Canad. Math. Bull. Vol. **53** (1), 2010 pp. 64–76
doi:10.4153/CMB-2010-011-1

On Antichains of Spreading Models of Banach Spaces

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Abstract. We show that for every separable Banach space X, either $\mathrm{SP}_{\mathrm{w}}(X)$ (the set of all spreading models of X generated by weakly-null sequences in X, modulo equivalence) is countable, or $\mathrm{SP}_{\mathrm{w}}(X)$ contains an antichain of the size of the continuum. This answers a question of S. J. Dilworth, E. Odell, and B. Sari.

1 Introduction

Let X be a separable Banach space, and denote by $\mathrm{SP}_{\mathrm{w}}(X)$ the set of all spreading models of X generated by weakly-null sequences in X, modulo equivalence. By $\leq$ we denote the usual relation on $\mathrm{SP}_{\mathrm{w}}(X)$ of domination. The study of the structure $(\mathrm{SP}_{\mathrm{w}}(X), \leq)$ was initiated by G. Androulakis, E. Odell, T. Schlumprecht and N. Tomczak-Jaegermann [1]. They showed, for instance, that $(\mathrm{SP}_{\mathrm{w}}(X), \leq)$ is a semilattice, *i.e.,* any two elements of $\mathrm{SP}_{\mathrm{w}}(X)$ admit a least upper bound. The question of determining which countable lattices can be realized as $(\mathrm{SP}_{\mathrm{w}}(X), \leq)$ for some separable Banach space X has been answered by S. J. Dilworth, E. Odell, and B. Sari [5].

This note is motivated by the following problem posed in [5, Problem 1.13].

Problem 1 *If $\mathrm{SP}_{\mathrm{w}}(X)$ is uncountable, must there exist $\{(x_n^\xi)_n : \xi < \omega_1\}$ in $\mathrm{SP}_{\mathrm{w}}(X)$ which is either strictly increasing with respect to ξ, or strictly decreasing, or consists of mutually incomparable elements?*

To state our first result, let us say that a seminormalized Schauder basic sequence $(x_n)_n$ in a Banach space X is *C-Schreier spreading* for some $C \geq 1$ (or simply Schreier spreading, if C is understood) if for every $k \in \mathbb{N}$ and every $k \leq n_0 < \cdots < n_k$ and $k \leq m_0 < \cdots < m_k$ we have that $(x_{n_i})_{i=0}^k$ is C-equivalent to $(x_{m_i})_{i=0}^k$. Observe that if $(x_n)_n$ is Schreier spreading, then there exists a unique spreading model (up to equivalence) generated by subsequences of $(x_n)_n$. Denote by $2^{<\mathbb{N}}$ the Cantor tree and let $\varphi\colon 2^{<\mathbb{N}} \to \mathbb{N}$ be the unique bijection satisfying $\varphi(s) < \varphi(t)$ if either $|s| < |t|$, or $|s| = |t| = n$ and $s <_{\mathrm{lex}} t$ (here $<_{\mathrm{lex}}$ stands for the usual lexicographical order on 2^n). We show the following.

Theorem 1.1 *Let X be a separable Banach space such that $\mathrm{SP}_{\mathrm{w}}(X)$ is uncountable. Then there exists a family $(x_t)_{t\in 2^{<\mathbb{N}}}$ in X and $C \geq 1$ such that the following hold.*

(i) *If $(t_n)_n$ is the enumeration of $2^{<\mathbb{N}}$ according to φ, then the sequence $(x_{t_n})_n$ is a seminormalized Schauder basic sequence.*

Received by the editors February 7, 2007.
Published electronically December 4, 2009.
AMS subject classification: **46B20**, 03E15.

(ii) *For every $\sigma \in 2^{\mathbb{N}}$, the sequence $(x_{\sigma|n})_n$ is weakly-null and C-Schreier spreading.*

(iii) *For every $\sigma, \tau \in 2^{\mathbb{N}}$ with $\sigma \neq \tau$, if $(y_n^{\sigma})_n$ and $(y_n^{\tau})_n$ are spreading models generated by subsequences of $(x_{\sigma|n})_n$ and $(x_{\tau|n})_n$ respectively, then $(y_n^{\sigma})_n$ and $(y_n^{\tau})_n$ are incomparable with respect to domination.*

Theorem 1.1 implies the following.

Corollary 1.2 *Let X be a separable Banach space such that $\mathrm{SP}_{\mathrm{w}}(X)$ is uncountable. Then $\mathrm{SP}_{\mathrm{w}}(X)$ contains an antichain of the size of the continuum.*

We notice that, independently, V. Ferenczi and C. Rosendal proved Corollary 1.2 under the additional assumption that X has separable dual [6].

In [1] (see also [5]), it was shown that $\mathrm{SP}_{\mathrm{w}}(X)$ can contain a strictly decreasing infinite sequence, yet no strictly increasing infinite sequence can be found in $\mathrm{SP}_{\mathrm{w}}(X)$. This is not, however, the case of the uncountable.

Theorem 1.3 *Let X be a separable Banach space.*

(i) *If $\mathrm{SP}_{\mathrm{w}}(X)$ contains a strictly decreasing sequence of length ω_1, then $\mathrm{SP}_{\mathrm{w}}(X)$ contains a strictly increasing sequence of length ω_1.*

(ii) *On the other hand, if $\mathrm{SP}_{\mathrm{w}}(X)$ does not contain a strictly increasing infinite sequence, then there exists a countable ordinal ξ_X such that $\mathrm{SP}_{\mathrm{w}}(X)$ does not contain strictly decreasing sequences of order type greater than ξ_X.*

It was shown [5, Theorem 3.7] that for every countable ordinal ξ there exists a separable Banach space X_ξ such that $(\mathrm{SP}_{\mathrm{w}}(X_\xi), \leq)$ does not contain a strictly increasing infinite sequence, yet $\mathrm{SP}_{\mathrm{w}}(X_\xi)$ contains a strictly decreasing sequence of order type ξ. Thus, the ordinal ξ_X obtained by part (i) of Theorem 1.3 is not uniformly bounded within the class of separable Banach spaces for which $\mathrm{SP}_{\mathrm{w}}(X)$ does not contain a strictly increasing infinite sequence.

