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STEINER TRIPLE SYSTEMS HAVING A PRESCRIBED NUMBER OF TRIPLES IN COMMON

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1. Introduction. A Steiner triple system (briefly *STS*) is a pair $(S, \mathscr{B})$ where S is a finite set and $\mathscr{B}$ is a collection of 3-subsets of S (called triples) such that every pair of distinct elements of S belongs to exactly one triple of $\mathscr{B}$. The number $|S|$ is called the order of $(S, \mathscr{B})$. It is well-known that there is an *STS* of order v if and only if $v \equiv 1$ or $3 \pmod 6$. Therefore in saying that a certain property concerning *STS* is true for all v it is understood that $v \equiv 1$ or $3 \pmod 6$. An *STS* of order v will sometimes be denoted by $STS(v)$.

Two Steiner triple systems $(S, \mathscr{B}_1)$ and $(S, \mathscr{B}_2)$ are said to *intersect in k triples* provided $|\mathscr{B}_1 \cap \mathscr{B}_2| = k$. If $k = 0$, $(S, \mathscr{B}_1)$ and $(S, \mathscr{B}_2)$ are said to be *disjoint*, and if $|\mathscr{B}_1 \cap \mathscr{B}_2| = 1$ they are said to be *almost disjoint*. The existence of a pair of disjoint $STS(v)$ of every order $v \geqq 7$ has been shown by J. Doyen in [**1**], and the existence of a pair of almost disjoint $STS(v)$ of every order $v \geqq 3$ has been shown by C. C. Lindner in [**6**]. Very little is known concerning the existence of *STS* intersecting in $k \geqq 2$ triples. The purpose of this paper is to give a complete solution to this problem.

2. Auxiliary constructions and basic lemmas. The number of triples in any $STS(v)$ will be denoted by t_v; i.e., $t_v = v(v-1)/6$. We set $I_v = \{0, 1, \ldots, t_v - 6, t_v - 4, t_v\}$; i.e., the set I_v contains all nonnegative integers not exceeding t_v with the exception of $t_v - 5$, $t_v - 3$, $t_v - 2$, and $t_v - 1$. Further, let $J[v]$ denote the set of all integers k such that there exists a pair of $STS(v)$ intersecting in k triples. The set $J[v]$ is easily determined for $v = 3$ and 7 and is well-known for $v = 9$ (see, e.g., [**5**]). We record this as our first lemma.

LEMMA 1. $J[3] = \{1\}$, $J[7] = \{0, 1, 3, 7\}$, $J[9] = \{0, 1, 2, 3, 4, 6, 12\}$.

A *partial triple system* is a pair $(P, \mathscr{Q})$ where P is a finite set and $\mathscr{Q}$ is a collection of 3-subsets of P such that every pair of distinct elements of P belongs to at most one triple of $\mathscr{Q}$. Two partial triple systems $(P, \mathscr{Q}_1)$ and $(P, \mathscr{Q}_2)$ are said to be *mutually balanced* if any given pair of distinct elements of P is contained in a triple of $\mathscr{Q}_1$ if and only if it is contained in a triple of $\mathscr{Q}_2$. Two mutually balanced partial triple systems are *disjoint* if they have no triple in common.

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LEMMA 2. *For any v and $n = 1, 2, 3$, or 5, $t_v - n \notin J[v]$.*

Proof. If two $STS(v)$ intersect in $t_v - n$ triples there are exactly n triples of one of the STS which do not occur in the other. Therefore the statement of our lemma is equivalent to saying that there are no disjoint mutually balanced partial triple systems containing 1, 2, 3, or 5 triples. It is seen instantly that this is so for $n = 1, 2, 3$, and a bit of reflection handles the case $n = 5$ without any undue difficulty.

COROLLARY 3. *For every v, $J[v] \subseteq I_v$.*

In what follows we will determine the set $J[v]$ for all v, showing, that apart from a few exceptions, $J[v] = I_v$. We will need the following two well-known constructions (for undefined graph-theoretical notions and standard notation, see [**4**]; cf. also [**7**]).

Construction A. Let $(S, \mathscr{B})$ be an $STS(v)$ where $S = \{a_1, a_2, \ldots, a_v\}$. Put $v + 1 = 2n$ and let $\mathscr{F} = \{F_i | i = 1, 2, \ldots, 2n - 1\}$ be a 1-factorization of K_{2n} with the vertex-set $V(K_{2n}) = T$ where $S \cap T = \emptyset$. Put $S^* = S \cup T$ and $\mathscr{B}^* = \mathscr{B} \cup \mathscr{C}$ where $\mathscr{C} = \{\{a_i, x, y\} | [x, y] \in F_i, i = 1, 2, \ldots, 2n - 1\}$. Then $(S^*, \mathscr{B}^*)$ is an $STS(2v + 1)$.

