $S$ lifts is $\tilde{G}/\operatorname{im} i_*$ and therefore $\operatorname{im} i_* = \pi_1(G(S))$.

Corollary 2.7. *The mapping $i_*$ is surjective if and only if $G(S) = G$, i.e. if $S$ does not lift into a non-trivial covering group of $G$.*

Corollary 2.8. *Let $q' : G' \to G$ be a covering of Lie groups, $D' := \ker q'$, $S' \subseteq G'$ the Lie semigroup with $\mathbf{L}(S') = \mathbf{L}(S)$, and $q'' : \tilde{G} \to G'$ the universal covering of $G'$. Then $q''(\operatorname{im} i_*) = D' \cap S'S'^{-1}$.*

*Proof.* This follows from the proof of Theorem 2.6.

Corollary 2.9. *If $\pi_1(S) = \{\mathbf{1}\}$, then $G(S) = \tilde{G}$, i.e. every simply connected generating Lie semigroup $S \subseteq G$ lifts into the universal covering group $\tilde{G}$ of $G$.*

The following diagram represents graphically most of the situation of the preceding discussion.

$$\begin{array}{ccccccc}
\pi_1(\overline{D_1 S_1}) & \longrightarrow & \pi_1(S) & \xrightarrow{i_*} & \pi_1(G(S)) & \longrightarrow & \pi_1(G) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \tilde{S} & \xrightarrow{\tilde{\imath}} & \tilde{G} & \xrightarrow{\mathrm{id}_{\tilde{G}}} & \tilde{G} \\
 & & \downarrow p & & \downarrow & & \downarrow q \\
 & & S & \longrightarrow & G(S) & \longrightarrow & G
\end{array}$$

Example 2.10. Let $G = \mathrm{Sl}(2, \mathbb{R})$ and

$$S := \mathrm{Sl}(2, \mathbb{R})^+ := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : a, b, c, d \geq 0 \right\}.$$

Then $S$ is a Lie semigroup and the mapping $\exp : \mathbf{L}(S) \to S$ is a homeomorphism [**5**, V.4.24]. Therefore $\pi_1(S) = \{\mathbf{1}\}$ and $G(S) = \mathrm{Sl}(2, \mathbb{R})^{\sim}$ (Corollary 2.9).

Theorem 2.11. *The homomorphism $\tilde{\imath} : \tilde{S} \to \tilde{G}$* (Lemma 1.12) *has the universal property of the free group on $\tilde{S}$, i.e. for every homomorphism $\alpha : \tilde{S} \to T$, where $T$ is a group, there exists a unique homomorphism $\alpha_1 : \tilde{G} \to T$ such that $\alpha_1 \circ \tilde{\imath} = \alpha$.*

*Proof.* Let $T$ be a group and $\alpha:\tilde{S}\to T$ a homomorphism. Then

$$\beta:\tilde{S}\to T\times\tilde{G},\qquad s\mapsto(\alpha(s),\tilde{\imath}(s))$$

is a homomorphism into a group such that

$$\ker\beta=\ker\alpha\cap\ker\tilde{\imath}\subseteq\ker\tilde{\imath}.$$

Therefore $\ker\beta$ is a discrete subgroup of $\pi_1(S)$. Set $S_1:=\tilde{S}/\ker\beta$ and $S_1^0:=\operatorname{int}(\tilde{S})/\ker\beta$. Then $S_1$ is a covering semigroup of $S$ which can be algebraically embedded in a group. Then $G(S_1^0)$, the free group on $S_1^0$, admits the structure of a Lie group [**5**, VII.3.27], and the corresponding universal homomorphism $\alpha_1:S_1^0\to G(S_1^0)$ is an embedding onto an open subsemigroup of $G(S_1^0)$. The homomorphism $\tilde{\imath}$ is constant on $\ker\beta$. Therefore it induces a homomorphism $p_1:S_1\to\tilde{G}$ which has an extension to a homomorphism $p_1':G(S_1^0)\to\tilde{G}$ with $p_1'\circ\alpha_1=p_1|_{S_1^0}$. We conclude that $p_1'$ is a surjective covering of Lie groups, because it is continuous on $S_1^0$ and therefore everywhere, and $p_1(S_1^0)$ is an open subset of $\tilde{G}$. But $\tilde{G}$ is simply connected and therefore $p_1'$ is an isomorphism. Hence $p_1|_{S_1^0}$ is injective and this proves that $\ker\beta=\ker\tilde{\imath}$, i.e. $\ker\tilde{\imath}\subseteq\ker\alpha$, whence $\alpha$ factors to a homomorphism $\alpha':\tilde{\imath}(\tilde{S})\to T$ with $\alpha'\circ\tilde{\imath}=\alpha$.