In the proofs of Theorem 1.1 and Theorem 1.3(i) we use the structural result obtained by B. Sari [16]. However, the central argument in the proof of Theorem 1.1 is essentially based on the work of Leo Harrington and Saharon Shelah on Borel orders. Deep as it is, the theory developed by Harrington and Shelah is highly sophisticated. In particular, all known proofs of their results use either effective descriptive set theory or forcing. However, for the proof of Theorem 1.1 we need only some instances of the theory and merely for F_σ orders. Thus, we have included "elementary" proofs of all the results that we need, making the paper essentially self-contained and accessible to anyone with basic knowledge of classical descriptive set theory. None of these proofs should be considered as a contribution to the field of Borel orders.

The paper is organized as follows. In Section 2 we state and prove all results on Borel orders that are needed for the proof of Theorem 1.1. In Section 3 we show that for every separable Banach space X the structure $(\mathrm{SP}_{\mathrm{w}}(X), \leq)$ can be realized as an F_σ order. In Section 4 we give the proof of Theorem 1.1 while the proof of Theorem 1.3 is given in Section 5.

Notations By $\mathbb{N} = \{0, 1, 2, \dots\}$ we denote the natural numbers, while $[\mathbb{N}]$ denotes the set of all infinite subsets of $\mathbb{N}$ (which is clearly a Polish subspace of $2^{\mathbb{N}}$). By $2^{<\mathbb{N}}$ we denote the set of all finite sequences of 0's and 1's (the empty sequence is

https://doi.org/10.4153/CMB-2010-011-1 Published online by Cambridge University Press

included). We view $2^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of extension. For every $t \in 2^{<\mathbb{N}}$, by $|t|$ we denote the *length* of t, *i.e.*, the cardinality of the set $\{s \in 2^{<\mathbb{N}} : s \sqsubset t\}$. For every $n \in \mathbb{N}$ we let $2^n = \{t \in 2^{<\mathbb{N}} : |t| = n\}$. If $s, t \in 2^{<\mathbb{N}}$, then by $s^\frown t$ we denote their concatenation. For every $\sigma \in 2^{\mathbb{N}}$ and every $n \geq 1$ we let $\sigma|n = \big(\sigma(0), \dots, \sigma(n-1)\big)$, while $\sigma|0 = (\varnothing)$.

If $(x_n)_n$ and $(y_n)_n$ are Schauder basic sequences in a Banach space X and $C \geq 1$, then we say that $(x_n)_n$ is C-dominated by $(y_n)_n$ (or simply dominated, if C is understood) if for every $k \in \mathbb{N}$ and every $a_0, \dots, a_k \in \mathbb{R}$ we have

$$\Big\| \sum_{n=0}^{k} a_n x_n \Big\| \leq C \Big\| \sum_{n=0}^{k} a_n y_n \Big\| .$$

We write $(x_n)_n \leq (y_n)_n$ to denote the fact that $(x_n)_n$ is dominated by $(y_n)_n$. All the other pieces of notation we use are standard as can be found, for instance, in [1,9,10].

2 Quasi-Orders and Borel Orders

A *quasi-order* is a set X with a binary relation $\leq$ on X that is reflexive and transitive. For $x, y \in X$ we let

(i) $x \equiv y \iff (x \leq y)$ and $(y \leq x)$,
(ii) $x < y \iff (x \leq y)$ and $(y \nleq x)$,
(iii) $x \perp y \iff (x \nleq y)$ and $(y \nleq x)$.

If $x, y \in X$ are as in case (iii) above, then we say that x and y are *incomparable*. An *antichain* is a subset of X consisting of pairwise incomparable elements. An ω_1*-chain* in X is a sequence $(x_\xi)_{\xi<\omega_1}$ in X such that either $x_\xi < x_\zeta$ for all $\xi < \zeta < \omega_1$ or $x_\xi < x_\zeta$ for all $\zeta < \xi < \omega_1$.

A *Borel order* is a quasi-order $(X, \leq)$ where X is Polish and $\leq$ is Borel in X^2. A Borel order is called *thin* if X does not contain a perfect set of pairwise incomparable elements. We will need the following lemma concerning the structure of F_σ thin orders.

Lemma 2.1 *Let X be a Polish space and $\leq$ an F_σ thin order on X. Then $(X, \leq)$ does not contain ω_1-chains.*

Lemma 2.1 is a very special case of a deep result due to L. Harrington and S. Shelah (see [7,8]) asserting that *no* Borel thin order contains ω_1-chains. We notice that, prior to [8], H. Friedman had shown that no Borel linear order contains ω_1-chains.

Proof of Lemma 2.1 Let $(F_n)_n$ be an increasing sequence of closed subsets of X^2 with $\leq\; = \bigcup_n F_n$. By symmetry, it is enough to show that if $(X, \leq)$ contains a strictly increasing sequence $(x_\xi)_{\xi<\omega_1}$, then there exists a perfect subset P of X such that $x \perp y$ for all $x, y \in P$ with $x \neq y$. Set $\Gamma = \{x_\xi : \xi < \omega_1\}$. Refining if necessary, we may assume that for every $\xi < \omega_1$ the point x_ξ is a condensation point of Γ. Let ρ be a compatible complete metric for X. By recursion on the length of sequences in $2^{<\mathbb{N}}$, we shall construct a family $(U_t)_{t\in 2^{<\mathbb{N}}}$ of open subsets of X such that the following are satisfied.

(i) For every $t \in 2^{<\mathbb{N}}$ we have $\overline{U}_{t^\frown 0}, \overline{U}_{t^\frown 1} \subseteq U_t$ and $\overline{U}_{t^\frown 0} \cap \overline{U}_{t^\frown 1} = \varnothing$.
(ii) For every $t \in 2^{<\mathbb{N}}$ with $|t| \geq 1$ we have $\rho - \operatorname{diam}(U_t) \leq \frac{1}{|t|}$.
(iii) For every $n \geq 1$ and every $t, s \in 2^n$ with $t \neq s$ we have $(U_t \times U_s) \cap F_n = \varnothing$ and $(U_s \times U_t) \cap F_n = \varnothing$.
(iv) For every $t \in 2^{<\mathbb{N}}$, $U_t \cap \Gamma \neq \varnothing$.