Before describing the second construction we need one more auxiliary device. An (A, k)-system is a set of k disjoint pairs (p_r, q_r) covering the elements of $\{1, 2, \ldots, 2k\}$ exactly once and such that $q_r - p_r = r$ for $r = 1, 2, \ldots, k$. Similarly, a (B, k)-system is a set of k disjoint pairs (p_r, q_r) covering the elements of $\{1, 2, \ldots, 2k - 1, 2k + 1\}$ exactly once and such that $q_r - p_r = r$ for $r = 1, 2, \ldots, k$. It is known (see, e.g., [**9**]) that an (A, k)-system exists if and only if $k \equiv 0$ or 1 (mod 4), and a (B, k)-system exists if and only if $k \equiv 2$ or 3 (mod 4). Observe that an (A, k)-system and a (B, k)-system are essentially the same things as what have been called in [**11**] a Skolem $(2, k)$-sequence and a hooked Skolem $(2, k)$-sequence, respectively.

Construction B. Let $(S, \mathscr{B})$ be an $STS(v)$, $v \geqq 7$, with $S = \{a_1, a_2, \ldots, a_v\}$. Let $U = \{b_1, b_2, \ldots, b_v\}$, $X = \{\infty_i | i = 1, 2, \ldots, 7\}$, and $(X, \mathscr{D})$ an $STS(7)$. Let $(v - 1)/2 = m$ and let $L = \{(p_r, q_r) | q_r - p_r = r, r = 1, 2, \ldots, m\}$ be an (A, m)-system or (B, m)-system according to whether $m \equiv 0, 1$ (mod 4) or $m \equiv 2, 3$ (mod 4). Set $Y = U \backslash W$ where $W = \{b_i | i = p_r$ or $q_r, r = 4, 5, \ldots, m, (p_r, q_r) \in L\}$. Let $Y = \{b_{j_i} | i = 1, 2, \ldots, 7\}$. Put $S^* = S \cup U \cup X$ and $\mathscr{B}^* = \mathscr{B} \cup \mathscr{D} \cup \mathscr{E} \cup \mathscr{F} \cup \mathscr{G}$ where

$$\mathscr{E} = \{\{\infty_i, a_k, b_{j_i+k-1}\} | i = 1, 2, \ldots, 7; k = 1, 2, \ldots, v\},$$

$$\mathscr{F} = \{\{a_k, b_{p_r+k-1}, b_{q_r+k-1}\} | k = 1, 2, \ldots, v; r = 4, 5, \ldots, m; (p_r, q_r) \in L\}$$

and $\mathscr{G} = \{\{b_i, b_{i+1}, b_{i+3}\} | i = 1, 2, \ldots, v\}$,

with subscripts reduced modulo v to the range $\{1, 2, \ldots, v\}$ whenever necessary. Then $(S^*, \mathscr{B}^*)$ is an $STS(2v + 7)$. (Cf. [**8**; **10**].)

LEMMA 4. *If* $k \in J[v]$ *then* $k + s(v + 1)/2 \in J[2v + 1]$ *for every* $s = 0, 1, 2, \ldots, v - 2, v$.

Proof. Let $(S, \mathscr{B}_1)$ and $(S, \mathscr{B}_2)$ be two $STS(v)$ intersecting in k triples and let S, T, S^*, and $\mathscr{F}$ be as in Construction A. Let α be any permutation of S fixing exactly s elements; obviously such an α exists for $s = 0, 1, 2, \ldots, v - 2, v$. Let now $\mathscr{C}$ be as in Construction A; i.e.,

$$\mathscr{C} = \{\{a_i, x, y\} | [x, y] \in F_i, i = 1, 2, \ldots, 2n - 1\}, \text{ and put}$$
$$\mathscr{C}_\alpha = \{\{a_i\alpha, x, y\} | [x, y] \in F_i, i = 1, 2, \ldots, 2n - 1\}.$$

Both $(S^*, \mathscr{B}_1 \cup \mathscr{C})$ and $(S^*, \mathscr{B}_2 \cup \mathscr{C}_\alpha)$ are $STS(2v + 1)$. Since each 1-factor of F contains $(v + 1)/2$ edges, $\mathscr{C}$ and $\mathscr{C}_\alpha$ have exactly $s(v + 1)/2$ triples in common so that

$$|(\mathscr{B}_1 \cup \mathscr{C}) \cap (\mathscr{B}_2 \cup \mathscr{C}_\alpha)| = k + s(v + 1)/2.$$

LEMMA 5. *For* $v \geqq 13$, $J[v] = I_v$ *implies* $J[2v + 1] = I_{2v+1}$.

Proof. Taking into account that $t_v - 6 \geqq (v + 1)/2$ for $v \geqq 9$ we obtain from Lemma 4, by putting consecutively $s = 0, 1, \ldots, v - 2$, that $k \in J[2v + 1]$ for $k = 0, 1, \ldots, t_{2v+1} - (v + 7)$ [since $t_v - 6 + (v - 2)(v + 1)/2 = t_{2v+1} - (v + 7)$]. On the other hand, $t_v \geqq v + 7$ for $v \geqq 13$ so that $t_{2v+1} - (v + 7) \geqq t_{2v+1} - t_v$, and applying Lemma 4 with $s = v$ gives $k \in J[2v + 1]$ for $k = t_{2v+1} - t_v, t_{2v+1} - t_v + 1, \ldots, t_{2v+1} - 6, t_{2v+1} - 4, t_{2v+1}$. Consequently, $J[2v + 1] = I_{2v+1}$.