It remains to prove that $\alpha'$ permits a continuation to a homomorphism $\alpha_1:\tilde{G}\to T$ with $\alpha_1\circ\tilde{\imath}=\alpha$. We use [**5**, VII.3.28] and Proposition 2.5 to see that $G(\tilde{\imath}(\tilde{S}))=\tilde{G}$. Then the universal property of $G(\tilde{\imath}(\tilde{S}))$ provides a continuation of $\alpha'$ to the whole group $\tilde{G}$.

Corollary 2.12. *Every quotient $\tilde{S}/D$ with $\ker i_*\nsubseteq D\subseteq\pi_1(S)$ is not algebraically embeddable in a group.*

Example 2.13. We assume the notation from Example 1.15.3. If $\mathbf{g}$ is a semisimple Lie algebra containing a pointed generating invariant cone $C$, then the centre of a maximal compactly embedded subalgebra $\mathbf{k}$ is non-trivial [**5**, III.4.7]. Therefore the centre of $\tilde{G}$ is infinite and $Z(G)\subseteq G_{\mathbb{C}}$ is finite. We conclude that $\tilde{G}\neq G$ and therefore that $\tilde{S}\neq S$ (Example 1.15.3). Now Corollary 2.12 shows that no quotient $\tilde{S}/D$, i.e. no non-trivial covering semigroup of $S$ is isomorphic to a subsemigroup of a group. The simplest example is the semigroup $S=\mathrm{Sl}(2,\mathbb{R})\exp(iC)\subseteq\mathrm{Sl}(2,\mathbb{C})$, where $\pi_1(S(\cong\mathbb{Z}$ and $\tilde{S}\cong\mathrm{Sl}(2,\mathbb{R})^{\sim}\mathrm{Exp}(iC)$. Another interesting example is *Howe's oscillator semigroup* (cf. [**4**]). Here $S=\mathrm{Sp}(n,\mathbb{R})\exp(iC)\subseteq\mathrm{Sp}(n,\mathbb{C})$, the group $\mathrm{Sp}(n,\mathbb{C})$ is simply connected, and $\pi_1(\mathrm{Sp}(n,\mathbb{R}))\cong\mathbb{Z}$. Consequently $\pi_1(S)\cong\mathbb{Z}$. The Oscillator semigroup is the double cover $\tilde{S}/\pi_1(S)^2$ of $S$. Its group of units is the well known metaplectic group $\mathrm{Mp}(n,\mathbb{R})$ which is a double cover of the symplectic group $H(S)=\mathrm{Sp}(n,\mathbb{R})$.

**3. Groups with directed orders.** In Section 2 we have considered the relations between the free group $G(S)$ over $S$ and the set homomorphism $i_*:\pi_1(S)\to\pi_1(G)$. Now we consider a particular class of generating Lie semigroups, namely those for which $G=S^{-1}S$. We show that this condition implies that $i_*$ is an isomorphism. Note that this is equivalent to $\tilde{S}\cong q^{-1}(S)\subseteq\tilde{G}$, where $q:\tilde{G}\to G$ is the universal covering of $G$ (Proposition 2.5).

