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Nilpotent Conjugacy Classes in p-adic Lie Algebras: The Odd Orthogonal Case

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Abstract. We will study the following question: Are nilpotent conjugacy classes of reductive Lie algebras over p-adic fields definable? By definable, we mean definable by a formula in Pas's language. In this language, there are no field extensions and no uniformisers. Using Waldspurger's parametrization, we answer in the affirmative in the case of special orthogonal Lie algebras $\mathfrak{so}(n)$ for n odd, over p-adic fields.

1 Historical Background: Motivic Representation Theory

In a lecture given at Orsay in 1995, M. Kontsevich introduced the concept of motivic integration. Since then it has become a tool of immense importance. The theory of motivic integration has been developed and extended by Jan Denef and Francois Loeser [1] and presented as *arithmetic motivic integration*. Their work strengthens the belief that all natural p-adic integrals are motivic.

A construction of Denef and Loeser [1] attaches a virtual Chow motive to formulae in Pas's language. We give a brief introduction to Pas's language in the next section. Virtual Chow motives are designed to be independent of the p-adic field.

This paper is a small part of an effort initiated by T. C. Hales [6] to relate various objects arising in representation theory of p-adic groups to geometry. In that approach, expressing the concepts of representation theory of p-adic groups in Pas's language is the first step towards the goal. It is conjectured that many basic objects in representation theory should be motivic in nature. If the conjecture is true, it will enable us to do computations without relying on the specific value of p [5, 6]. In his paper, Hales [6] achieves the goal for orbital integrals of p-adics by showing that under general conditions p-adic orbital integrals of definable functions are represented by virtual Chow motives. In her thesis, J. Gordon [2] proves that character values of a class of depth-zero representations of symplectic groups ($SP(2n)$) and special orthogonal groups ($SO(2n + 1)$) over p-adic fields can be represented as virtual Chow motives.

We call a mathematical object *definable* if it can be described (defined) by a formula in Pas's language. As we describe in the next section, this language makes no reference to the specific value of p [10]. As a result, we can give a field-independent description of nilpotent conjugacy classes. Moreover, the objects found in our proofs will be formulae in this language of (somewhat new entities called) *virtual sets*. We would like to point out that the word *virtual* in this context has no connection with its use in *virtual Chow motive*.

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Nilpotent conjugacy classes are extremely important objects in the study of p-adic groups. They appear prominently in the Shalika germ expansion [13,16]. Along with these orbits, if other components of the expansion are shown to be definable, then it would imply that Shalika germs exist independently of primes.

In this paper, we show that nilpotent conjugacy classes of $\mathfrak{so}(n)$, for n odd, are indeed definable in Pas's language. Why do we consider only this case? Can this method be generalized to other reductive Lie algebras, or for that matter, to the even orthogonal case? First of all, our treatment of $\mathfrak{so}(n)$ relies on Waldspurger's parametrization [17]. He gives combinatorial data as parameters for the nilpotent conjugacy classes of $\mathfrak{so}(n)$ for odd n, but excludes the case where n is even. Secondly, there are no field extensions in Pas's language. At best, a finite field extension can be viewed as a vector space, but it is not possible to extend the valuation to the vector space.

In Sections 1 and 2, we give a brief overview of Pas's language and the use of *virtual sets* in that context. Section 3 gives a somewhat tedious but detailed list of all the required formulae. Section 4 deals exclusively with the formulation of the statement of the main theorem and its proof.

2 Introduction: Pas's Language

In this paper we will be dealing with local fields. A *local field* is locally compact; it is complete with respect to its valuation and has a finite residue field. Since we desire a field-independent description, we find it convenient to use Pas's language, which allows us to exploit the structure of a local field without referring to its individual features, such as uniformiser of the valuation [10].

Pas's language is a *first order language* with three sorts of variables: variables for the elements of the valued field ($\mathbb{F}$), variables for the elements of the residue field ($\overline{\mathbb{F}}$) and variables for elements of the value group (Γ). It contains symbols for standard field operations in the valued field and in the residue field (*i.e.,* addition and multiplication) along with symbols for the usual operation (only addition) in the value group (Γ). In addition, both field sorts have a symbol for equality ($=$). The value sort has symbols $\leq$, $\geq$ and $\equiv_n$ for congruence modulo each non-zero $n \in \mathbb{N}$. With $\mathbb{Z}$ as a structure for Γ, these symbols have the usual meaning.

Let $\mathbb{L}_{\mathbb{F}}$ be the language of fields for the field sort ($\mathbb{F}$-sort) and $\mathbb{L}_{\overline{\mathbb{F}}}$ the language of fields for the residue field sort ($\overline{\mathbb{F}}$-sort). For the value group sort, let $\mathbb{L}_{\Gamma}$ be the language of ordered Abelian groups with an element ∞ on top given by

$$\mathbb{L}_{\Gamma\infty} = \{+, 0, 1, \infty, \leq\}.$$

Then the following is Pas's language $\mathcal{L}$:

$$\mathcal{L} = (\mathbb{L}_{\mathbb{F}}, \mathbb{L}_{\overline{\mathbb{F}}}, \mathbb{L}_{\Gamma}, \mathrm{val}, \overline{\mathrm{ac}}).$$

Note With $\mathbb{Z}$ as a structure for the value group, $\mathbb{Q}_p$ is a structure for the language $\mathcal{L}$. (See §2.1.2.)

Moreover, in the valued field sort, there are symbols 0 and 1, respectively, for the additive and multiplicative identities. Using these, we formally add symbols denoting other integers to this language.

Example 2.1 Let $P(t)$ be a formula in Pas's language with t as a free variable, and $P(-1)$ is the abbreviation for

$$\exists x\, P(x) \wedge (x + 1 = 0).$$

The language contains symbols for existential ($\exists$) and universal ($\forall$) quantifiers for each sort. In particular, we have six symbols;

$$\forall_{\mathbb{F}}, \quad \forall_{\overline{\mathbb{F}}}, \quad \forall_{\Gamma}, \quad \exists_{\mathbb{F}}, \quad \exists_{\overline{\mathbb{F}}}, \quad \exists_{\Gamma}.$$

Whether the quantifiers range over the field sort, the residue field sort or the value group sort will generally be clear from the context. If there is a possibility of confusion, we will attach the respective sort symbol to the quantifier as shown above. Once the *sort* of variable symbols used is clear, we will use them in a way that indicates that meaning.

Pas's language also has standard symbols for logical disjunction ($\vee$), conjunction ($\wedge$) and negation ($\neg$). In addition, we use the following standard logical abbreviations for implication ($\Rightarrow$), biconditional ($\Leftrightarrow$), and exclusive or ($\veebar$), respectively:

$$\phi \Rightarrow \psi \text{ for } \neg\phi \vee \psi, \quad \phi \Leftrightarrow \psi \text{ for } (\phi \Rightarrow \psi) \ \wedge \ (\psi \Rightarrow \phi), \quad \phi \veebar \psi \text{ for } \neg(\phi \Leftrightarrow \psi).$$

The restriction of Pas's language to the residue field sort coincides with the first order language of rings. [4]

Pas's language includes a function symbol "val" for the valuation map from the valued field to the value group and another function symbol for an angular component "$\overline{\text{ac}}$" from the valued field to the residue field. We will explain the role of these symbols in the next section after we introduce structures for this language.

2.1 Pas's Structures

We make a distinction between the variable symbols used and their interpretation. Here we discuss structures (in the model theoretic sense) for Pas's language $\mathcal{L}$ [10]. We will state explicitly the conditions on these structures.