LEMMA 6. *Let* $v \geqq 7$. *If* $k \in J[v]$ *then* $k + s(v + 7)/2 + \delta v + \gamma \in J[2v + 7]$ *for every* $s = 0, 1, 2, \ldots, v - 2, v$; $\delta = 0, 1$; $\gamma = 0, 1, 3, 7$.

Proof. Let $(S, \mathscr{B}_1)$ and $(S, \mathscr{B}_2)$ be two $STS(v)$ intersecting in k triples and let S, U, X, and S^* be as in Construction B. Let α be any permutation of S fixing exactly s elements; i.e., $s \in \{0, 1, 2, \ldots, v - 2, v\}$. With $\mathscr{E}$ and $\mathscr{F}$ as in Construction B, let $\mathscr{E}_\alpha$ and $\mathscr{F}_\alpha$ denote the set of triples obtained from $\mathscr{E}$ and de, respectively, by replacing every a_k, $k = 1, 2, \ldots, v$, by $a_k\alpha$. Further, denote

$$\mathscr{G}_\delta = \begin{cases} \mathscr{G} & \text{if } \delta = 1, \text{ and} \\ \{\{b_i, b_{i+2}, b_{i+3}\} | i = 1, 2, \ldots, v\} & \text{if } \delta = 0. \end{cases}$$

Let further $(X, \mathscr{D}_1)$ and $(X, \mathscr{D}_2)$ be two $STS(7)$ intersecting in γ triples. Set $\mathscr{B}_1^* = \mathscr{B}_1 \cup \mathscr{D}_1 \cup \mathscr{E} \cup \mathscr{F} \cup \mathscr{G}$ and $\mathscr{B}_2^* = \mathscr{B}_2 \cup \mathscr{D}_2 \cup \mathscr{E}_\alpha \cup \mathscr{F}_\alpha \cup \mathscr{G}_\delta$. Then both $(S^*, \mathscr{B}_1^*)$ and $(S^*, \mathscr{B}_2^*)$ are $STS(2v + 7)$. Since there are exactly 7 triples of $\mathscr{E}$ and exactly $(v - 7)/2$ triples of $\mathscr{F}$ containing a fixed element a_k, we have

$$|(\mathscr{E} \cup \mathscr{F}) \cap (\mathscr{E}_\alpha \cup \mathscr{F}_\alpha)| = s(v + 7)/2.$$

Further, $|\mathscr{G} \cap \mathscr{G}_\delta| = \delta v$ and $|\mathscr{D}_1 \cap \mathscr{D}_2| = \gamma$ so that

$$|\mathscr{B}_1^* \cap \mathscr{B}_2^*| = k + s(v+7)/2 + \delta v + \gamma.$$

LEMMA 7. *For* $v \geqq 15$, $J[v] = I_v$ *implies* $J[2v+7] = I_{2v+7}$.

Proof. Taking into account that $t_v - 6 \geqq (v+7)/2$ for $v \geqq 13$, we obtain from Lemma 6 by putting consecutively $s = 0, 1, \ldots, v-2$ that $k \in J[2v+7]$ for $k = 0, 1, \ldots, t_{2v+7} - (v+19)$ [since $t_v - 6 + (v-2)(v+7)/2 + v + 1 = t_{2v+7} - (v+19)$]. On the other hand, $t_v \geqq v + 19$ for $v \geqq 15$ so that $t_{2v+7} - (v+19) \geqq t_{2v+7} - t_v$, and using now Lemma 6 with $s = v$ gives $k \in J[2v+7]$ for $k = t_{2v+7} - t_v, t_{2v+7} - t_v + 1, \ldots, t_{2v+7} - 6, t_{2v+7} - 4, t_{2v+7}$. Thus $J[2v+7] = I_{2v+7}$.

3. The sets $J[v]$ for small v. To obtain the results of this section we will need the following lemma.

LEMMA 8. *If* $k \in J[v]$, *then* $t_u - t_v + k \in J[u]$ *for every* $u \geqq 2v+1$.

Proof. Let $(S, \mathscr{B})$ be an $STS(v)$. In [**2**], J. Doyen and R. M. Wilson have shown that any $STS(v)$ can be embedded into an $STS(u)$ for every $u \geqq 2v+1$. Let $(S^*, \mathscr{B} \cup \mathscr{C})$ be an $STS(u)$ containing $(S, \mathscr{B})$ as a subsystem and let $(S, \mathscr{B}_1)$ and $(S, \mathscr{B}_2)$ be two $STS(v)$ intersecting in k triples. Then $(S^*, \mathscr{B}_1 \cup \mathscr{C})$ and $(S^*, \mathscr{B}_2 \cup \mathscr{C})$ are two $STS(u)$ intersecting in $t_u - t_v + k$ triples.

LEMMA 9. $J[13] = I_{13} \backslash \{15, 17, 19\}$.