Definition 3.1. Let $S\subseteq G$ be a Lie semigroup. Then we define a left invariant quasiorder $\leq_S$ on $G$ by

$$g\leq_S g'\Leftrightarrow g'\in gS.$$

LEMMA 3.2. *The quasiorder $\leq_S$ is directed (filtered) iff $G = SS^{-1}$ ($G = S^{-1}S$).*

*Proof.* If $\leq_S$ is directed and $g \in G$, then there exists $s \in G$ such that $\mathbf{1} \leq_S s$ and $g \leq_S s$. Hence $s \in S$ and $g \in sS^{-1}$. If, conversely, $G = SS^{-1}$ and $g, g' \in G$, then $g^{-1}g' = s_1 s_2^{-1}$. Thus $g, g' \leq_S g's_2 = gs_1$. The proof for the second statement is similar.

Note that $\leq_S$ is filtered iff $\leq_{S^{-1}}$ is directed.

LEMMA 3.3. *If $S$ is generating, the quasiorder $\leq_S$ is filtered, and $K \subseteq G$ is compact, then $K$ is bounded from below, i.e. there exists $g_0 \in G$ such that $K \subseteq g_0 S$.*

*Proof.* It is clear that $K \subseteq \bigcup_{g \in G} g \operatorname{int}(S)$. Let $K \subseteq \bigcup_{i=1}^{n} g_i \operatorname{int}(S)$. Then there exists $g_0 \in G$ such that $g_0 \leq_S g_1, \ldots, g_n$. Therefore $g_i \operatorname{int}(S) \subseteq g_0 \operatorname{int}(S)$ for $i = 1, \ldots, n$ and consequently $K \subseteq g_0 \operatorname{int}(S)$.

THEOREM 3.4. *Let $S \subseteq G$ be a generating Lie semigroup such that $(G, \leq)$ is filtered (directed). Then the homomorphism $i_*: \pi_1(S) \to \pi_1(G)$, which is induced by the inclusion $i: S \to G$, is an isomorphism.*

*Proof.* Let $[\gamma] \in \pi_1(S)$, where $\gamma: [0, 1] \to S$ is a continuous mapping with $\gamma(0) = \gamma(1) = \mathbf{1}$. Suppose that $i_*([\gamma]) = [i \circ \gamma] = 1$. Then there exists a continuous mapping $F: [0, 1] \times [0, 1] \to G$ such that

$$F(0, t) = \gamma(t), F(1, t) = \mathbf{1} \quad \text{and} \quad F(s, 0) = F(s, 1) = \mathbf{1}.$$

According to Lemma 3.3 there exists $g_0 \in G$ such that $K := F([0, 1] \times [0, 1]) \subseteq g_0 \operatorname{int}(S)$. In particular we have that $\mathbf{1} \in g_0 \operatorname{int}(S)$, i.e. $g_0^{-1} \in \operatorname{int}(S)$. Hence there exists a continuous path $\alpha: [0, 1] \to S$ such that $\alpha(0) = \mathbf{1}$ and $\alpha(1) = g_0^{-1}$ (Proposition 1.2). Now

$$[\alpha \diamond g_0^{-1}\gamma \diamond \hat{\alpha}] = [\alpha \diamond g_0^{-1} \diamond \hat{\alpha}] = [\alpha][\alpha]^{-1} = 1$$

in $\pi_1(S)$ because $(s, t) \mapsto g_0^{-1}F(s, t)$ is a deformation of $g_0^{-1}\gamma$ to the constant path $g_0^{-1}$ in $S$. Now

$$t \mapsto \alpha\,|_{[0,t]} \diamond \alpha(t)\gamma \diamond \hat{\alpha}|_{[0,t]}$$

defines a continuous deformation of the path $\alpha \diamond g_0^{-1}\gamma \diamond \hat{\alpha}$ to $\gamma$. Thus $[\gamma] = 1$ in $\pi_1(S)$. So we have proved that $i_*$ is injective.