Assuming that the construction has been carried out, we set

$$P = \bigcup_{\sigma \in 2^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} U_{\sigma|n}.$$

By (i) and (ii) above, we see that P is a perfect subset of X. Moreover, using (iii), it is easy to check that P is in addition an antichain.

We proceed to the construction. For $n = 0$, we set $U_{(\varnothing)} = X$. Let $\xi < \zeta < \omega_1$. Then $x_\xi < x_\zeta$, and so $x_\zeta \nleq x_\xi$. In particular, $(x_\zeta, x_\xi) \notin F_1$. Hence, there exist V^0, W^0 open subsets of X such that $x_\zeta \in V^0$, $x_\xi \in W^0$, and $(V^0 \times W^0) \cap F_1 = \varnothing$. Notice that both $V^0 \cap \Gamma$ and $W^0 \cap \Gamma$ are uncountable. So we may select $\eta < \theta < \omega_1$ such that $x_\eta \in V^0$ and $x_\theta \in W^0$. As $x_\theta \nleq x_\eta$, we find V^1, W^1 open subsets of V^0 and W^0, respectively such that $x_\theta \in W^1$, $x_\eta \in V^1$, and $(W^1 \times V^1) \cap F_1 = \varnothing$. Notice that conditions (iii) and (iv) above are satisfied for V^1 and W^1 except, possibly, (i) and (ii). Thus, refining, we find $U_{(0)}$ and $U_{(1)}$ open subsets of V^1 and W^1 respectively such that conditions (i)–(iv) are satisfied. For the general step we proceed similarly. ∎

For more information on the structure of Borel thin orders we refer to the work of A. Louveau [11], and A. Louveau and J. Saint Raymond [12]. For applications of the theory of Borel orders to Banach space theory we refer to the work of C. Rosendal [15].

We will also need the following special case of the theorem of J. H. Silver [17] on the number of equivalence classes of co-analytic equivalence relations. The proof given below is an adaptation of Louveau's approach on Silver's theorem via the so-called Gandy–Harrington topology (see [13]) in an easier setting.

Lemma 2.2 *Let X be a Polish space and $\sim$ an F_σ equivalence relation on X. Then, either the equivalence classes of $\sim$ are countable, or there exists a Cantor set $P \subseteq X$ consisting of pairwise inequivalent elements.*

Proof Let $\mathcal{B} = (U_n)_n$ be a countable basis of X. For every closed subset F of X let

$$D(F) = F \setminus \bigcup \{U_n \in \mathcal{B} : \exists x \in F \text{ with } U_n \cap F \subseteq [x]\},$$

where $[x] = \{y \in X : x \sim y\}$. That is, $D(F)$ results by removing from F all basic relatively open subsets of F which are contained in a single equivalence class. Clearly $D(F)$ is closed and $D(F) \subseteq F$. By transfinite recursion, we define a decreasing sequence $(X_\xi)_{\xi<\omega_1}$ of closed subsets of X as follows. We set $X_0 = X$, $X_{\xi+1} = D(X_\xi)$ and $X_\lambda = \bigcap_{\xi<\lambda} X_\xi$ if λ is limit. There exists $\xi_0 < \omega_1$ such that $X_{\xi_0} = X_{\xi_0+1}$.

Case 1: $X_{\xi_0} = \varnothing$. Notice that for every $\xi < \xi_0$ the set $X_\xi \setminus X_{\xi+1}$ is contained in at most countably many equivalence classes. As $X_{\xi_0} = \varnothing$, we see that

$$X = \bigcup_{\xi<\xi_0} X_\xi \setminus X_{\xi+1}.$$

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Hence, this case implies that the equivalence classes of $\sim$ are countable.

Case 2: $X_{\xi_0} \neq \varnothing$. We set $Y = X_{\xi_0}$ and $\sim' = \sim \cap Y^2$. Clearly $\sim'$ is F_σ in Y^2. We claim that $\sim'$ is meager in Y^2. By the Kuratowski–Ulam theorem (see [9, Theorem 8.41]), it is enough to show that for every $x \in Y$ the set

$$[x]' = \{y \in Y : x \sim' y\} = \{y \in Y : x \sim y\}$$

is meager. Notice that $[x]'$ is F_σ in Y. So if $[x]'$ were not meager, then there would have existed $U_n \in \mathcal{B}$ such that $U_n \cap Y \subseteq [x]'$. This implies that $D(X_{\xi_0}) \subsetneq X_{\xi_0}$, which is a contradiction. Thus, $\sim'$ is meager in Y^2 as claimed. It follows by a classical result of Mycielski (see [9, Theorem 19.1]) that there exists a Cantor set $P \subseteq Y$ such that $x \nsim' y$ for all $x, y \in P$ with $x \neq y$. This clearly implies that $x \nsim y$ for all $x, y \in P$ with $x \neq y$. ■

3 Coding $(\mathrm{SP_w}(X), \leq)$ as an F_σ Order

Let X be a separable Banach space. Our aim is to show that the quasi-order $(\mathrm{SP_w}(X), \leq)$ can be realized as an F_σ order. This is done in a rather standard and natural way.

Let U be the universal space of A. Pelczynski for unconditional basic sequences (see [14]). That is, U has an unconditional Schauder basis $(u_n)_n$ and for any other unconditional Schauder basic sequence $(y_n)_n$ in some Banach space Y there exists $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]$ such that $(y_n)_n$ is equivalent to $(u_{l_n})_n$. In what follows, for every $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]$ by $(u_n)_{n\in L}$ we denote the subsequence $(u_{l_n})_n$ of $(u_n)_n$ determined by L. Define $\leq$ in $[\mathbb{N}] \times [\mathbb{N}]$ by

$$L \leq M \iff (u_n)_{n\in L} \text{ is dominated by } (u_n)_{n\in M}.$$

Clearly $\leq$ is a quasi-order. Let $\sim$ be the associated equivalence relation (*i.e.*, $L \sim M$ if and only if $L \leq M$ and $M \leq L$) and let $<$ be the strict part of $\leq$, *i.e.*, $L < M$ if and only if $L \leq M$ and $M \nleq L$. Notice that $L \sim M$ if and only if the sequences $(u_n)_{n\in L}$ and $(u_n)_{n\in M}$ are equivalent as Schauder basic sequences. We have the following easy fact whose proof is sketched for completeness.