Proof. It follows from [**1**] and [**6**] that $0, 1 \in J[13]$. An example in [**3**, p. 237], shows $22 \in J[13]$, and trivially $26 \in J[13]$. Let $S = \{1, 2, \ldots, 13\}$ and let $\mathscr{B}_i$, $i = 1, 2, \ldots, 8$, be the sets of 26 triples given in Table 1 written as columns (for brevity all brackets are omitted). Then $(S, \mathscr{B}_i)$, $i = 1, 2, \ldots, 8$, are $STS(13)$, and we have:

$$\begin{array}{llll}
|\mathscr{B}_4 \cap \mathscr{B}_5| = 2, & |\mathscr{B}_2 \cap \mathscr{B}_4| = 3, & |\mathscr{B}_3 \cap \mathscr{B}_5| = 4, & |\mathscr{B}_2 \cap \mathscr{B}_3| = 5, \\
|\mathscr{B}_4 \cap \mathscr{B}_7| = 6, & |\mathscr{B}_3 \cap \mathscr{B}_7| = 7, & |\mathscr{B}_1 \cap \mathscr{B}_4| = 8, & |\mathscr{B}_2 \cap \mathscr{B}_6| = 9, \\
|\mathscr{B}_2 \cap \mathscr{B}_7| = 10, & |\mathscr{B}_1 \cap \mathscr{B}_3| = 11, & |\mathscr{B}_5 \cap \mathscr{B}_8| = 12, & |\mathscr{B}_5 \cap \mathscr{B}_6| = 13, \\
|\mathscr{B}_1 \cap \mathscr{B}_6| = 14, & |\mathscr{B}_1 \cap \mathscr{B}_2| = 16, & |\mathscr{B}_1 \cap \mathscr{B}_7| = 18, & |\mathscr{B}_1 \cap \mathscr{B}_8| = 20.
\end{array}$$

In order to complete the proof, assume $(S, \mathscr{C}_1)$ and $(S, \mathscr{C}_2)$ to be a pair of $STS(13)$ intersecting in 19 triples. Then there exist disjoint mutually balanced partial triple systems $(P, \mathscr{Q}_1)$ and $(P, \mathscr{Q}_2)$ with $P \subseteq S$, $\mathscr{Q}_i \subseteq \mathscr{C}_i$, and $|\mathscr{Q}_i| = 7$, $i = 1, 2$. It follows that $|P| = 7$, and consequently $(P, \mathscr{Q}_1)$ and $(P, \mathscr{Q}_2)$ are $STS(7)$. However, an $STS(7)$ cannot be embedded into an $STS(13)$ and therefore $19 \notin J[13]$. It can be shown in a similar fashion (cf. also Lemma 10 below) that $17 \notin J[13]$ and $15 \notin J[13]$, although in the latter case there exist two essentially different pairs of disjoint mutually balanced partial triple systems with 11 triples (neither of which, however, can be embedded into an $STS(13)$). This completes the proof of the lemma.

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TABLE 1

B_1			B_2			B_3			B_4			B_5			B_6			B_7			B_8		
1	2	3	1	2	3	1	2	4	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
1	4	5	1	4	6	1	3	9	1	4	5	1	4	8	1	4	5	1	4	5	1	4	5
1	6	7	1	5	7	1	5	8	1	6	7	1	5	12	1	6	7	1	6	7	1	6	7
1	8	9	1	8	10	1	6	7	1	8	9	1	6	10	1	8	9	1	8	9	1	8	10
1	10	11	1	9	12	1	10	11	1	10	11	1	7	11	1	10	11	1	10	13	1	9	12
1	12	13	1	11	13	1	12	13	1	12	13	1	9	13	1	12	13	1	11	12	1	11	13
2	4	6	2	4	5	2	3	10	2	4	13	2	4	5	2	4	8	2	4	6	2	4	6
2	5	7	2	6	7	2	5	7	2	5	12	2	6	7	2	5	12	2	5	7	2	5	7
2	8	10	2	8	9	2	6	8	2	6	11	2	8	9	2	6	13	2	8	10	2	8	9
2	9	12	2	10	11	2	9	12	2	7	8	2	10	11	2	7	11	2	9	12	2	10	11
2	11	13	2	12	13	2	11	13	2	9	10	2	12	13	2	9	10	2	11	13	2	12	13
3	4	8	3	4	8	3	4	8	3	4	8	3	4	6	3	4	6	3	4	8	3	4	8
3	7	11	3	5	12	3	5	11	3	5	10	3	5	7	3	5	7	3	5	12	3	5	12
3	5	12	3	6	10	3	6	12	3	6	12	3	8	10	3	8	10	3	6	13	3	6	13
3	9	13	3	7	11	3	7	13	3	7	13	3	9	12	3	9	12	3	7	11	3	7	11
4	7	9	3	9	13	4	5	12	3	9	11	3	11	13	3	11	13	3	9	10	3	9	10
4	10	13	4	7	9	4	6	10	4	6	10	4	7	9	4	7	9	4	7	9	4	7	9
4	11	12	4	10	13	4	7	11	4	7	11	4	10	13	4	10	13	4	10	11	4	10	13
5	8	11	4	11	12	4	9	13	4	9	12	4	11	12	4	11	12	4	12	13	4	11	12
5	6	13	5	6	13	5	6	13	5	6	8	5	6	13	5	6	10	5	6	10	5	6	10
5	9	10	5	8	11	5	9	10	5	7	9	5	8	11	5	8	11	5	8	11	5	8	11
6	9	11	5	9	10	6	9	11	5	11	13	5	9	10	5	9	13	5	9	13	5	9	13
6	8	12	6	8	12	7	8	9	6	9	13	6	8	12	6	8	12	6	8	12	6	8	12
7	8	13	6	9	11	7	10	12	7	10	12	6	9	11	6	9	11	6	9	11	6	9	11
7	10	12	7	8	13	8	10	13	8	10	13	7	8	13	7	8	13	7	8	13	7	8	13
3	6	10	7	10	12	8	11	12	8	11	12	7	10	12	7	10	12	7	10	12	7	10	12