2.1.1 Conditions on Pas's Structures

Definition 2.2 An SPL is a structure **R** for Pas's language that consists of the following:

- A structure for the field sort $(\mathbb{F}, +_{\mathbb{F}}, -_{\mathbb{F}}, \cdot_{\mathbb{F}}, 0_{\mathbb{F}}, 1_{\mathbb{F}})$, where $\mathbb{F}$ is the domain for the field sort, and $\mathbb{F}$ is assumed to be a valued field of characteristic 0.
- A structure for the residue field sort $(\overline{\mathbb{F}}, +_{\overline{\mathbb{F}}}, -_{\overline{\mathbb{F}}}, \cdot_{\overline{\mathbb{F}}}, 0_{\overline{\mathbb{F}}}, 1_{\overline{\mathbb{F}}})$, where $\overline{\mathbb{F}}$ is assumed to be a finite field.

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- For the value group sort: $(\mathbb{Z}, +, 0, \infty, \geq)$.
- The valuation function, val, on $\mathbb{F}$. (See §2.1.3.)
- An angular component map, $\overline{\mathrm{ac}}$, on $\mathbb{F}$. (See §2.1.3.)

Remark 2.3 We mention in passing that in his paper [10], Pas placed an additional condition that $\mathbb{F}$ be Henselian. It was required for the quantifier elimination proved in that paper. This condition is not used in this paper.

Let $\mathbf{R}$ be the domain of the structure. A structure with domain $\mathbf{R}$ attaches a set $A(\mathbf{R})$ to every virtual set A and an interpretation $\theta^{\mathbf{R}}$ to every formula θ. (See 2.2.)

Since the three sorts of this language are fields, finite fields and Abelian groups, respectively, the language comes equipped with field and group axioms. Thus we have the theories of fields and Abelian groups at our disposal. In the following sections we prove some theorems where we will need to make use of the theory of fields. We use the notation (even though R is a structure and not a model)

$$R \models \phi$$

to indicate that a formula ϕ in Pas's language holds in SPL $\mathbf{R}$. An example of an SPL is a p-adic field.

2.1.2 p-Adic Fields

Definition 2.4 Let $\mathbb{Q}$ denote the field of rational numbers and p a prime integer. Then the *p-adic norm* $|\cdot|_p$ is defined as follows: Given $x \in \mathbb{Q}^{\times}$, there is a unique $r, m, n \in \mathbb{Z}$ such that $(m, n) = 1$ and $p \not| m$, $p \not| n$ and $x = p^r \frac{m}{n}$. Then $|x|_p = p^{-r}$. Set $|0|_p = 0$.

Definition 2.5 The completion of $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$ is denoted by $\mathbb{Q}_p$, and $\mathbb{Q}_p$ is called a p-adic field.

Thus, any non-zero element of $\mathbb{Q}_p$ can be written as a power series in p.

Example 2.6 In $\mathbb{Q}_5$, $37 = 2 \times 5^0 + +2 \times 5^1 + 1 \times 5^2$.

Note Any finite extension of $\mathbb{Q}_p$ is also called a p-adic field.

2.1.3 Function Symbols: $\overline{\mathrm{ac}}$ and val

Here we explain the role played by the function symbols $\overline{\mathrm{ac}}$ and val.

Let $\mathbb{F}$ be a valued field with valuation $\mathrm{val}\colon \mathbb{F} \to \mathbb{Z} \cup \{\infty\}$. We write

$$\mathfrak{o} = \{x \in \mathbb{F} / \ \mathrm{val}(x) \geq 0\} \quad \text{and} \quad \mathfrak{p} = \{x \in \mathbb{F} / \ \mathrm{val}(x) > 0\}$$

for the valuation ring and valuation (maximal) ideal, respectively. Denote the residue field $\mathfrak{o}/\mathfrak{p}$ by $\overline{\mathbb{F}}$. The set of units of $\mathfrak{o}$ is denoted by $\mathfrak{u}$, *i.e.*, $\mathfrak{u} = \{x \in \mathfrak{o} \mid \mathrm{val}(x) = 0\}$.

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Definition 2.7 An *angular component* map modulo $\mathfrak{p}$ on $\mathbb{F}$ is a map

$$\overline{\mathrm{ac}}\colon \mathbb{F} \to \overline{\mathbb{F}}$$
$$x \mapsto \overline{\mathrm{ac}}(x)$$

such that

- $\overline{\mathrm{ac}}(0) = 0$;
- the restriction of $\overline{\mathrm{ac}}$ to $\mathbb{F}^*$ is a multiplicative morphism from $\mathbb{F}^*$ to $\overline{\mathbb{F}}^*$;
- the restriction of $\overline{\mathrm{ac}}$ to $\mathfrak{u}$ coincides with the canonical projection from $\mathfrak{o}$ to $\mathfrak{p}$.

To illustrate how the functions val and $\overline{\mathrm{ac}}$ work, consider the following example.

Example 2.8 Let $\mathbb{F}$ be the field $\mathbb{Q}_5$. Recall that every non-zero element in $\mathbb{Q}_5$ is of the form $\sum_{i=N}^{\infty} a_i 5^i$, where N is an integer, $a_i \in \{0, 1, 2, 3, 4\}$ and $a_N \neq 0$. Then $\mathrm{val}(\sum_{i=N}^{\infty} a_i t^i) = N$, $\overline{\mathrm{ac}}(\sum_{i=N}^{\infty} a_i t^i) = a_N$. So from Example 2.6 we have $\mathrm{val}(37) = 0$ and $\overline{\mathrm{ac}}(37) = 2$.

This language is highly restrictive, with no notion of sets. More specifically, the set membership predicate $\in$ is absent. We introduce *virtual sets* into the language by means of various logical formulae. The notion of virtual sets is similar to what Quine [11] refers to as *virtual classes*.[1]

2.2 Virtual Sets

A virtual set is a *construct* of the form $\{x : \phi(x)\}$, where ϕ is a formula in Pas's language with free variables $x_1, x_2, \ldots, x_n$ and x is a multi-variable symbol

$$x = (x_1, x_2, \ldots, x_n).$$

In this case, we say that the variable symbol x has *length* n. We write

$$y \in \{x : \phi(x)\} \quad \text{for } \phi(y). \tag{1}$$

Thus a serviceable "$\in$" of ostensible class membership can be introduced as a purely notational adjunct [12]. The whole combination $y \in \{x : \phi(x)\}$ reduces to $\phi(y)$, so there remains no trace of the existence of a class $\{x : \phi(x)\}$. We could rephrase $y \in \{x : \phi(x)\}$ by $(\exists x)\big((x = y) \wedge \phi(x)\big)$, but we prefer to view $\in$ and class abstraction as fragments of the entire combination of (1).

When we write $x \in V$, we mean $V(x)$. (This is an extension of the notation $\phi(x)$.) It is also to be understood that the length of x is the same as the number of free variables used in the formula defining V.

The virtual set theory shares some aspects of set theory. We note that the usual set operations union, intersection and a notion of subset are present. If A and B are virtual sets defined by formulae $\phi(x)$ and $\psi(x)$, respectively, then

[1] Quine states, "…classes are freed of any deceptive hint of tangibility, there is no reason to distinguish them from *properties*. It matters little whether we read $x \in y$ as 'x is a member of the class y' or 'x has the property y'" [11, p. 120].

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- $A \cup B$ is a virtual set defined by $\{x : \phi(x) \vee \psi(x)\}$.
- $A \cap B$ is a virtual set defined by $\{x : \phi(x) \wedge \psi(x)\}$.
- We say that A is a subset of B and denote it by $A \subset B$, where $A \subset B$ is an abbreviation of the formula $\forall x(\phi(x) \Rightarrow \psi(x))$. Since x is a multi-variable symbol, $\forall x$ is a quantified n-tuple.

Note Although a set can be a member of another set, a virtual set cannot be a member of another virtual set. Thus $A \in B$ is not permissible.