Fact 3.1 *Both $\leq$ and $\sim$ are F_σ.*

Proof It is enough to show that $\leq$ is F_σ. For every $K \in \mathbb{N}$ with $K \geq 1$ let $\leq_K$ be the relation on $[\mathbb{N}] \times [\mathbb{N}]$ defined by

$$L \leq_K M \iff (u_n)_{n\in L} \text{ is } K\text{-dominated by } (u_n)_{n\in M}.$$

It is easy to see that $\leq_K$ is closed in $[\mathbb{N}] \times [\mathbb{N}]$. As $\leq$ is the union of $\leq_K$ over all $K \geq 1$, the result follows. ■

Our coding of $(\mathrm{SP_w}(X), \leq)$ as an F_σ order will be done via the following lemma.

Lemma 3.2 *Let X be a separable Banach space. Then there exists $A_X \subseteq [\mathbb{N}]$ analytic such that the following are satisfied.*

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(i) *For every $(y_n)_n \in \mathrm{SP}_{\mathrm{w}}(X)$ there exists $L \in A_X$ such that $(y_n)_n$ is equivalent to $(u_n)_{n\in L}$.*
(ii) *For every $L \in A_X$ there exists $(y_n)_n \in \mathrm{SP}_{\mathrm{w}}(X)$ such that $(u_n)_{n\in L}$ is equivalent to $(y_n)_n$.*

Proof Recall that a sequence $(x_n)_n$ in X is said to be *Cesaro summable* if

$$\lim_{n\to\infty} \frac{x_0 + \cdots + x_{n-1}}{n} = 0.$$

Let SPC be the subset of $X^{\mathbb{N}}$ defined by

$(x_n)_n \in \mathrm{SPC} \iff (x_n)_n$ is seminormalized, Schauder basic, Cesaro summable and C-Schreier spreading for some $C \geq 1$.

It is easy to check that SPC is a Borel subset of $X^{\mathbb{N}}$ (actually, it is $F_{\sigma\delta}$). Consider the subset A of $[\mathbb{N}]$ defined by

$L \in A \iff$ if $L = \{l_0 < l_1 < \cdots\}$, then $\exists (x_n)_n \in X^{\mathbb{N}} \exists \theta \geq 1$ with $\big[(x_n)_n \in \mathrm{SPC}$ and $\big(\forall k\, \forall k \leq n_0 < \cdots < n_k$ we have $(x_{n_i})_{i=0}^k \overset{\theta}{\sim} (u_{l_i})_{i=0}^k\big)\big]$.

As SPC is Borel in $X^{\mathbb{N}}$, it is easy to see that the set A is analytic. Denote by $(e_n)_n$ the standard basis of ℓ_1. Let us isolate the following property of the set A.

(P) If $L \in A$, then the sequence $(u_n)_{n\in L}$ is not equivalent to $(e_n)_n$. This follows from the fact that every sequence $(x_n)_n$ belonging to SPC is a Cesaro summable Schauder basic sequence.

The proof of the lemma will be finished once we show the following.

Claim 1 *Let $(y_n)_n \in \mathrm{SP}_{\mathrm{w}}(X)$ which is not equivalent to $(e_n)_n$. Then there exists $L \in A$ such that $(y_n)_n$ is equivalent to $(u_n)_{n\in L}$. Conversely, for every $L \in A$ there exists $(y_n)_n \in \mathrm{SP}_{\mathrm{w}}(X)$ which is not equivalent to $(e_n)_n$ and such that $(u_n)_{n\in L}$ is equivalent to $(y_n)_n$.*

Proof of Claim 1 Let $(y_n)_n \in \mathrm{SP}_{\mathrm{w}}(X)$ not equivalent to $(e_n)_n$ and let $(x_n)_n$ be a seminormalized weakly-null sequence in X that generates it. By passing to a subsequence, we may assume that $(x_n)_n$ is a seminormalized, C-Schreier spreading (for some $C \geq 1$) Schauder basic sequence. As $(y_n)_n$ is not equivalent to $(e_n)_n$, by a result of H. P. Rosenthal we see that $(x_n)_n$ has a subsequence $(x_{n_k})_k$ which is additionally Cesaro summable (see [3, Theorem II.9.8]). Hence $(x_{n_k})_k \in \mathrm{SPC}$. As $(x_{n_k})_k$ still generates $(y_n)_n$ as a spreading model, we easily see that there exists $L \in A$ such that $(u_n)_{n\in L}$ is equivalent to $(y_n)_n$.

Conversely, let $L \in A$. We pick $(x_n)_n \in \mathrm{SPC}$ witnessing that $L \in A$. By property (P) above, we have that $(u_n)_{n\in L}$ is not equivalent to $(e_n)_n$. Now we claim that $(x_n)_n$ is weakly-null. Assume not. Then there exist $M = \{m_0 < m_1 < \cdots\} \in [\mathbb{N}]$, $x^* \in X^*$ and $\varepsilon > 0$ such that $x^*(x_{m_n}) > \varepsilon$ for every $n \in \mathbb{N}$ (notice also that $m_n \geq n$). Let

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$K \geq 1$ be the basis constant of $(x_n)_n$. Let also $C \geq 1$ be such that $(x_n)_n$ is C-Schreier spreading. Observe that for every $n \in \mathbb{N}$ we have

$$\Big\| \frac{x_0 + \cdots + x_{2n-1}}{2n} \Big\| \geq \frac{1}{2(K+1)} \Big\| \frac{x_n + \cdots + x_{2n-1}}{n} \Big\| \\ \geq \frac{1}{2C(K+1)} \Big\| \frac{x_{m_n} + \cdots + x_{m_{2n-1}}}{n} \Big\| \geq \frac{\varepsilon}{2C(K+1)},$$

which implies that $(x_n)_n$ is not Cesaro summable, a contradiction. Thus, $(x_n)_n$ is weakly-null. Let $(y_n)_n$ be a spreading model generated by a subsequence of $(x_n)_n$. Then $(y_n)_n \in \mathrm{SP_w}(X)$. Invoking the definition of the set A again, we see that $(y_n)_n$ is equivalent to $(u_n)_{n\in L}$. This yields additionally that $(y_n)_n$ is not equivalent to $(e_n)_n$. The claim is proved. ■