Let $[\gamma] \in \pi_1(G)$ and $\gamma: [0, 1] \to G$ with $\gamma(0) = \gamma(1) = \mathbf{1}$. Then, by the same argument as above, there exists a $g_0 \in G$ such that $\gamma([0, 1]) \subseteq g_0 \operatorname{int}(S)$. By the same construction as in the first part of the proof we find a path $\alpha: [0, 1] \to S$ such that

$$[\gamma] = [\alpha \diamond g_0^{-1}\gamma \diamond \hat{\alpha}] \in i_*(\pi_1(S)).$$

If $(G, \leq_S)$ is directed, the first part of the proof shows that the inclusion $S^{-1} \to G$ induces an isomorphism $\pi_1(S^{-1}) \to \pi_1(G)$. Then we apply the inversion $G \to G$, $g \mapsto g^{-1}$ to see that the inclusion $S \to G$ also induces an isomorphism $\pi_1(S) \to \pi_1(G)$.

COROLLARY 3.5. *If $S \subseteq G$ is a Lie semigroup such that $G = SS^{-1}$, then $G(S) = G$.*

*Proof.* This follows from $\operatorname{im} i_* = \pi_1(G(S)) = \pi_1(G)$ (Theorem 2.6, Theorem 3.4).

REMARK 3.6. Let $S$ be an abstract subsemigroup of an abstract group $G$ such that $G = SS^{-1}$. Then $G(S) = G$ holds in this general context. The simple proof can be found in [**1**, p. 36]. It does not depend on the results of this paper.

REMARK 3.7. Note that it was already clear from Proposition 2.3 that $G(S) = S_1S_1^{-1}$ implies that $G(S) = G$ (cf. [**12**, Proposition 3.6]). But that $G = SS^{-1}$ implies even that $G(S) = S_1S_1^{-1}$ because $S_1S_1^{-1}$ then is a subgroup of $G(S)$ [**12**, Proposition 1.2].

COROLLARY 3.8. *If $G$ is a simply connected Lie group and $S \subseteq G$ a generating Lie semigroup such that $\leq_S$ is filtered or directed, then $S$ is simply connected.*

REMARK 3.9. We remark that this corollary is an answer to Problem PVII.2 in [**5**] and also to Problem 3.1 in [**3**, p. 122]): Is a Lie subsemigroup of a simply connected Lie group simply connected? This corollary gives a sufficient condition for simply connectedness. That a generating Lie semigroup $S$ in a simply connected Lie group $G$ is not always simply connected can be seen with Example 2.13, where we have the Lie semigroup $\mathrm{Sl}(2, \mathbb{R})\exp(iC)$ in the simply connected group $\mathrm{Sl}(2, \mathbb{C})$.

REMARK 3.10. Note that $\leq_S$ is in particular directed if $S$ is invariant. For weaker conditions on $S$ which guarantee that $\leq_S$ is directed, see [**10**, Remark II.12].

THEOREM 3.11. (Ruppert) *If $S$ is a generating Lie semigroup in the connected solvable Lie group $G$, then $G(S) = G$.*

*Proof.* This is Theorem 3.7 in [**12**].

THEOREM 3.12. (Ruppert) *If $S$ is a generating Lie semigroup in the connected Lie group $G$ with compact or nilpotent Lie algebra, then $G = SS^{-1} = S^{-1}S$, and in particular $G(S) = G$.*

*Proof.* The nilpotent case is [**12**, Proposition 1.5]. So we may assume that $\mathbf{L}(G)$ is compact. Then the commutator group $G'$ is a compact semisimple Lie group, and $S_1 := SG'/G'$ is a generating Lie semigroup in the abelian Lie group $G/G'$. Now the assertion follows from [**12**, Corollary 2.5] and the observation that $G/G' = S_1S_1^{-1} = S_1^{-1}S_1$.