Here are two examples of virtual sets:

- The ring of integers $\mathfrak{o}$ of any valued field is a virtual set defined thus:

$$\{x \in \mathbb{F} : \mathrm{val}(x) \geq 0\};$$

- The maximal ideal $\mathfrak{p}$ in $\mathfrak{o}$ is a virtual set defined thus:

$$\{x \in \mathbb{F} : \mathrm{val}(x) > 0\}.$$

We conclude this section with one more definition. Let $\Psi(x, y)$ be a formula with free variables $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. We define a *virtual set with parameters y* by

$$u \in \{x : \Psi(x, y)\} \quad \text{for } \Psi(u, y),$$

where $u = (u_1, u_2, \dots, u_n)$

One should note that the quantifiers are *not* allowed to range over virtual sets. Hence, there is no such expression as $\forall V$ where V is a virtual set.

Remark 2.9 In Section 3.2 we prove some facts in linear algebra using virtual sets defined by formulae in Pas's language. Many of the proofs are classical, and at times, instead of giving the entire proof, we say " …the rest of the proof is classical." However, caution must be exercised in making such statements. It may not always be possible to lift proofs from classical mathematics and fit them into Pas's language. Virtual set theory is more restrictive than set theory. Concepts and objects of set theory may not always have virtual set analogues. For example, the empty set has no virtual counterpart. Since our quantifiers do not range over formulae, there is no effective way to define an empty set.

Remark 2.10 In the most recent version of motivic integration, Cluckers and Loeser[2] avoid some of the aforementioned difficulties by using a category-theoretic construct called *definable subassignments*. Their setting admits a good dimension theory and makes a general integration version possible.

[2] R. Cluckers and F. Loeser, *Constructible motivic functions and motivic integration.* In preparation.

3 A List of Formulae in Pas's Language

3.1 Introduction

We wish to speak about linear algebra in this language, so we will start with vectors. By a *vector* x we mean an n-tuple $(x_1, x_2, \dots, x_n)$ where the x_i are variable symbols of either the valued field sort or the residue field sort. Hence, when we say $\forall x (x \in V)$ we really mean $\forall x_1, \forall x_2, \dots, \forall x_n ((x_1, x_2, \dots, x_n) \in V)$,

Example 3.1 If x is an n-tuple $(x_1, x_2, \dots, x_n)$ of variables of the field sort and y is an n-tuple $(y_1, y_2, \dots, y_n)$ of variables of the field sort as well, then $x + y$, too, is an n-tuple $(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$.

Example 3.2 If the context is an $n \times n$ matrix, we will use variable symbols x_{ij}, y_{ij}, and so forth rather than labelling the n^2 entries in a sequence $x_1, x_2, \dots, x_{n^2}$.

Example 3.3 If X is an $n \times n$ matrix (x_{ij}) of variable symbols of the (say) valued field sort, then $\exists_{\mathbb{F}} X$ is an abbreviation of $\exists_{\mathbb{F}} x_{11}, \exists_{\mathbb{F}} x_{12}, \dots, \exists_{\mathbb{F}} x_{nn}$.

And finally, we define an operation on matrices.

Definition 3.4 If A is an $n \times n$ matrix (a_{ij}) of variable symbols of the valued field sort and B is an $m \times m$ matrix (b_{ij}) of variable symbols of the valued field sort, then $A \oplus B$ is an $(n+m) \times (n+m)$ matrix $(a \oplus b)_{ij}$ where

$$(a \oplus b)_{ij} = \begin{cases} a_{ij} & \text{if } 1 \leq i, j \leq n, \\ b_{i-n\, j-n} & \text{if } n+1 \leq i, j \leq n+m, \\ 0 & \text{otherwise.} \end{cases}$$

The following section gives a long list of formulae. While the list seems tedious, it contains formulae for all the objects needed in the proof of our main result. We hope that this will allow us to present a short and clean proof.

3.2 Formulae

Formula 1. If V is a non-empty virtual set, let $\mathrm{Lin}(V)$ be the formula:

$$\underline{0} \in V \ \wedge\ \forall \lambda_1, \forall \lambda_2 \forall x_1, \forall x_2\ (x_1, x_2 \in V \Rightarrow \lambda_1 x_1 + \lambda_2 x_2 \in V).$$

Here λ_1 and λ_2 are variable-symbols of the valued field sort (or residue field sort) and x_1 and x_2 are vectors of variable-symbols of the valued field sort (or residue field sort). We use $\mathrm{Lin}(V)$ to define a *virtual vector space* over the valued field (or the residue field, respectively).

Definition 3.5 Let T_R be the theory consisting of sentences that are true for all SPL **R**. If $T_R \models \mathrm{Lin}(V)$, then we say that V is a virtual vector space.

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(In the first order language of rings, our structure would be a ring. In that case, $\mathrm{Lin}(V)$ would assert that V is a module.)

Formula 2. Let $\text{Lin-ind}(e_1, \dots, e_n, V)$ be the formula

$$\forall \lambda_1, \lambda_2, \dots, \lambda_n \Big(\sum_{i=1}^{n} \lambda_i e_i = 0 \Big) \Rightarrow (\lambda_1 = \cdots = \lambda_n = 0) \\ \wedge\ V(e_1)\ \wedge\ V(e_2)\ \wedge\ \dots\ \wedge\ V(e_n).$$

This formula asserts the linear independence of vectors $e_1, e_2, \dots, e_n$ in V, where V is a virtual set with M free variables and e_i are vectors of length M each consisting of terms.

Formula 3. Let $\text{Lin-comb}(e_1, e_2, \dots e_m, u)$ be the formula

$$\exists \lambda_1, \dots, \lambda_m \ (u = \sum_{i=1}^{m} \lambda_i e_i).$$

This formula states that u is a linear combination of $e_1, e_2, \dots, e_m$.

Formula 4. Let $\mathrm{Span}(e_1, e_2, \dots, e_m, V)$ be the formula

$$\forall v \big(V(v) \Leftrightarrow \text{Lin-comb}(e_1, e_2, \dots e_m, v) \big) .$$

This states that V is the span of vectors $(e_1, e_2, \dots, e_m)$.

Formula 5. Let $\mathrm{Basis}(e_1, e_2, \dots, e_m, V)$ be the formula

$$\text{Lin-ind}(e_1, e_2, \dots, e_m, V)\ \wedge\ \mathrm{Span}(e_1, e_2, \dots, e_m, V).$$

This formula states that $(e_1, e_2, \dots, e_m)$ is a basis for V.

Formula 6. For m, a fixed natural number, let $\mathrm{Dim}(m, V)$ be the formula

$$\exists e_1, e_2, \dots, e_m \,\mathrm{Basis}(e_1, e_2, \dots, e_m, V).$$

We wish to point out that here m is not a variable in Pas's language.

Formula 7. At times we will need to say that a vector space has odd (respectively even) dimension. We will be dealing with only finite dimensional vector spaces so, *a priori*, there will be an upper bound n on the dimension.

- Let $\text{Odd-Dim}(n, V)$ be the formula

  $$\mathrm{Dim}(1, V)\ \vee \mathrm{Dim}(3, V) \vee \cdots \vee \mathrm{Dim}(2k-1, V) \text{ where } n-1 \le 2k-1 \le n.$$

  The formula asserts that the virtual set V is a vector space of odd dimension that is less than or equal to n.

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- Let Even-Dim(n, V) be the formula

$$\mathrm{Dim}(0, V) \vee \mathrm{Dim}(2, V) \vee \cdots \vee \mathrm{Dim}(2k, V) \text{ where } n-1 \le 2k \le n.$$

Formula 8. Let Int-comb$(e_1, e_2, \dots e_m, u)$ be the formula

$$\exists \lambda_1 \dots \lambda_m\ (\mathrm{val}(\lambda_i) \ge 0) \ \wedge\ u = \sum_{i=1}^{m} \lambda_i e_i.$$

This formula asserts that u is an integral combination of vectors $(e_1, e_2, \dots, e_m)$.