If $(e_n)_n \notin \mathrm{SP_w}(X)$, then we set $A_X = A$. If $(e_n)_n \in \mathrm{SP_w}(X)$, then we set $A_X = A \cup \{L \in [\mathbb{N}] : (u_n)_{n\in L} \sim (e_n)_n\}$. Clearly A_X is analytic and, by Claim 1, A_X is as desired. The lemma is proved. ■

4 Proof of Theorem 1.1

Let X be a separable Banach space such that $\mathrm{SP_w}(X)$ is uncountable. Let A_X be the analytic subset of $[\mathbb{N}]$ obtained by Lemma 3.2. We fix $\Phi\colon \mathbb{N}^{\mathbb{N}} \to [\mathbb{N}]$ continuous with $\Phi(\mathbb{N}^{\mathbb{N}}) = A_X$. We define $\precsim$ on $\mathbb{N}^{\mathbb{N}}$ by

$$\alpha \precsim \beta \iff \Phi(\alpha) \leq \Phi(\beta).$$

By Fact 3.1 and the continuity of Φ, we see that $\precsim$ is an F_σ quasi-order on the Baire space $\mathbb{N}^{\mathbb{N}}$.

Lemma 4.1 *Let X be a separable Banach space such that $\mathrm{SP_w}(X)$ is uncountable, and consider the F_σ quasi-order $(\mathbb{N}^{\mathbb{N}}, \precsim)$. Then either*

(i) *$(\mathbb{N}^{\mathbb{N}}, \precsim)$ is not thin, or*
(ii) *$(\mathbb{N}^{\mathbb{N}}, \precsim)$ contains a strictly increasing sequence of length ω_1.*

Proof Let $\cong$ be the equivalence relation associated with $\precsim$, *i.e.*, $\alpha \cong \beta$ if $\alpha \precsim \beta$ and $\beta \precsim \alpha$. Notice that

$$\alpha \cong \beta \iff \Phi(\alpha) \sim \Phi(\beta)$$

for every $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. Also observe that $\cong$ is an F_σ equivalence relation. As $\mathrm{SP_w}(X)$ is uncountable, we see that $\cong$ has uncountably many equivalence classes. Thus, by Lemma 2.2, there exists a Cantor set $P \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\alpha \ncong \beta$ for every $\alpha, \beta \in P$ with $\alpha \neq \beta$. Fix a homeomorphism $h\colon 2^{\mathbb{N}} \to P$. Let $<_{\mathrm{lex}}$ be the (strict) lexicographical ordering on $2^{\mathbb{N}}$. For every $Q \subseteq 2^{\mathbb{N}}$, denote by $[Q]^2$ the set of unordered pairs of elements of Q. Consider the following subsets $\mathcal{I}$ and $\mathcal{D}$ of $[2^{\mathbb{N}}]^2$ defined by

$$\{\sigma, \tau\} \in \mathcal{I} \iff \text{ if } \sigma <_{\mathrm{lex}} \tau, \text{ then } h(\sigma) \precsim h(\tau),$$
$$\{\sigma, \tau\} \in \mathcal{D} \iff \text{ if } \sigma <_{\mathrm{lex}} \tau, \text{ then } h(\tau) \precsim h(\sigma).$$

It is easy to check that both $\mathcal{I}$ and $\mathcal{D}$ are Borel in $[2^{\mathbb{N}}]^2$, in the sense that the sets

$$\mathcal{I}^* = \{(\sigma,\tau) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : \{\sigma,\tau\} \in \mathcal{I}\} \text{ and } \mathcal{D}^* = \{(\sigma,\tau) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : \{\sigma,\tau\} \in \mathcal{D}\}$$

are both Borel subsets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. By a result of F. Galvin (see [9, Theorem 19.7]), there exists $Q \subseteq 2^{\mathbb{N}}$ perfect such that one of the following cases occur.

Case 1:$[Q]^2 \subseteq \mathcal{I}$. We fix a sequence $(\sigma_n)_n$ in Q that is increasing with respect to $<_{\text{lex}}$. Then $h(\sigma_n) \precsim h(\sigma_m)$ for all $n < m$. As $h(Q) \subseteq P$ and P consists of inequivalent elements with respect to $\cong$, we see that the sequence $(h(\sigma_n))_n$ is strictly increasing. This yields that $(\mathrm{SP_w}(X), \leq)$ contains a strictly increasing sequence. By a result of B. Sari [16], we conclude that $\mathrm{SP_w}(X)$ must contain a strictly increasing sequence of length ω_1. This clearly implies that $(\mathbb{N}^{\mathbb{N}}, \precsim)$ contains a strictly increasing sequence of length ω_1, *i.e.*, part (ii) of the lemma is valid.

Case 2:$[Q]^2 \subseteq \mathcal{D}$. Let $(\tau_n)_n$ be a sequence in Q which is decreasing with respect to $<_{\text{lex}}$. Arguing as in Case 1 above, we see that the sequence $(h(\tau_n))_n$ is strictly increasing. So this case also implies part (ii) of the lemma.