LEMMA 10. $J[15] = I_{15} \backslash \{26\}$.

Proof. Applying Lemma 4 to $J[7]$ we get $k \in J[15]$ for all $k \in I_{15}$ except for $k = 2, 6, 10, 14, 18, 22, 24, 25, 26$. Let $S = \{a_1, a_2, \ldots, a_7\}$, $T = \{1, 2, \ldots, 8\}$, and let $\mathscr{F} = \{F_i | i = 1, 2, \ldots, 7\}$ be the 1-factorization of K_8 with $V(K_8) = T$ given by:

$$F_1 = \{[1, 2], [3, 4], [5, 6], [7, 8]\},$$
$$F_2 = \{[1, 3], [2, 4], [5, 7], [6, 8]\},$$
$$F_3 = \{[1, 4], [2, 3], [5, 8], [6, 7]\},$$
$$F_4 = \{[1, 5], [2, 6], [3, 7], [4, 8]\},$$
$$F_5 = \{[1, 6], [2, 5], [3, 8], [4, 7]\},$$
$$F_6 = \{[1, 7], [2, 8], [3, 5], [4, 6]\},$$
$$F_7 = \{[1, 8], [2, 7], [3, 6], [4, 5]\}.$$

Let α be a permutation of the set $\{4, 5, 6, 7\}$ fixing exactly s elements (i.e., $s = 0, 1, 2$ or 4), and let $\mathscr{G} = \{G_i | i = 1, 2, \ldots, 7\}$ be another 1-factorization of K_8 on T given by:

$$G_1 = \{[1, 4], [2, 3], [5, 6], [7, 8]\},$$
$$G_2 = \{[1, 2], [3, 4], [5, 7], [6, 8]\},$$
$$G_3 = \{[1, 3], [2, 4], [5, 8], [6, 7]\}, \text{ and}$$
$$G_i = F_i, \text{ for } i = 4, 5, 6, 7.$$

Let $(S, \mathscr{B}_1)$ and $(S, \mathscr{B}_2)$ be two disjoint $STS(7)$ and let

$$\mathscr{C}_1 = \{\{a_i, x, y\} | [x, y] \in F_i, i = 1, 2, \ldots, 7\}, \text{ and}$$
$$\mathscr{C}_2 = \{\{a_i, x, y\} | [x, y] \in G_i, i = 1, 2, \ldots, 7\}.$$

Then the two $STS(15)$ $(S \cup T, \mathscr{B}_1 \cup \mathscr{C}_1)$ and $(S \cup T, \mathscr{B}_2 \cup \mathscr{C}_2)$ intersect in $4s + 6$ triples. Hence $6, 10, 14, 22 \in J[15]$.

Let $\mathscr{H} = \{H_i | i = 1, 2, \ldots, 7\}$ be another 1-factorization of K_8 on T given by $H_i = F_i$ for $i = 1, 2, 3, 6, 7$, and

$$H_4 = \{[1, 6], [2, 5], [3, 7], [4, 8]\},$$
$$H_5 = \{[1, 5], [2, 6], [3, 8], [4, 7]\},$$

and let

$$\mathscr{C}_3 = \{\{a_i, x, y\} | [x, y] \in H_i, i = 1, 2, \ldots, 7\}.$$

Then the two $STS(15)$ $(S \cup T, \mathscr{B}_1 \cup \mathscr{C}_1)$ and $(S \cup T, \mathscr{B}_2 \cup \mathscr{C}_3)$ intersect in 18 triples so that $18 \in J[15]$. If $(S, \mathscr{B}_1)$ and $(S, \mathscr{B}_3)$ are two $STS(7)$ intersecting in 3 triples then $(S \cup T, \mathscr{B}_1 \cup \mathscr{C}_1)$ and $(S \cup T, \mathscr{B}_3 \cup \mathscr{C}_2)$ (with $s = 4$) intersect in 25 triples so that $25 \in J[15]$. Let $J = \{J_i | i = 1, 2, \ldots, 7\}$ be another 1-factorization of K_8 on T given by $J_1 = G_1, J_2 = F_4, J_3 = F_5, J_4 = F_6, J_5 = F_7, J_6 = G_2, J_7 = G_3$, and let