Formula 9. Let Int-basis$(e_1, \dots, e_n, L)$ be the formula

$$\text{Lin-ind}(e_1, \dots, e_n) \ \wedge\ (\forall w \in L\, \text{Int-comb}(e_1, \dots, e_n, w)).$$

Formula 10. Let $V = U \oplus W$ be the formula

$$(W \subset V) \ \wedge\ (U \subset V) \ \wedge\ (U \cap W = \underline{0}) \ \wedge\ \big(\forall v \in V (\exists w \in W, u \in U\ (v = u + w))\big).$$

This formula states that V is the direct sum of U and W. (The lack of conditions on U, W and V is intentional. This decomposition allows us to talk about direct sums of lattices, vector spaces or modules.)

Formula 11. Let Q-space$(U, V/W)$ be the formula

$$\mathrm{Lin}(V) \ \wedge\ \mathrm{Lin}(W) \ \wedge\ \mathrm{Lin}(U) \ \wedge\ V = U \oplus W.$$

Observe that the quotient of a vector space by a subspace is identified with its complement in the decomposition.

Remark 3.6 Henceforth, objects defined on quotient spaces will be identified with objects on complements.

Formula 12. Let Q-Basis$(e_1, \dots, e_n, U, V/W)$ be the formula

$$\text{Q-space}(U, V/W) \ \wedge\ \mathrm{Basis}(e_1, e_2, \dots, e_n, U).$$

This says that the vectors $e_1, \dots, e_n$ form a basis for the quotient space $V/W = U$.

Formula 13. A *lattice* in a linear space V is an integral-span of a basis of V. Let Lattice(L, V) be the formula

$$\mathrm{Lin}(V) \ \wedge\ (L \subset V) \ \wedge\ \exists e_1, \dots, e_n \\ \big(\mathrm{Basis}(e_1, \dots, e_n, V) \ \wedge\ \forall w\, (w \in L \Longleftrightarrow \text{Int-comb}(e_1, \dots, e_n, w))\big).$$

This asserts that the virtual set L is a lattice in V.

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Formula 14. Let lattice$(e_1, e_2, \dots, e_m)$ be the virtual set:

$$\{u \mid \exists_{\mathbb{F}} \alpha_1, \dots, \alpha_m \operatorname{val}(\alpha_i) \geq 0 (u = \sum_{i=1}^{m} \alpha_i e_i)\}.$$

Remark 3.7 What is the difference between this formula and the earlier one? In the previous formula we assert that L, a known virtual set, is a lattice; whereas, in this formula we construct a lattice. It seems as though we are splitting hairs here, but we are not. This allows us to use (say) L as an abbreviation for a virtual set. The formula $L = \text{lattice}(e_1, e_2, \dots, e_m)$ will thus mean "Label this particular virtual set as L."

Formula 15. Similarly, let vectorspace$(e_1, e_2, \dots, e_m)$ be the virtual set

$$\{u \mid \exists_{\mathbb{F}} \alpha_1, \dots, \alpha_m (u = \sum_{i=1}^{m} \alpha_i e_i)\}.$$

Formula 16. Let L, $\tilde{L}$ and V be virtual sets. Let J be an M by M matrix of terms, where M is the number of free variables in V. The formula Dual-lattice$(L, \tilde{L}, J, V)$ is

$$\text{Lattice}(L, V) \ \wedge \ (\tilde{L} \subset V) \ \wedge \\ \forall w \in V \big(w \in \tilde{L} \iff \big(\forall v (v \in L \Rightarrow \operatorname{val}({}^t v J w) \ \geq 0) \big) \big) \, .$$

This asserts that $\tilde{L}$ is the dual of lattice L with respect to matrix J.

Formula 17. Let sym-bil-nd(J, V) denote the formula

$$\exists e_1, \dots, e_n \big(\text{Lin}(V) \ \wedge \ \text{Basis}(e_1, \dots, e_n, V) \ \wedge \ \det(A) \neq 0 \ \wedge \ (A_{ij} = A_{ji}) \big) \, ,$$

where $A_{ij} = {}^t e_i J e_j$. Here, e_i are vectors of variable symbols of length M, and J is an M by M matrix of terms, where M is the number of free variables in V.

Lemma 3.8 *Under these definitions, a* dual-lattice *is a lattice. More precisely, if R is an* SPL*, then*

$$R \models \text{sym-bil-nd}(J, V) \Longrightarrow \big(\text{Dual-lattice}(L, \tilde{L}, J, V) \Rightarrow \text{lattice}(\tilde{L}, V) \big) \, ,$$

where J is an $M \times M$ matrix of terms, V is a virtual set with M free variables, L is a virtual set with M free variables, and $\tilde{L}$ is a virtual set with M free variables.

Proof Let R be an SPL. Then

$$R \models \text{sym-bil-nd}(J, V) \\ \Longrightarrow \exists e_1, \dots, e_n \, \text{Lin}(V) \ \wedge \ \det(A) \neq 0 \ \wedge \ A_{ij} = A_{ji} 1 \leq i \leq n, 1 \leq j \leq n,$$

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where $A_{ij} = {}^t e_i J e_j$. The proof is constructive in the sense that using the basis $\{e_1, \dots, e_n\}$ for lattice L, we will produce a basis $\{e'_1, \dots, e'_n\}$ such that $\tilde{L}$ is a lattice with respect to this basis. In other words, we will show that $R \models \exists e'_1, \dots, e'_n$ such that $\text{Basis}(e'_1, \dots, e'_n, V) \ \wedge \ (\forall w(w \in \tilde{L} \Leftrightarrow \text{Int-comb}(e'_1, \dots, e'_n, w))$. Refer to Formula 13.

Define e'_i as follows.

$$(2) \qquad e'_i = \sum_{j=1}^{M} \alpha_{ij} e_j \quad \text{such that } {}^t e'_i J e_j = \delta_{ij}.$$

We need to show that

$$\text{Basis}(e'_1, e'_2, \dots, e'_n, V) \ \wedge \ \forall w \in V(w \in \tilde{L} \Leftrightarrow \text{Int-comb}(e'_1, \dots, e'_n, w)).$$

To say that these e'_i's exist and are unique is equivalent to saying that the α_{ij}'s exist and are unique.

For each i, (2) gives a system of n linear equations in n variables. Since $A_{ij} = {}^t e_i J e_j$ is a square non-degenerate matrix (*i.e.*, $\det(A_{ij}) \neq 0$), the α_{ij}'s exist and are unique. Thus the e'_i's are uniquely defined and form a basis of V. The rest of the proof is classical. ■

Formula 18. A lattice L is said to be *almost self dual* if the following hold:

$$\mathfrak{p}\tilde{L} \subset L \subset \tilde{L}.$$

While $A \subset B$ is a formula in Pas's language, a comment is needed on the meaning of $\mathfrak{p}\tilde{L}$. It is the following virtual set: $\{v \in V : \exists \alpha \in \mathfrak{p}, \exists w \in \tilde{L}(v = \alpha w)\}$. Let $\text{ASD}(L, J, V)$ be the following formula:

$$\text{Lin}(V) \ \wedge \ \text{lattice}(L, V) \ \wedge \ \text{Dual-lattice}(\tilde{L}, L, V, J) \ \wedge \ (L \subset \tilde{L}) \ \wedge \ (\mathfrak{p}\tilde{L} \subset L).$$

Formula 19. We will need a formula for lattices of quotient spaces. Recalling that we identify quotients of vector spaces with orthogonal complements, we will let $\text{Q-Lattice}(L, U, V/W)$ denote the formula

$$\text{Q-space}(U, V/W) \ \wedge \ \text{lattice}(L, U).$$

Formula 20. Let $\text{Gram}_{ij}(e_1, e_2, \dots, e_m, J)$ be the entry ${}^t e_i J e_j$. Here J is an $M \times M$ matrix of terms.

Formula 21. Let $\text{Gram-det}(e_1, e_2, \dots, e_m, J)$ be the determinant of matrix $({}^t e_i J e_j)$.