Case 3:$[Q]^2 \cap (\mathcal{I} \cup \mathcal{D}) = \varnothing$. We set $R = h(Q)$. Clearly R is a perfect subset of $\mathbb{N}^{\mathbb{N}}$. It is easy to check that if $\alpha, \beta \in R$ with $\alpha \neq \beta$, then α and β are incomparable with respect to $\precsim$. Hence R is a perfect antichain of $(\mathbb{N}^{\mathbb{N}}, \precsim)$, *i.e.*, $(\mathbb{N}^{\mathbb{N}}, \precsim)$ is not thin. Thus, this case implies part (i) of the lemma. ■

Lemma 4.2 *Let X be a separable Banach space such that $\mathrm{SP_w}(X)$ is uncountable. Then there exists a Cantor set $P \subseteq A_X$ consisting of pairwise incomparable elements with respect to domination.*

Proof Assume, towards a contradiction, that such a Cantor set P does not exist. This easily implies that $(\mathbb{N}^{\mathbb{N}}, \precsim)$ is a thin quasi-order. By Lemma 4.1, we see that $(\mathbb{N}^{\mathbb{N}}, \precsim)$ is an F_σ thin order that contains an ω_1-chain. But this possibility is ruled out by Lemma 2.1. Having arrived at the desired contradiction, the lemma is proved. ■

Remark 1 We notice that Lemma 3.2 and Lemma 4.2 immediately yield that if X is a separable Banach space such that $\mathrm{SP_w}(X)$ is uncountable, then $\mathrm{SP_w}(X)$ must contain an antichain of the size of the continuum.

We are ready to proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1 Let $P \subseteq A_X$ be the Cantor set obtained by Lemma 4.2. By passing to a perfect subset of P if necessary, we may assume that

(A) for every $L \in P$ the sequence $(u_n)_{n\in L}$ is not equivalent to the standard basis of ℓ_1.

We will construct the family $(x_t)_{t\in 2^{<\mathbb{N}}}$ by "pulling back" inside X the spreading models coded by P. To this end, let $(d_m)_m$ be a countable dense subset of X. Let SPC be the Borel subset of $X^{\mathbb{N}}$ defined in the proof of Lemma 3.2. Consider the subset G of $P \times [\mathbb{N}]$ defined by

$$(L,M) \in G \iff \begin{array}{l} \text{if } L = \{l_0 < l_1 < \cdots\} \text{ and } M = \{m_0 < m_1 < \cdots\}, \text{ then} \\ \big[L \in P \text{ and } (d_{m_n})_n \in \mathrm{SPC} \text{ and } \big(\exists \theta \geq 1 \forall k \forall k \leq n_0 < \cdots < \\ n_k \text{ we have } (d_{m_{n_i}})_{i=0}^k \overset{\theta}{\sim} (u_{l_i})_{i=0}^k\big)\big]. \end{array}$$

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Let us gather some of the properties of the set G.

(P1) The set G is Borel.

(P2) For every $(L, M) \in G$ and every N infinite subset of M, if $(y_n)_n$ is a spreading model generated by a subsequence of $(d_m)_{m\in N}$, then $(y_n)_n$ is equivalent to $(u_n)_{n\in L}$.

(P3) For every $L \in P$ there exists $M \in [\mathbb{N}]$ such that $(L, M) \in G$.

(P4) For every $(L, M) \in G$, the sequence $(d_m)_{m\in M}$ is weakly-null.

Properties (P1) and (P2) are rather straightforward consequences of the definition of the set G. Property (P3) follows by assumption (A) above, the fact that P is a subset of A_X and a standard perturbation argument. Property (P4) has already been verified in the proof of Lemma 3.2.

As G is a Borel subset of $P \times [\mathbb{N}]$, by (P3) above and the Yankov–von Neumann uniformization theorem (see [9, Theorem 18.1]), there exists a map $f\colon P \to [\mathbb{N}]$ that is measurable with respect to the σ-algebra generated by the analytic sets and such that $(L, f(L)) \in G$ for every $L \in P$. Notice that the map f must be one-to-one. Invoking the classical fact that analytic sets have the Baire property, by [9, Theorem 8.38] and by passing to a perfect subset of P, we may assume that f is actually continuous. Moreover, by passing to a further perfect subset of P if necessary, we may also assume that there exist $j_0, k_0 \in \mathbb{N}$ such that for every $L \in P$, the sequence $(d_m)_{m\in f(L)}$ is j_0-Schreier spreading and satisfies $\frac{1}{k_0} \le \|d_m\| \le k_0$ for every $m \in f(L)$.

The function f is one-to-one and continuous. Hence, identifying every element of $[\mathbb{N}]$ with its characteristic function (*i.e.*, an element of $2^{\mathbb{N}}$), we see that the set $f(P)$ is a perfect subset of $2^{\mathbb{N}}$. Recall that by $\varphi\colon 2^{<\mathbb{N}} \to \mathbb{N}$ we denote the canonical bijection described in the introduction. By recursion on the length of finite sequences in $2^{<\mathbb{N}}$, we may easily select a family $(m_s)_{s\in 2^{<\mathbb{N}}}$ in $\mathbb{N}$ with the following properties.

(P5) For every $s_1, s_2 \in 2^{<\mathbb{N}}$ we have $\varphi(s_1) < \varphi(s_2)$ if and only if $m_{s_1} < m_{s_2}$.

(P6) For every $\sigma \in 2^{\mathbb{N}}$, setting $M_\sigma = \{m_{\sigma|n} : n \in \mathbb{N}\} \in [\mathbb{N}]$, there exists a unique $L_\sigma \in P$ such that $M_\sigma \subseteq f(L_\sigma)$.

We set $x_s = d_{m_s}$ for every $s \in 2^{<\mathbb{N}}$. We observe that $\frac{1}{k_0} \le \|x_s\| \le k_0$ for all $s \in 2^{<\mathbb{N}}$. We also notice that for every $\sigma \in 2^{\mathbb{N}}$, the sequence $(x_{\sigma|n})_n$ is j_0-Schreier spreading.

Now let $s \in 2^{<\mathbb{N}}$ with $|s| = k$ and $\sigma \in 2^{\mathbb{N}}$ with $\sigma|k = s$. By properties (P4) and (P6), we see that the sequence $(x_{\sigma|n})_{n>k}$ is weakly-null. Using this observation and the classical procedure of Mazur for constructing Schauder basic sequences (see [10]), we may select a family $(s_t)_{t\in 2^{<\mathbb{N}}}$ in $2^{<\mathbb{N}}$ such that, setting $x_t = x_{s_t}$ for every $t \in 2^{<\mathbb{N}}$, the following are satisfied.