THEOREM 3.13. *Let $S$ be a generating Lie semigroup in $G$ and suppose that one of the following conditions is satisfied:*

1) *$G$ is nilpotent.*
2) *$\mathbf{L}(G)$ is a compact Lie algebra.*
3) *$S$ is invariant in $G$.*

*Then $i_* : \pi_1(S) \to \pi_1(G)$ is an isomorphism.*

4) *If $G$ is solvable then $i_*$ is surjective.*

*Proof.* This follows from Remark 3.10, Theorem 3.11, Theorem 3.12, and Theorem 3.4.

EXAMPLE 3.14. We have seen in Example 2.10 that there exist Lie semigroups $S \subseteq G$ with $G(S) \neq G$ and therefore homomorphisms $\alpha : S \to T$, where $T$ is a group, such that $\alpha$ has no extension to $G$. This does not occur when $\mathbf{L}(G)$ is a compact Lie algebra. So one is led to the question whether this is true or not if $T$ is a compact group. But even in this particular case there exist counterexamples as the following construction, due to Karl H. Hofmann, shows.

Let $S \subseteq G$ be a generating Lie semigroup with $G(S) \neq G$ (Example 2.10) and let $D \subseteq G(S)$ denote the kernel of the covering $G(S) \to G$. In Example 2.10 we have that $D \cong \pi_1(\mathrm{Sl}(2, \mathbb{R})) \cong \mathbb{Z}$. Next we take a semidirect product $G_1 = H \rtimes K$, where $K$ is a compact group, $S_1 \subseteq G_1$ is a generating Lie semigroup, and there exists a homomorphism $f: D \to Z(G_1)$ such that $f(D) \not\subseteq H$. The simplest example for such a group is $G_1 = \mathbb{R} \times S^1$, the cylinder, where $S_1 = \exp W_1$ for a generating pointed cone $W_1 \subseteq \mathbf{L}(G_1) = \mathbb{R} \oplus \mathbb{R}$ with $W_1 \cap (\{0\} \oplus \mathbb{R}) = \{0\}$. Note that every non-trivial abelian group, in particular $D$, permits a non-trivial homomorphism to $S^1$.

Now we set $G_2 := G(S) \times G_1$ and $S_2 := S \times S_1$, where $S$ is identified with its lift in $G(S)$ (Theorem 2.1). We define a homomorphism

$$\pi: G_2 \to K, (g, h, k) \mapsto k$$

and set $D' := \{(d, f(d)) : d \in D\}$. Then $\pi(D') \neq \{\mathbf{1}\}$ because $f(D) \not\subseteq H$. We claim that $S_2 S_2^{-1} \cap D' = \{\mathbf{1}\}$. If $(s_1, s_2)(s_1', s_2')^{-1} \in D'$, then $s_1 s_1'^{-1} \in SS^{-1} \cap D = \{\mathbf{1}\}$ (Proposition 2.3). Hence $s_1 = s_1'$. Therefore

$$(s_1, s_2)(s_1', s_2')^{-1} = (\mathbf{1}, s_2 s_2'^{-1}) \in D'$$

implies that $s_2 s_2'^{-1} = f(\mathbf{1}) = \mathbf{1}$, i.e. $s_2 = s_2'$ and the claim follows.

Now we set $G_3 := G_2/D'$ and write $q: G_2 \to G_3$ for the quotient homomorphism. Then $S_3 := q(S_2)$ is a generating Lie semigroup $G_3$ which lifts to $S_2$ in $G_2$ (Proposition 2.3). Let

$$\alpha := (q|_{S_2})^{-1} : S_3 \to S_2.$$

Then $\pi \circ \alpha: S_3 \to K$ is a homomorphism of $S_3$ into the compact group $K$ and $\pi \circ \alpha$ permits no continuation to $G_3$. In fact, if $\beta: G_3 \to K$ is such a continuation, then

$$\beta \circ q(s_2) = \pi \circ \alpha \circ q(s_2) = \pi(s_2) \quad \forall s_2 \in S_2.$$

Hence $\beta \circ q = \pi$ and $\pi(\ker q) = \pi(D') = \{\mathbf{1}\}$, a contradiction.

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