Formula 22. Let $\Theta(sq, J, V)$ be the formula

$$\forall e_1, \dots, e_n$$
$$\big(\text{Basis}(e_1, \dots, e_n, V) \ \Rightarrow \big(\exists \xi \in \overline{\mathbb{F}} \ \xi \neq 0 \ \wedge \ \xi^2 = \text{ac}(\text{Gram-det}(e_1, \dots, e_n, J))\big)\big).$$

This states that the Gram-determinant of the quadratic form on V, given by matrix J is a square class in the residue field.

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Formula 23. Let $\Theta(\text{nsq}, J, V)$ be the formula

$$\forall e_1, \dots, e_n$$
$$\big(\text{Basis}(e_1, \dots, e_n, V) \Rightarrow \big(\nexists \xi \in \overline{\mathbb{F}}\ \xi \neq 0 \ \wedge\ \xi^2 = \text{ac}(\text{Gram-det}(e_1, \dots, e_n, J))\big)\big).$$

This states that the Gram-determinant of the quadratic form on V given by matrix J is a non-square class in the residue field.

Formula 24. Let Q-dim(L_1, L_2, V, k) be the formula

$$(L_1 \subset L_2) \ \wedge\ \text{Lin}(V) \ \wedge\ \text{lattice}(L_1, V) \ \wedge\ \text{lattice}(L_2, V)$$
$$\wedge\ \exists e_1, \dots, e_n \in V \big(\forall_{\mathbb{F}} \alpha \ \text{val}(\alpha) = 1(\text{Int-basis}(e_1, \dots, e_n, L_2))$$
$$\Rightarrow \text{Int-basis}(\alpha e_1, \dots, \alpha e_k, e_{k+1}, \dots, e_n, L_1)\big).$$

This formula asserts that the dimension of the vectorspace L_2/L_1 (over the residue field) is k.

Formula 25. As in Formula 7, we will write formulae stating that the dimension of the aforesaid quotient is odd (respectively even).

- Let Odd-Qdim(n, L_1, L_2, V) be the formula
$$\text{Q-dim}(L_1, L_2, V, 1) \vee \text{Q-dim}(L_1, L_2, V, 3) \vee \cdots \vee \text{Q-dim}(L_1, L_2, V, 2k-1),$$
where $n - 1 \leq 2k - 1 \leq n$.
- Let Even-Qdim(n, L_1, L_2, V) be the formula
$$\text{Q-dim}(L_1, L_2, V, 0) \vee \text{Q-dim}(L_1, L_2, V, 2) \vee \cdots \vee \text{Q-dim}(L_1, L_2, V, 2k),$$
where $n - 1 \leq 2k \leq n$.

Formula 26. Let Anisotropic$(e_1, e_2, \dots, e_m, J, V)$ be the formula

$$\text{Lin}(V) \ \wedge\ \text{Lin-ind}(e_1, e_2, \dots, e_m, V)$$
$$\wedge\ \forall \lambda_1, \dots, \lambda_m \Big({}^t\Big(\sum_{i=1}^{m} \lambda_i e_i\Big) J \Big(\sum_{i=1}^{m} \lambda_i e_i\Big) = 0\Big) \Rightarrow (\lambda_1 = \cdots = \lambda_m = 0).$$

Recall that in Lin-ind$(e_1, e_2, \dots, e_m, V)$, the e_i are vectors of terms, V is a virtual set with M free variables, and J is an $M \times M$ matrix of terms. This formula states that if V is a vector space and if J is the matrix of a quadratic form on V, then the linearly independent vectors $\{e_1, \dots, e_m\}$ span a subspace of the anisotropic kernel of V.

Formula 27. Let Dim-aniso(m, J, V) be the formula

$$\exists e_1, e_2, \dots, e_m \ \text{Anisotropic}(e_1, e_2, \dots, e_m, J, V)$$
$$\wedge\ \nexists e_1, e_2, \dots, e_{m+1} \ \text{Anisotropic}(e_1, e_2, \dots, e_{m+1}, J, V).$$

This asserts that m is the dimension of the anisotropic kernel of V.

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Formula 28. Let Iso-aniso(V, J_V, W, J_W) be the formula

$$\exists e_{v_1}, \ldots, e_{v_m}, e_{w_1}, \ldots, e_{w_m} \big(\text{Anisotropic}(e_{v_1}, \ldots, e_{v_m}, J_V, V)$$
$$\wedge \ \text{Anisotropic}(e_{w_1}, \ldots, e_{w_m}, J_W, W) \ \wedge \ {}^t e_{v_i} J_V e_{v_i} = {}^t e_{w_i} J_W e_{w_i} \forall i 1 \leq i \leq m \big)$$
$$\wedge \ \nexists e_{v_1}, \ldots, e_{v_m}, e_{v_{m+1}}, e_{w_1}, \ldots, e_{w_m}, e_{w_{m+1}} \text{Anisotropic}(e_{v_1}, \ldots, e_{v_{m+1}}, J_V, V)$$
$$\wedge \ \text{Anisotropic}(e_{w_1}, \ldots, e_{w_{m+1}}, J_W, W).$$

This formula asserts that the vector spaces have isomorphic anisotropic kernels under their respective quadratic forms.

Formula 29. Now we would like to be able to talk about direct sums of vector spaces formed by annexing 2 arbitrary vector spaces. Let e be a vector of terms of length n. Let f be a vector of terms of length m. We construct a vector of terms of length $n + m$ by concatenating e with f. We denote this by $e \oplus f$. Thus, if e is given by $(e_1, e_2, \ldots, e_n)$ and f by $(f_1, f_2, \ldots, f_m)$, then $e \oplus f$ is given by

$$(e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_m),$$

where the free variables e_i are distinct from the free variables f_j.

Moreover, if e and h have length n and f and k have length m, then define

$$(e \oplus f) + (h \oplus k) := (e + h) \oplus (f + k).$$

Let Dir-sum(V, W, U) denote the formula

$$\text{Lin}(V) \ \wedge \ \text{Lin}(W) \ \wedge \ \big(\forall f (f \in U) \Longleftrightarrow \big(\exists f_v \in V \, \exists f_w \in W (f = f_v \oplus f_w)\big)\big).$$

Lemma 3.9 *The direct sum of two vector spaces is a vector space. More precisely, let R be an* SPL. *Then* $R \models \text{Dir-sum}(V, W, U) \Rightarrow \text{Lin}(U)$.

Proof Now the symbol $\lambda(e \oplus f)$ will denote a vector of terms of length $n + m$ where the first n terms are that of the vector λe (scalar multiplication by the field constant λ) and the remaining n terms are those of the vector λf.

$$\begin{aligned} R \models \quad & \forall f \forall e (f \in U \ \wedge \ e \in U \ \wedge \ \text{Dir-sum}(V, W, U)) \\ & \Rightarrow \big((\exists f_v \exists f_w f_v \in V, f_w \in W (f = f_v \oplus f_w)) \\ & \qquad \wedge \ (\exists e_v \exists e_w e_v \in V, e_w \in W (e = e_v \oplus e_w)) \big) \\ & \Rightarrow \forall \lambda_1 \forall \lambda_2 \Big(\lambda_1 f + \lambda_2 e = \lambda_1 (f_v \oplus f_w) + \lambda_2 (e_v \oplus e_w) \\ & \qquad = \lambda_1 f_v \oplus \lambda_1 f_w + \lambda_2 e_v \oplus \lambda_2 e_w \\ & \qquad = (\lambda_1 f_v + \lambda_2 e_v) \oplus (\lambda_1 f_w + \lambda_2 e_w) \Big), \end{aligned}$$

$$\text{Lin}(V) \Rightarrow \lambda_1 f_v + \lambda_2 e_v \in V,$$
$$\text{Lin}(W) \Rightarrow \lambda_1 f_w + \lambda_2 e_w \in W \Rightarrow \lambda_1 f + \lambda_2 e \in U \Rightarrow \text{Lin}(U).$$

■

4 Nilpotent Orbits in p-Adic Lie Algebras: The Odd Orthogonal Case

We follow closely Waldspurger's treatment of the parametrization of nilpotent orbits in the classical p-adic Lie algebras [17]. Since it is essential for our purpose, Section 4.1 is nearly a verbatim quote from [17, I.5, I.6, I.7].