$$\mathscr{C}_4 = \{\{a_i, x, y\} | [x, y] \in J_i, i = 1, 2, \ldots, 7\}.$$

Then $(S \cup T, \mathscr{B}_1 \cup \mathscr{C}_1)$ and $(S \cup T, \mathscr{B}_2 \cup \mathscr{C}_4)$ intersect in 2 triples so that $2 \in J[15]$. Finally, let $K = \{K_i | i = 1, 2, \ldots, 7\}$ be another 1-factorization of K_8 on T given by:

$$K_1 = \{[1, 3], [2, 4], [5, 6], [7, 8]\},$$

$$K_2 = G_2, K_i = F_i \text{ for } i = 3, 4, 5, 6, 7,$$

and let

$$\mathscr{C}_5 = \{\{a_i, x, y\} | [x, y] \in K_i, i = 1, 2, \ldots, 7\}.$$

Then $(S \cup T, \mathscr{B}_1 \cup \mathscr{C}_1)$ and $(S \cup T, \mathscr{B}_2 \cup \mathscr{C}_5)$ intersect in 24 triples so that $24 \in J[15]$.

In order to complete the proof, assume $(S, \mathscr{B}_1)$ and $(S, \mathscr{B}_2)$ to be a pair of $STS(15)$ intersecting in 26 triples. Then there exist disjoint mutually balanced partial triple systems $(P, \mathscr{Q}_1)$ and $(P, \mathscr{Q}_2)$ with $P \subseteq S$, $\mathscr{Q}_i \subseteq \mathscr{B}_i$, and $|\mathscr{Q}_i| = 9$, $i = 1, 2$. It follows easily that $|P| = 9$, and elementary considerations show that there is essentially only one pair of disjoint mutually balanced partial triple systems $(P, \mathscr{Q}_i)$ with $|P| = |\mathscr{Q}_i| = 9$ neither of which can be embedded into an $STS(15)$. Thus $26 \notin J[15]$ and the proof is complete.

LEMMA 11. $J[19] = I_{19}$.

Proof. Applying Lemma 4 to $J[9]$ we get $k \in J[19]$ for every $k \in I_{19}$ except for $k = 40, 42, 43, 44, 50$ and 53. Since $0, 3 \in J[7]$ applying Lemma 8 with $k = 0$ and 3, $v = 7$, and $u = 19$ gives $50, 53 \in J[19]$.

Let $T = \{1, 2, \ldots, 10\}$ and let $\mathscr{F} = \{F_i | i = 1, 2, \ldots, 9\}$ be a 1-factorization of K_{10} on T containing a sub-1-factorization of K_4 on $\{1, 2, 3, 4\}$. Let, without loss of generality, $[1, 2], [3, 4] \in F_1$, $[1, 3], [2, 4] \in F_2$, and $[1, 4], [2, 3] \in F_3$. Let $\mathscr{G} = \{G_i | i = 1, 2, \ldots, 9\}$ be a 1-factorization of K_{10} on T such that $[1, 2], [3, 4] \in G_2$, $[1, 3], [2, 4] \in G_1$, and for all other edges $[x, y]$, $[x, y] \in G_i$ if and only if $[x, y] \in F_i$. Let $\mathscr{H} = \{H_i | i = 1, 2, \ldots, 9\}$ be a 1-factorization of K_{10} on T such that $[1, 2], [3, 4] \in H_2$, $[1, 3], [2, 4] \in H_3$, $[1, 4], [2, 3] \in H_1$, and for all other edges $[x, y]$, $[x, y] \in H_i$ if and only if $[x, y] \in F_i$. Let $S = \{a_1, a_2, \ldots, a_9\}$ and, as in Construction A, define three sets of triples $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ as follows:

$$\mathscr{C}_1 = \{\{a_i, x, y\} | [x, y] \in F_i, i = 1, 2, \ldots, 9\},$$

$$\mathscr{C}_2 = \{\{a_i, x, y\} | [x, y] \in G_i, i = 1, 2, \ldots, 9\}, \text{ and}$$

$$\mathscr{C}_3 = \{\{a_i, x, y\} | [x, y] \in H_i, i = 1, 2, \ldots, 9\}.$$

Clearly, $|\mathscr{C}_1 \cap \mathscr{C}_2| = 41$ and $|\mathscr{C}_1 \cap \mathscr{C}_3| = 39$. Let $(S, \mathscr{B}_1)$ and $(S, \mathscr{B}_2)$ be two $STS(9)$ intersecting in k triples. For $k = 1$, $(S \cup T, \mathscr{B}_1 \cup \mathscr{C}_1)$ and $(S \cup T, \mathscr{B}_2 \cup \mathscr{C}_3)$ intersect in 40 triples, and for $k = 1, 2, 3$, $(S \cup T, \mathscr{B}_2 \cup \mathscr{C}_1)$

and $(S \cup T, \mathscr{B}_2 \cup \mathscr{C}_2)$ intersect in 42, 43 and 44 triples, respectively. Thus 40, 42, 43, 44 $\in J[19]$ which completes the proof.