(P7) For every $t_1, t_2 \in 2^{<\mathbb{N}}$ we have that $s_{t_1} \sqsubset s_{t_2}$ if and only if $t_1 \sqsubset t_2$. Moreover, $|s_{t_1}| < |s_{t_2}|$ if and only if $\varphi(s_1) < \varphi(s_2)$.

(P8) If $(t_n)_n$ is the enumeration of $2^{<\mathbb{N}}$ according to φ, then the sequence $(x_{t_n})_n$ is Schauder basic.

It is easy to verify that the family $(x_t)_{t\in 2^{<\mathbb{N}}}$ has all properties stated in Theorem 1.1. ■

Remark 2 We would like to note a few things on the richness of the structure $(\mathrm{SP_w}(X), \le)$ when $\mathrm{SP_w}(X)$ is uncountable. Let X be a separable Banach space and

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assume that there exist $C \geq 1$ and a family $\{(y_n^\xi)_n : \xi < \omega_1\}$ of mutually inequivalent spreading models generated by weakly-null sequences in X such that for every $\xi < \zeta < \omega_1$ either the sequence $(y_n^\xi)_n$ is C-dominated by $(y_n^\zeta)_n$ or vice versa. By Lemma 3.2, there exist $K \geq 1$ and $U \subseteq A_X$ uncountable such that the following hold. For every $L, M \in U$ either $(u_n)_{n\in L}$ is K-dominated by $(u_n)_{n\in M}$ or vice versa, and moreover, for every $L \in U$ there exists a unique ordinal $\xi_L < \omega_1$ such that $(u_n)_{n\in L}$ is equivalent to $(y_n^{\xi_L})_n$. Let $\overline{U}$ be the closure of U in $[\mathbb{N}]$ and set $F = \overline{U} \cap A_X$. Then F is an uncountable analytic set. Consider the following symmetric relation $\approx_K$ in $[\mathbb{N}] \times [\mathbb{N}]$ defined by

$$L \approx_K M \Longleftrightarrow \text{either } (u_n)_{n\in L} \text{ is } K\text{-dominated by } (u_n)_{n\in M} \text{ or vice versa.}$$

It is easy to see that $\approx_K$ is closed in $[\mathbb{N}] \times [\mathbb{N}]$. By the choice of U, we have $L \approx_K M$ for every $L, M \in U$. As $\approx_K$ is closed, we see that $L \approx_K M$ for every $L, M \in \overline{U}$. In particular, $L \approx_K M$ for every $L, M \in F$. Notice that $U \subseteq F$, and so the relation $\sim$ of equivalence restricted on F has uncountably many equivalence classes. By Lemma 2.2, there exists a perfect subset P of F such that for every $L, M \in P$ the sequences $(u_n)_{n\in L}$ and $(u_n)_{n\in M}$ are not equivalent[1]. Thus, we have shown the following.

Proposition 4.3 *Let X be a separable Banach space and assume that there exist $C \geq 1$ and a family $\{(y_n^\xi)_n : \xi < \omega_1\}$ of mutually inequivalent spreading models generated by weakly-null sequences in X such that for every $\xi < \zeta < \omega_1$ either the sequence $(y_n^\xi)_n$ is C-dominated by $(y_n^\zeta)_n$ or vice versa. Then $(\mathrm{SP_w}(X), \leq)$ contains a linearly ordered subset of the size of the continuum.*

Related to Proposition 4.3, the following question is open to us. Let X be a separable Banach space and assume that $\mathrm{SP_w}(X)$ is uncountable. Does $(\mathrm{SP_w}(X), \leq)$ contain a linearly ordered subset of the size of the continuum, or at least uncountable?

5 Proof of Theorem 1.3

(i) First we need to recall some standard facts (see [9, p. 351]. Let S be a set and $\prec$ a strict, well-founded (binary) relation on S. This is equivalent to asserting that there is no infinite decreasing chain $\cdots \prec s_1 \prec s_0$. By recursion on $\prec$, we define the *rank* function $\rho_\prec : S \to \mathrm{Ord}$ of $\prec$ by the rule

$$\rho_\prec(s) = \sup\{\rho_\prec(x) + 1 : x \prec s\}.$$

In particular, $\rho_\prec(s) = 0$ if and only if s is minimal. The *rank* $\rho(\prec)$ of $\prec$ is defined by $\rho(\prec) = \sup\{\rho_\prec(s) + 1 : s \in S\}$.

We are ready to proceed to the proof. So, let X be a separable Banach space such that $\mathrm{SP_w}(X)$ contains a strictly decreasing sequence of length ω_1. Let A_X be the analytic subset of $[\mathbb{N}]$ obtained by Lemma 3.2. Consider the following relation $\prec$ on $[\mathbb{N}]$ defined by

$$L \prec M \Leftrightarrow (L \in A_X) \text{ and } (M \in A_X) \text{ and } (M < L).$$

[1] This does not follow directly by Lemma 2.2 as F is not Polish. One has to observe that F is the continuous surjective image of $\mathbb{N}^{\mathbb{N}}$ and use an argument as in the beginning of Section 4.

That is, $\prec$ is the relation $>$ (the reverse of $<$) restricted on $A_X \times A_X$. Clearly $\prec$ is analytic (as a subset of $[\mathbb{N}] \times [\mathbb{N}]$). Let $\{(y_n^\xi)_n : \xi < \omega_1\}$ be a strictly decreasing sequence in $\mathrm{SP_w}(X)$. By Lemma 3.2, for every $\xi < \omega_1$ we may select $L_\xi \in A_X$ such that $(u_n)_{n\in L_\xi}$ is equivalent to $(y_n^\xi)_n$. It follows that $L_\xi < L_\zeta$ if and only if $\zeta < \xi$.