4.1 Parametrization of Nilpotent Orbits

Let $\mathbb{F}$ be a p-adic field with $\mathbb{F}_q$ as its residue field. Let $\mathfrak{g}$ be the Lie Algebra $\mathfrak{so}(r)$ with r odd, and let $X \in \mathfrak{g}$ be a nilpotent element. The following discussion is restricted to the odd orthogonal case.

Let (V, q_V) be the underlying vector space of $\mathfrak{g}$ with the q_V as the quadratic form in the definition of $\mathfrak{g}$. Let the set of partitions of r be denoted by $P(r)$. Now consider the subset of $P(r)$ consisting of partitions $\Lambda = (\lambda_j)$ of r with the following property.

$(*)$ In the orthogonal case, for any even $i \geq 2$, $c_i(\Lambda)$ is even, where $c_i(\Lambda)$ denotes the number of λ_j that equal i.

We can associate with X a partition Λ of r: for all integers $i \geq 1$, $c_i(\Lambda)$ is the number of Jordan blocks of X of length i in the natural matrix representation. This partition automatically satisfies the above condition. Such a Λ will be our first parameter for the conjugacy class of X. The remaining parameters are given by the following construction:

(3) $$V_i = \ker(X^i)/[\ker(X^{i-1}) + X\ker(X^{i+1})],$$

for all $i \geq 1$, i odd.

Define the quadratic form $\tilde{q}_i$ on $\ker(X^i)$, for all odd i by

(4) $$\tilde{q}_i(v, v') = (-1)^{[\frac{i-1}{2}]} q_V(X^{i-1}(v), v')$$

where $[\,\cdot\,]$ denotes the integer part of the fraction. (We ignore even values of i, since they do not enter the parametrization in the orthogonal case.)

Passing to a quotient, we get a non-degenerate form q_i on V_i. Moreover, in the orthogonal case, the forms q_i satisfy the condition [See 4.2]:

$$\bigoplus_{i \text{ odd}} q_i \sim_a q_V.$$

The relation $\sim_a$ indicates that the two forms have the same anisotropic kernel.

The family $(\Lambda, (q_i))$ parameterizes the conjugacy class of X. In turn, the set $\{(\Lambda, (q_i))\}$ parameterizes the nilpotent conjugacy classes, where Λ is a partition of r satisfying $(*)$.

Now we need invariants for the isomorphism class of (V_i, q_i). In the orthogonal case, these invariants are d_i (the dimension of V_i) and the quantities [17, I.3]

$$(d_i', d_i'', \eta_i', \eta_i'') \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{F}_q^*/\mathbb{F}_q^{*2})^2.$$

These quantities are defined as follows: for each odd i, start with a lattice L (*i.e.*, an $\mathfrak{o}_F$-module, where $\mathfrak{o}_F$ is the valuation ideal of $\mathbb{F}$) in V_i that generates V_i over $\mathbb{F}$. Define its dual thus:

$$\tilde{L} = \{v \in V_i : \forall w \in L, q_i(v, w) \in \mathfrak{o}_F\}. \tag{5}$$

A lattice is said to be *almost self-dual* if

$$\tilde{L} \supset L \supset \mathfrak{p}_F \tilde{L}. \tag{6}$$

Such L determines two vector spaces over $\mathbb{F}_q$:

$$l' = L/\mathfrak{p}_F\tilde{L}, \quad l'' = \tilde{L}/L. \tag{7}$$

Furthermore, they are equipped with quadratic forms that are of the same type as q_V with values in $\mathbb{F}_q$ defined by

$$q_{l'}(\overline{v}, \overline{w}) = \overline{q_i(v, w)} \quad \text{for } v, w \in L, \tag{8}$$

$$q_{l''}(\overline{v}, \overline{w}) = \overline{\varpi_F q_i(v, w)} \quad \text{for } v, w \in \tilde{L}, \tag{9}$$

where ϖ is any uniformiser of the valuation on $\mathbb{F}$.

Now we are in the realm of finite fields, and we can make use of the following fact.

- The isomorphism class of $(V', q_{V'})$, defined over $\mathbb{F}_q$, is determined [15, IV.1.7] by the quantities $d(V') \in \mathbb{N}$ and $\eta(q_{V'}) \in \mathbb{F}_q^*/\mathbb{F}_q^{*2}$, where $\eta(q_{V'})$ is the image of $(-1)^{[\frac{d(V')}{2}]} \det(q_{V'})$ in $\mathbb{F}_q^*/\mathbb{F}_q^{*2}$.

Let $\eta_i' = \eta(q_{l'})$ and $\eta_i'' = \eta(q_{l''})$. The invariants of $(l', q_{l'})$ and $(l'', q_{l''})$ are $(d(l'), \eta_i')$ and $(d(l''), \eta_i'')$, respectively. In the orthogonal case, the anisotropic kernels of $q_{l'}$ and $q_{l''}$ do not depend on L. These kernels, together with dimension d_i, the dimension of V_i, determine the isomorphism class of (V_i, q_{V_i}). For the anisotropic kernel, we only need to worry about the reduction of the dimensions of these vectorspaces mod $2\mathbb{Z}$ [14, pp. 11–18]. Let d_i' (respectively d_i'') be the reduction of $d(l')$ (respectively $d(l'')$) in $\mathbb{Z}/2\mathbb{Z}$. They satisfy the condition $d_i' + d_i'' \equiv d_i \bmod 2\mathbb{Z}$.

We now state a theorem for the orthogonal case.

Theorem 4.1 (Waldspurger [17, I.3, I.6]) *Let $\mathbb{F}$ be a finite extension of the field $\mathbb{Q}_p$ with $\overline{\mathbb{F}}$ as its residue field. Let V be a vector space over $\mathbb{F}$ with* $\dim V = d$*, where d is odd and $p \geq 3d + 1$. Let $J = (J_{ij})$ where*

$$J_{ij} = \begin{cases} 1 & \textit{if } i + j = d + 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Let $\mathfrak{g}$ = Lie algebra (V, J). Let $\Sigma = (\Lambda, (d_i', d_i'', \eta_i', \eta_i''))$.

Then the set of nilpotent conjugacy classes are in bijection with the set $\{\Sigma\}$ and are denoted by N_Σ, where

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- *$\Lambda \in P(d)$ is a partition of d satisfying the condition $\forall i \in 2\mathbb{Z}\ c_i(\Lambda) \in 2\mathbb{Z}$.*
- *$\forall i \notin 2\mathbb{Z}$, if $c_i(\Lambda) \neq 0$, we have $d_i' \in \mathbb{Z}/2\mathbb{Z}$, $d_i'' \in \mathbb{Z}/2\mathbb{Z}$ and $d_i' + d_i'' \equiv c_i(\Lambda) \mod (2\mathbb{Z})$.*
- *$\eta_i' \in \{s, ns\}$ and $\eta_i'' \in \{s, ns\}$, where s and ns denote square classes and non-square classes in the field $\overline{\mathbb{F}}$, respectively.*

Furthermore, we have

$$(*, \Sigma) \qquad \bigoplus_{i \text{ odd}} q_i \sim_a q_V,$$

where the relation $\sim_a$ indicates that the two forms have the same anisotropic kernel.

Proof See Waldspurger [17]. ∎

4.2 The Relation $\bigoplus_{i \text{ odd}} q_i \sim_a q_V$

This is quoted verbatim from J. L. Waldspurger's personal notes. Let $H = F \times F$ and q_H be the quadratic form on H given by $q_H((x, y)(x', y')) = xy' + yx'$. Let V be a finite dimensional vectorspace on F equipped with a non-degenerate quadratic form. Then there exists an orthogonal decomposition

$$(V, q_V) = (V_a, q_{V_a}) \oplus (H, q_H) \oplus \cdots \oplus (H, q_H),$$

where q_{V_a} is anisotropic. The equivalence class of (V_a, q_{V_a}) is well determined.