LEMMA 12. $J[21] = I_{21}$.

Proof. Taking into account that if $v = 7$ then $(U, \mathscr{G})$, where U and $\mathscr{G}$ are as in Construction B, is itself an $STS(7)$, Lemma 6 can be modified to read in this particular case: "*If* $k \in J[7]$ *then* $k + 7s + \delta + \gamma \in J[21]$ *for* $s = 0, 1, 2, 3, 4, 5, 7, \delta = 0, 1, 3, 7, \gamma = 0, 1, 3, 7$." Applying this to $J[7]$ we get $k \in J[21]$ for every $k \in I_{21}$ except for $k = 47$ and 61. Since $3 \in J[9]$ applying Lemma 8 with $k = 3, v = 9, u = 21$ gives $61 \in J[21]$. Further, let $A = \{a_1, a_2, \ldots, a_9\}$, $B = \{b_1, b_2, \ldots, b_9\}$, $X = \{x, y, z\}$, $S = A \cup B \cup X$, and let $(A, \mathscr{D}_1)$ and $(A, \mathscr{D}_2)$ be two $STS(9)$ intersecting in 3 triples. Let further $\mathscr{E}$ and $\mathscr{F}$ be the following sets of triples:

$$\mathscr{E} = \{\{x, y, z\}, \{b_1, b_4, b_7\}, \{b_2, b_5, b_8\}, \{b_3, b_6, b_9\}\}, \text{ and}$$
$$\mathscr{F} = \{\{x, a_i, b_i\}, \{y, a_i, b_{i+1}\}, \{z, a_i, b_{i+2}\}, \{a_i, b_{i+3}, b_{i+5}\}, \{a_i, b_{i+4}, b_{i+8}\}, \{a_i, b_{i+6}, b_{i+7}\} | i = 1, 2, \ldots, 9\}$$

where the subscripts in $\mathscr{F}$ are reduced modulo 9 to the range $\{1, 2, \ldots, 9\}$. It is seen easily that $(S, \mathscr{D}_i \cup \mathscr{E} \cup \mathscr{F})$ is an $STS(21)$ (where $i = 1$ or 2).

Further, let P_1 and P_2 be the following sets of 7 pairs each:

$$P_1 = \{\{x, a_1\}, \{y, a_9\}, \{z, a_8\}, \{a_2, b_6\}, \{a_3, b_9\}, \{a_4, b_2\}, \{a_5, b_8\}\}, \text{ and}$$
$$P_2 = \{\{x, a_4\}, \{y, a_3\}, \{z, a_2\}, \{a_1, b_6\}, \{a_5, b_9\}, \{a_8, b_2\}, \{a_9, b_8\}\}.$$

Let T_1 and T_2 be the following two sets of 14 triples each:

$$\mathscr{T}_1 = \{\{b_1, u, v\}, \{b_4, w, t\} | \{u, v\} \in P_1, \{w, t\} \in P_2\}, \text{ and}$$
$$\mathscr{T}_2 = \{\{b_1, w, t\}, \{b_4, u, v\} | \{u, v\} \in P_1, \{w, \mathrm{t}\} \in P_2\}.$$

Clearly, $(S, \mathscr{T}_1)$ and $(S, \mathscr{T}_2)$ are disjoint mutually balanced partial triple systems and $\mathscr{T}_1 \subseteq \mathscr{F}$. Therefore the two triple systems $(S, \mathscr{D}_1 \cup \mathscr{E} \cup \mathscr{F})$ and $(S, \mathscr{D}_2 \cup \mathscr{E} \cup (\mathscr{F} \backslash \mathscr{T}_1) \cup \mathscr{T}_2)$ intersect in $3 + 4 + 54 - 14 = 47$ triples. Thus $47 \in J[21]$ and the proof of the lemma is complete.

LEMMA 13. $J[25] = I_{25}$.

Proof. Applying Lemma 6 to $J[9]$ we get $k \in J[25]$ for every $k \in I_{25}$ except for $k = 96$. Since $3 \in J[7]$ applying Lemma 8 with $k = 3, v = 7$, and $u = 25$ gives $96 \in J[25]$.

LEMMA 14. $J[27] = I_{27}$.

Proof. Applying Lemma 4 to $J[13]$ we obtain $k \in J[27]$ for every $k \in I_{27}$ except for $k = 106, 108$ and 110. Since $0 \in J[7]$ applying Lemma 8 with $k = 0$, $v = 7$, and $u = 27$ gives $110 \in J[27]$. Since $1, 3 \in J[9]$ applying Lemma 8 with $k = 1$ and 3, $v = 9$, and $u = 27$ gives $106, 108 \in J[27]$.

LEMMA 15. $J[v] = I_v$ *for* $v = 31, 33$ *and* 37.