Assume, towards a contradiction, that $\mathrm{SP_w}(X)$ does not contain a strictly increasing sequence of length ω_1. Then, by the result of Sari [16] already quoted in the proof of Theorem 1.1, $\mathrm{SP_w}(X)$ does not contain a strictly increasing sequence of length ω. It follows that $\prec$ is a well-founded relation on $[\mathbb{N}]$ which is in addition analytic. By the Kunen–Martin theorem (see [9, Theorem 31.5]), we see that $\rho(\prec)$ is a countable ordinal, say ξ_0. For every $\eta < \xi_0$ let

$$A_X^\eta = \{L \in A_X : \rho_\prec(L) = \eta\}.$$

As $\rho_\prec(L) < \xi_0$ for every $L \in A_X$ we see that $A_X = \bigcup_{\eta<\xi_0} A_X^\eta$. Moreover, for every $L, M \in A_X^\eta$ we have that either $L \sim M$ or $L \perp M$. That is, we have partitioned the quotient $A_X/\!\sim$ into countably many antichains. As the family $\{L_\xi : \xi < \omega_1\}$ is uncountable, we see that there exist $\xi, \zeta < \omega_1$ with $\xi \neq \zeta$ and $\eta < \xi_0$ such that $L_\xi, L_\zeta \in A_X^\eta$. But this is clearly impossible. Having arrived at the desired contradiction, the proof of part (i) is completed.

(ii) Again we need to discuss some standard facts. Let R be a binary relation on $\mathbb{N}$, *i.e.,* $R \subseteq \mathbb{N} \times \mathbb{N}$. By identifying R with its characteristic function, we view every binary relation on $\mathbb{N}$ as an element of $2^{\mathbb{N}\times\mathbb{N}}$. Let LO be the subset of $2^{\mathbb{N}\times\mathbb{N}}$ consisting of all (strict) linear orderings on $\mathbb{N}$. It is easy to see that LO is a closed subset of $2^{\mathbb{N}\times\mathbb{N}}$ (see also [9, p. 212]). For every $\alpha \in \mathrm{LO}$ and every $n, m \in \mathbb{N}$ we write

$$n <_\alpha m \iff \alpha(n, m) = 1.$$

Let WO be the subset of LO consisting of all well orderings on $\mathbb{N}$. For every $\alpha \in \mathrm{WO}$, $|\alpha|$ stands for the unique ordinal which is isomorphic to $(\mathbb{N}, <_\alpha)$. We will need the following boundedness principle for WO (see [9, p. 240]): if B is an analytic subset of WO, then $\sup\{|\alpha| : \alpha \in B\} < \omega_1$.

We proceed to the proof of part (ii). Let X be a separable Banach space. Let A_X be the analytic subset of $[\mathbb{N}]$ obtained by Lemma 3.2. Consider the following subset O_X of LO defined by

$$\alpha \in \mathrm{O}_X \iff \exists (L_n)_n \in ([\mathbb{N}])^{\mathbb{N}} \text{ with } \big[(\forall n\, L_n \in A_X) \text{ and } [\forall n, m\, (n <_\alpha m \iff L_n > L_m)]\big].$$

As A_X is analytic, it easy to check that O_X is an analytic subset of LO.

Claim 2 *The set* $\mathrm{SP_w}(X)$ *does not contain a strictly increasing sequence if and only if* $\mathrm{O}_X \subseteq \mathrm{WO}$.

Proof of Claim 2 First assume that there exists $\alpha \in \mathrm{O}_X$ with $\alpha \notin \mathrm{WO}$. By definition, there exists a sequence $(L_n)_n$ in A_X such that for all $n, m \in \mathbb{N}$ we have

$$n <_\alpha m \iff L_n > L_m.$$

As $\alpha \notin \mathrm{WO}$, there exists a sequence $(n_i)_i$ in $\mathbb{N}$ such that $n_{i+1} <_\alpha n_i$ for all $i \in \mathbb{N}$. It follows that $(L_{n_i})_i$ is a strictly increasing sequence, which clearly implies that $\mathrm{SP}_{\mathrm{w}}(X)$ contains a strictly increasing sequence.

Conversely, assume that $\mathrm{SP}_{\mathrm{w}}(X)$ contains a strictly increasing sequence. Hence, we may find a sequence $(L_n)_n$ in A_X such that $L_n < L_m$ if and only if $n < m$. Let $\alpha \in \mathrm{LO}$ be defined by

$$n <_\alpha m \iff n > m \;(\iff L_n > L_m).$$

Then $\alpha \in \mathrm{O}_X$ and $\alpha \notin \mathrm{WO}$. ■

Now, let X be a separable Banach space that does not contain a strictly increasing sequence. By Claim 2, we see that the set O_X is an analytic subset of WO. Hence, by boundedness, we see that

$$\sup\{|\alpha| : \alpha \in \mathrm{O}_X\} = \xi_X < \omega_1.$$

We claim that ξ_X is the desired ordinal. Indeed, let ξ be a countable ordinal and $\{(y_n^\zeta)_n : \zeta < \xi\}$ a strictly decreasing sequence in $\mathrm{SP}_{\mathrm{w}}(X)$. By Lemma 3.2, we may find $(L_\zeta)_{\zeta<\xi}$ in A_X which is strictly decreasing. Fix a bijection $e\colon \mathbb{N} \to \{\zeta : \zeta < \xi\}$ and define $\alpha \in \mathrm{WO}$ by

$$n <_\alpha m \iff e(n) < e(m) \;(\iff L_{e(n)} > L_{e(m)}).$$

It follows that $\alpha \in \mathrm{O}_X$, and so, $\xi = |\alpha| \le \xi_X$. ■

Remark 3 Denote by SB the standard Borel space of all separable Banach spaces as it is discussed in [2, 4, 9]. Consider the subset NCI of SB defined by

$$X \in \mathrm{NCI} \iff \mathrm{SP}_{\mathrm{w}}(X) \text{ does not contain a strictly increasing infinite sequence.}$$

It can be shown, using some results from [5], that the set NCI is co-analytic non-Borel in SB. Moreover, there exists a co-analytic rank $\phi\colon \mathrm{NCI} \to \omega_1$ on NCI such that for every $X \in \mathrm{NCI}$ we have

$$\sup\{|\alpha| : \alpha \in \mathrm{O}_X\} \le \phi(X),$$

where O_X is as in the proof of Theorem 1.3(ii) (for the definition of co-analytic ranks we refer to [9], while for applications of rank theory to Banach space theory we refer to [2]).

Acknowledgments I would like to thank Spiros A. Argyros for many discussions on the subject as well as for his comments on the paper.

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