Definition 4.2 We say that $(V, q_V) \sim_a (V', q_{V'})$ if $(V_a, q_{V_a}) \cong (V_a', q_{V_a'})$

Now suppose V is a finite dimensional space over $\mathbb{F}$ equipped with a nondegenerate quadratic form q_V. Let X be a nilpotent element of the orthogonal Lie algebra of (V, q_V). Then there exists an orthogonal decomposition

$$(V, q_V) \cong \bigoplus_{j \in J} (V_j, q_{V_j}),$$

such that each V_j is stable under X; denote the restriction of X on V_j by X_j.

Recall that the family $(\Lambda, (q_i))$ parameterizes the conjugacy class of X. For odd i, the form q_i is equivalent to $\sum_j a_j x_j^2$. For even i, the anisotropic kernel is zero, for odd i the anisotropic kernel is of the form ax^2 where a is a non-zero element of the field $\mathbb{F}$. Hence, q_v is $\sim_a$ to the form $\sum_j a_j x_j^2$ summed over the j in the orthogonal decomposition. This is nothing but the condition $\bigoplus_{i \text{ odd}} q_i \sim_a q_V$.

4.3 Definability of Nilpotent Conjugacy Classes in $\mathfrak{so}(n)$, n Odd: The Main Theorem

We will now show that the conjugacy classes parameterized by the set $\{\Sigma\}$ and the condition $(*, \Sigma)$ are definable. Recall that $\Sigma = (\Lambda, (d_i', d_i'', \eta_i', \eta_i''))$.

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4.3.1 A Brief Outline

What are we trying to do here? In the odd orthogonal case, the nilpotent conjugacy classes are uniquely parameterized by the family $(\Lambda, (V_i, q_i))$. (Refer to equations (3) and (4) and the subsequent comment in §4.1.) Each (V_i, q_i) is uniquely determined by the 4-tuple $(d_i', d_i'', \eta_i', \eta_i'')$, where

- $d_i' = \bar{0}$ (respectively $\bar{1}$) means that the dimension of the vectorspace $l_i' = L/\mathfrak{p}\tilde{L}$ (over the residue field) is even (respectively odd). In our case, L is any *almost self dual* on the quotient space V_i given by equation (3).
- $d_i'' = \bar{0}$ (respectively $\bar{1}$) means that the dimension of the vectorspace $l_i'' = \tilde{L}/L$ (over the residue field) is even (respectively odd).
- $\eta_i' =$ sq (respectively nsq) means that the Gram-determinant of the quadratic form on l_i' given by equation (8) is a square (respectively non-square) in the residue field.
- $\eta_i'' =$ sq (respectively nsq) means that the Gram-determinant of the quadratic form on l_i'' given by equation (9) is a square (respectively non-square) in the residue field.

In the proof, we fix $n = \mathrm{Dim}(V, \mathbb{F})$ and select a partition of n satisfying the condition $c_i(\Lambda) \in 2\mathbb{Z}$ for all $i \in 2\mathbb{Z}$. For each i such that $c_i(\Lambda) \neq 0$, select a 4-tuple for the parameters $(d_i', d_i'', \eta_i', \eta_i'')$ from the set $\{\bar{0}, \bar{1}\} \times \{\bar{0}, \bar{1}\} \times \{\text{sq}, \text{nsq}\} \times \{\text{sq}, \text{nsq}\}$. We claim that there is a formula in Pas's language for each of the aforementioned four statements and for the condition $(*, \Sigma)$. (This condition is satisfied by the quadratic forms and quotient spaces (V_i, q_i) and (V, q) considered here.)

Finally, the main claim is that the virtual set cut out by these formulae is either empty or a nilpotent conjugacy class. The definition of the parameters indicates that there are $2^4 = 16$ possible choices for the 4-tuple $(d_i', d_i'', \eta_i', \eta_i'')$. Some of these will be ruled out by the condition $d_i' + d_i'' \equiv c_i(\Lambda) (\bmod 2\mathbb{Z})$, but many options remain. It will be extremely cumbersome to write out all these options together. Hence, we will state as clearly as possible how they are to be pieced together instead of presenting a long formula containing concatenated conjunctions and disjunctions.

4.3.2 The Statement

Theorem 4.3

(i) *For* $\Sigma = (\Lambda, (d_i', d_i'', \eta_i', \eta_i''))$, $S_d = \{\Sigma\}$ *is a finite, field-independent set and there exists a formula* ϕ_Σ *in Pas's language for each* $\Sigma \in S_d$.

(ii) *(*,*Σ*) can be expressed by a formula* $\phi_{*,\Sigma}$ *in Pas's language.*

(iii) *Let* $\mathbb{F}$ *be a* p*-adic field (see* 2.1.2*) such that its residue field* $\overline{\mathbb{F}}$ *is finite. Let* V *be a virtual set such that* $\mathrm{Lin}(V) \wedge \mathrm{Dim}(d, V)$ *holds. Let* J *be a matrix of terms satisfying the condition* $J_{ij} = J_{ji}$*. Let* $\mathbf{g}$ *be the virtual set* $\{Y : {}^tYJ + YJ = 0\}$*. (Here* Y *is a matrix of terms of the valued field sort.) Then*

$$\{X \in \mathfrak{g} : \phi_\Sigma(X) \ \wedge \ \phi_{*,\Sigma}(X)\} \tag{10}$$

is either empty or a nilpotent conjugacy class in $\mathfrak{g}$*.*

(iv) *For each* $\mathbb{F}$*, every nilpotent class appears exactly once in this set.*

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Proof For convenience of notation, define

$$\tilde{P}(d) = \{\Lambda \in P(d) : \forall i \in 2\mathbb{Z}, c_i(\Lambda) \in 2\mathbb{Z} \ \wedge \ \forall i \notin 2\mathbb{Z}, c_i(\Lambda) \neq 0\}.$$

Step 1: Any integer d has a finite number of partitions and thus, only a finite number of them appear as Λ in the set $\{\Sigma\}$. The partitions depend only on d and not on the field.

Step 2: Let $\Lambda \in \tilde{P}(d)$. There is a unique J_Λ (the Jordan block matrix) associated with the partition Λ. Let $\mathrm{J}\Lambda(X)$ denote the formula:

$$(J\Lambda(X)) \qquad \exists (g_{ij})_{1\leq i,j\leq d} (g_{ij})X = J_\Lambda(g_{ij}) \ \wedge \ \det(g_{ij}) \neq 0.$$

This states that X is conjugate to J_Λ.

Now Λ is fixed for the rest of the proof.

Step 3: For each $i \notin 2\mathbb{Z}$ such that $c_i(\Lambda) \neq 0$, the following are virtual sets with a parameter X ranging over $n \times n$ matrices:

$$K_i := K_i(X) := \ker(X^i) := \{v \in V \mid X^i(v) = 0\}$$

for all $i \in 2\mathbb{Z}$ with $c_i(\Lambda) \neq 0$,

$$W_i := W_i(X) := \{y \in V \mid \Phi(y, X)\},$$

where $\Phi(y, X)$ is the formula

$$\exists y_1, y_2, u_2, (y = y_1 + y_2 \ \wedge \ X^{i-1}(y_1) = 0 \ \wedge \ X(u_2) = y_2 \ \wedge \ X^{i+1}(u_2) = 0).$$

The virtual set W_i replaces the space $[\ker(X^{i-1}) + X\ker(X^{i+1})]$ in Section 4.1.

Now i is fixed until the last step. Thus $c_i(\Lambda)$ is fixed; call it c_i.