Proof. Applying Lemma 4 to $J[15]$ we get $k \in J[31]$ for every $k \in I_{31}$ except for $k = 146$. Applying Lemma 8 with $k = 3, v = 9$, and $u = 31$ gives $146 \in J[31]$. Applying Lemma 6 to $J[13]$ we get $k \in J[33]$ for every $k \in I_{33}$ except for $k = 167$. Applying Lemma 8 with $k = 3, v = 9$, and $u = 33$ gives $167 \in J[33]$. Applying Lemma 6 to $J[15]$ we get $k \in J[37]$ for every $k \in I_{37}$ except for $k = 213$. Applying Lemma 8 with $k = 3, v = 9$, and $u = 37$ gives $213 \in J[37]$.

4. Main results.

THEOREM 16. *For every* $v \geqq 19$, $J[v] = I_v$.

Proof. For $v = 19, 21, 25, 27, 31, 33$ and 37 our statement follows from Lemmas 11–15. Assume therefore $v \geqq 39$, and assume that for all $w < v\, (w \geqq 19)$, $J[w] = I_w$. If $v \equiv 3$ or $7 \pmod{12}$ then $(v - 1)/2 \equiv 1$ or $3 \pmod 6$ and $(v - 1)/2 \geqq 19$. Therefore $J[(v - 1)/2] = I_{(v-1)/2}$ and by Lemma 5, $J[v] = I_v$ as well. If $v \equiv 1$ or $9 \pmod{12}$ then $(v - 7)/2 \equiv 1$ or $3 \pmod 6$ and $(v - 7)/2 \geqq 19$. Therefore $J[(v - 7)/2] = I_{(v-7)/2}$ and by Lemma 7, $J[v] = I_v$ as well.

Let k be a nonnegative integer. Define c_k to be the smallest integer such that for all $v \geqq c_k$, $k \in J[v]$. Clearly, Theorem 16 shows that c_k exists for all non-negative integers k, and, in fact, the following theorem giving the values of c_k for all k is an easy consequence of Theorem 16 and the results of Section 3.

THEOREM 17. *Let* $\overset{*}{\urcorner}\, x\, \overset{*}{\ulcorner}$ *denote the least integer* $\equiv 1$ *or* $3 \pmod 6$ *not less than* x. *Then*

$$c_k = \overset{*}{\urcorner}\, \tfrac{1}{2}(1 + \sqrt{1 + 24k})\, \overset{*}{\ulcorner} + \delta_k$$

where

$$\delta_k = \begin{cases} 6 & \text{if } k = 0, 5, 7 \text{ or } 26, \\ 4 & \text{if } k = 8 \text{ or if } k = 6t^2 + 5t - a \text{ for some positive integer } t \text{ and} \\ & a = 0, 1, 2, 4, k \neq 7, \\ 2 & \text{if } k = 15, 17 \text{ or } 19 \text{ or if } k = 6t^2 + t - a \text{ for some positive integer } t \\ & \text{and } a = 1, 2, 3, 5, k \neq 5, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Let us remark in conclusion that we have considered pairs of $STS(v)$ regardless of whether they are isomorphic or not. By analogy with $J[v]$, one could define $J^*[v]$ to be the set of all integers k such that there exists a pair of *isomorphic* $STS(v)$ intersecting in k triples. Trivially $J^*[v] = J[v]$ for $v = 3, 7, 9$. On the other hand, for every $v \geqq 13$, two $STS(v)$ intersecting in $t_v - 4$ triples are necessarily non-isomorphic so that $t_v - 4 \notin J^*[v]$. Thus, for every $v \geqq 13$, $J^*[v]$ is a proper subset of $J[v]$. To determine the sets $J^*[v]$ for $v \geqq 13$ remains an open problem.

REFERENCES

1. J. Doyen, *Constructions of disjoint Steiner triple systems*, Proc. Amer. Math. Soc. *32* (1972), 409–416.

2. J. Doyen and R. M. Wilson, *Embeddings of Steiner triple systems*, Discrete Math. *5* (1973), 229–239.
3. M. Hall, Jr., *Combinatorial Theory* (Ginn-Blaisdell, Waltham, Mass., 1967).
4. F. Harary, *Graph Theory* (Addison-Wesley, Reading, Mass., 1969).
5. E. S. Kramer and D. M. Mesner, *Intersections among Steiner systems*, J. Combinatorial Theory, Ser. A *16* (1974), 273–285.
6. C. C. Lindner, *Construction of Steiner triple systems having exactly one triple in common*, Can. J. Math. *26* (1974), 225–232.
7. C. C. Lindner, E. Mendelsohn and A. Rosa, *On the number of 1-factorizations of the complete graph*, to appear, J. Comb. Theory (B).
8. C. C. Lindner and A. Rosa, *On the existence of automorphism-free Steiner triple systems*, J. Algebra, *34* (1975),430–443.
9. A. Rosa, *Remark on cyclic Steiner triple systems* (in Slovak), Mat.-fyz.časopis *16* (1966), 285–290.
10. ——— *Steiner triple systems and their chromatic number*, Acta Fac. Rerum Natur. Univ. Comenian. Math. *24* (1970), 159–174.
11. D. P. Roselle, *Distributions of integers into s-tuples with given differences*, Proc. Manitoba Conference on Numerical Mathematics, Winnipeg 1971, 31–42.

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