Step 4: We need a formula for the set of elements in $\{\Sigma_d\}$ that correspond to $(d_i', d_i'', \eta_i', \eta_i'')$. This lengthy construction is divided into five substeps. To keep us on track, we will give appropriate parallel references to Waldspurger's treatment from Section 4.1. In the final formula, all the quantities will be bound by appropriate quantifiers.

Step 4a: First, we need to cut out a formula that gives an almost self-dual lattice in $V_i = K_i/W_i$. Note that we will use the labels V_i, K_i and W_i in the sense of formula 14 in Section 3.2. $(\text{Q-space}(V_i, K_i/W_i) \ \wedge \ \text{Basis}(e_{i_1}, \dots, e_{i_{c_i}}, V_i) \ \wedge \ \text{ASD}(L_i, {}^tX^{i-1}J, V_i))$ where $K_i = K_i(X)$, $V_i = \text{vectorspace}(e_{i_1}, \dots, e_{i_{c_i}})$ and $L_i = \text{lattice}(e_{i_1}, \dots, e_{i_{c_i}})$ Call this formula $\phi_i^{(1)}(X, e_{i_1}, \dots, e_{i_{c_i}})$.

Note The i refers to the fixed i and the superscript (1) refers to Step 4a.

Step 4b: Now we need to cut out a formula stating that the dimension of the quotient space l_i' is even, (odd) respectively. Recall that this number is bounded above by c_i. This would be: Even-Qdim$(c_i, \mathfrak{p}\tilde{L}_i, L_i, V_i)$, respectively Odd-Qdim$(c_i, \mathfrak{p}\tilde{L}_i, L_i, V_i)$), where $\tilde{L}_i = \{u \mid \mathrm{val}({}^te_{i_j}{}^tX^{i-1}Ju) \geq 0\}$ for $j = 1, \ldots, c_i$, and L_i, V_i are as in Step 4a. Call these formulae $\phi^{(2)}_{i,\epsilon}(X, e_{i_1}, \ldots, e_{i_{c_i}})$. Here ϵ refers to "odd" or "even".

Step 4c: Now suppose that the value selected (at random) for the parameter η_i' is sq (respectively nsq). This is given be the following formula: $\Theta(sq, {}^tX^{i-1}J, V_i)$ (respectively $\Theta(\mathrm{nsq}, {}^tX^{i-1}J, V_i)$). Piecing together one formula each from steps 4b and 4c gives the pair (d_i', η_i'). Call these formulae $\phi^{(3)}_{i,\epsilon}(X, e_{i_1}, \ldots, e_{i_{c_i}})$. Here ϵ refers to "square" or "non-square".

Now we need to construct formulae for the pair (d_i'', η_i''). Recall, d_i'' is the dimension of the vector space $l'' = \tilde{L}/L$ modulo $2\mathbb{Z}$. (See 4.1(7))

Step 4d: The formula for $d_i'' = \overline{0}$ (respectively $\overline{1}$) is Even-Qdim$(c_i, L_i, \tilde{L}_i, V_i)$ (respectively Odd-Qdim$(c_i, L_i, \tilde{L}_i, V_i)$), where V_i, L_i and $\tilde{L}_i$ are as above. Call these formulae $\phi^{(4)}_{i,\epsilon}(X)$. Here ϵ refers to "odd" or "even".

Step 4e: The formula for $\eta_i'' = \mathrm{sq}$ (respectively nsq) is given by:

$$\forall e_1', \ldots, e_{c_i}'(\mathrm{Basis}(e_1', \ldots, e_{c_i}', V_i) \Longrightarrow \exists \eta \in \mathfrak{o} \ \wedge \ \exists \xi \in \overline{\mathbb{F}}^*$$

such that

$$\mathrm{val}(\eta) = c_i + \mathrm{val}(\text{Gram-det}(e_1', \ldots, e_{c_i}', {}^tX^{i-1}J)$$
$$\wedge \ \xi^2 = \mathrm{ac}(\eta) \ \wedge \ \mathrm{ac}(\eta) = \mathrm{ac}(\text{Gram-det}(e_1', \ldots, e_{c_i}', {}^tX^{i-1}J))$$

where V_i is as above.

The formula for nsq follows similarly. Call these formulae $\phi^{(4)}_{i,\epsilon}(X)$. Here ϵ refers to "square" or "non-square". Piecing together one formula each from Steps 4d and 4e gives the pair (d_i'', η_i'').

Step 5: Finally, we show that the condition $(*, \Sigma)$ is definable. Note that if $(*, \Sigma)$ is not satisfied by the parameters, then the parameters give an empty conjugacy class. Now recall that $(*, \Sigma)$, *i.e.*, $\bigoplus_{i \text{ odd}} q_i \sim_a q_V$ is a concise notation for $((V_1 \oplus V_3 \oplus \cdots \oplus V_j)_a, (q_1 \oplus q_3 \oplus \cdots \oplus q_j)_a) \cong (V_a, q_a)$ where j is the largest odd integer less than or equal to d for which $c_j(\Lambda) \neq 0$ and the subscript a refers to the anisotropic part. This is given by the formula [refer to formula 28]

$$\text{Iso-aniso}(V_1 \oplus V_3 \oplus \cdots \oplus V_j, J \oplus {}^tX^2J \oplus \cdots \oplus {}^tX^{j-1}J, V, J),$$

where $V_i = \mathrm{vectorspace}(e_{i_1}, \ldots, e_{i_{c_i}})$

Step 6: How do we piece all this together to present a virtual set in the form given by equation (10)? Recall $\Sigma = (\Lambda, (d_i', d_i'', \eta_i', \eta_i''))$. Now, for each $\Lambda \in \tilde{P}(d)$, consider

$$\{X \in \mathfrak{g} \mid \exists_{\mathbb{F}} e_{i_1}, \ldots, e_{i_{c_i}} \phi_\Sigma(X, e_{i_1}, \ldots, e_{i_{c_i}}) \ \wedge \ \phi_{*,\Sigma}(X)\}, \tag{11}$$

where $1 \leq i \leq n$, and i ranges over all odd numbers appearing in the partition Λ. (For brevity, we use the notation $i \in \Lambda$ to indicate this condition on i.)

- $\phi_\Sigma(X, e_{i_1}, \dots, e_{i_{c_i}})$ is the conjunction

$$J\Lambda(X) \bigwedge \Big(\bigwedge_{i \in \Lambda} \phi_i(X) \Big),$$

where $\phi_i(X, e_{i_1}, \dots, e_{i_{c_i}})$ stands for

$$\phi_i^{(1)}(X, e_{i_1}, \dots, e_{i_{c_i}}) \ \wedge\ \phi_{i,\epsilon}^{(2)}(X, e_{i_1}, \dots, e_{i_{c_i}}) \ \wedge\ \phi_{i,\epsilon}^{(3)}(X, e_{i_1}, \dots, e_{i_{c_i}}) \\ \wedge\ \phi_{i,\epsilon}^{(4)}(X) \ \wedge\ \phi_{i,\epsilon}^{(5)}(X),$$

combining the formulae from step 4a and one each (for the choice of ϵ) from steps 4b to 4e.
- $\phi_{*,\Sigma}(X)$ is the formula

$$\text{Iso-aniso}(V_1 \oplus V_3 \oplus \cdots \oplus V_j, J \oplus {}^t X^2 J \oplus \dots \oplus {}^t X^{j-1} J, V, J).$$

In conclusion, the virtual set given by equation (11) is either empty or a nilpotent conjugacy class in $\mathfrak{g}$. This gives definability in the orthogonal case. ∎

5 Concluding Remarks

The use of Pas's language to reformulate p-adic representation theory gives rise to many important directions for research. Is this language powerful enough to express other representation theoretic objects? One would naturally be interested in extending this language to determine if Cartan subgroups, conjugacy classes of Cartan subgroups are definable. If an effective way to define field extensions could be found, we could use them to show that nilpotent conjugacy classes over unitary Lie algebras are definable.

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