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# TWO APPLICATIONS OF HOMOLOGY DECOMPOSITIONS

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We show that a map of rational spaces (see Definition 1) induces a map of homology sections at each stage, and that the $k'$-invariants are mapped naturally. This is used to characterize rational spaces in which all (matric) Massey products vanish as wedges of rational spheres, and yields the precise Eckmann-Hilton dual of a result of M. Dyer [7]. Berstein's result on co-$H$ spaces [3] is also deduced. These results form a part of the author's doctoral dissertation at Cornell University written under Professor I. Berstein, to whom I express my sincere thanks for his patient help and encouragement. Extensions and counterexamples will appear in a future paper.

The proof is dual to Kahn's corresponding result for Postnikov systems. Since fundamental classes are *not* uniquely defined in our case, some care is called for. For these reasons, we only show that the $k'$-invariants are mapped nicely. Complete details may be found in [**17**].

In [**5**], an example of a space $X$ is given such that (a) $H_*(X, \mathbf{Z})$ has torsion and (b) $X$ admits two homology decompositions, the sections of which do not have the same homotopy type.

*Definition* 1. A *rational space* will denote a pointed, simply connected, connected space having the homotopy type of a $CW$ complex, and such that $H_n(X, \mathbf{Z})$ is a finite dimensional vector space over $\mathbf{Q}$, the rationals, for each $n > 0$.

For example the rationalization (in the sense of Sullivan [**16**]) of a simply connected $CW$ complex of finite type is a rational space.

Let $G$ be an abelian group. Recall that a Moore space $K'(G, n)$ denotes a space with abelian fundamental group and with a single non-vanishing homology group viz. $G$ in dimension $n$. We can take $K'(G, n)$ to be a $CW$ complex, and the homotopy type of $K'(G, n)$ is uniquely determined for $n \geqq 2$. Notice that $\Sigma K'(G, n - 1)$ is also a $K'(G, n)$ and thus $K'(G, n)$ is a co-$H$ space for $n \geqq 2$.

*Definition* 2 [**8**]. Let $n \geqq 2$ be an integer. The $n$th homotopy group of any pointed space $X$ with coefficients in $G$ is defined by

$$\pi_n(G, X) = [\Sigma K'(G, n - 1), X].$$

(Based homotopy classes are understood.) We refer the reader to [**8**] for the definition of relative homotopy groups $\pi_n(G; X, A)$.

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THEOREM 3 [**8**]. *There is a natural exact sequence*

$$0 \to \operatorname{Ext}(G, \pi_{n+1}(X, A) \to \pi_n(G; X, A)) \xrightarrow{\eta} \operatorname{Hom}(G, \pi_n(X, A)) \to 0$$

*where* $\eta([f])$ *is the composite*

$$H_n(K'(G, n)) \xrightarrow{h^{-1}} \pi_n(K'(G, n)) \xrightarrow{f_\#} \pi_n(X),$$

*where* $f_\#$ *denotes the map induced by* $[f] \in \pi_n(G; X, A)$ *and* $h$ *denotes a Hurewicz homomorphism.*

It follows easily from Theorem 3 that a homomorphism $\varphi: G \to G'$ induces a map $\tilde{\varphi}: K'(G, n) \to K'(G', n)$ and thus a "coefficient homomorphism" $\tilde{\varphi}^c: \pi_n(G', X) \to \pi_n(G, X)$ for any space $X$. Further if $G$ is a vector space over $\mathbf{Q}$, then $\eta$ induces an isomorphism

$$\pi_n(G; X, A) \xrightarrow{\cong} \operatorname{Hom}(G, \pi_n(X, A)).$$

We recall from [**8**] that a simply connected $CW$ complex admits a homology decomposition. In the next result, we do not need the fact that $H_n(X, \mathbf{Z})$ is finite dimensional for each $n$.

THEOREM 4. *Let* $f: X \to Y$ *be a (based) map of rational spaces and let homology decompositions* $\{X^r, k_r'(X)\}$, $\{Y^r, k_r'(Y)\}$ *be given. Then there are diagrams:*

(a)
$$\begin{tikzcd}
X^2 \arrow[r, "f^2"] \arrow[d, hook] \arrow[ddd, bend right=60, "i^2"'] & Y^2 \arrow[d, hook] \arrow[ddd, bend left=60, "i^2"] \\
X^3 \arrow[r, "f^3"] \arrow[d, dashed] \arrow[dd, bend right, "i^3"'] & Y^3 \arrow[d, dashed] \arrow[dd, bend left, "i^3"] \\
\vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

*such that each rectangle is strictly commutative, and each outer triangle is homotopy commutative;*

(b)
$$\begin{tikzcd}
K'(H_r(X), r-1) \arrow[r, "k_r'(X)"] \arrow[d, "\tilde{f}_*"'] & X^{r-1} \arrow[d, "f^{r-1}"] \\
K'(H_r(Y), r-1) \arrow[r, "k_r'(Y)"] & Y^{r-1}
\end{tikzcd}$$

*which is homotopy commutative, written* $f^{r-1}{}_\# k_r'(X) = \tilde{f}_*{}^c k_r'(Y)$.

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*Proof.* We content ourselves with proving (b). Since $X/X^p$ is $p$-connected, it has a fundamental class $j(X) \in \pi_{p+1}(H_{p+1}(X), X/X^p)$. Choose $\xi(X)$ so that $\epsilon_1\xi(X) = j(X)$ and let $k_p'(X) = \partial\xi(X)$ as in the following diagram:

$$\begin{tikzcd}
j(X) \in \pi_{p+1}(H_{p+1}(X), X/X^p) \arrow[d, "(f/f^p)_\#"] & \pi_{p+1}(H_{p+1}(X); X, X^p) \arrow[l, "\epsilon_1"'] \arrow[r, "\partial"] \arrow[d, "(f, f_p)_\#"] & \pi_p(H_{p+1}(X), X^p) \arrow[d, "f^p{}_\#"] \\
\pi_{p+1}(H_{p+1}(X), Y/Y^p) & \pi_{p+1}(H_{p+1}(X); Y, Y^p) \arrow[l, "\bar\epsilon_1"'] \arrow[r, "\bar\partial"] & \pi_p(H_{p+1}(X), Y^p) \\
j(Y) \in \pi_{p+1}(H_{p+1}(Y), Y/Y^p) \arrow[u, "f_*{}^c"'] & \pi_{p+1}(H_{p+1}(Y); Y, Y^p) \arrow[l, "\epsilon_1"'] \arrow[r, "\partial"] \arrow[u, "f_*{}^c"'] & \pi_p(H_{p+1}(Y), Y^p). \arrow[u, "f_*{}^c"']
\end{tikzcd}$$

The diagram commutes by naturality and $\epsilon_1$ is onto (see [**8**, Chapter 7]). We claim $\bar\epsilon_1$ is an isomorphism: since Ext $(H_{p+1}(X), G) = 0$ for any group $G$, we get a commutative diagram (in which the upper vertical homomorphisms are isomorphisms by Theorem 3)

$$\begin{tikzcd}
\pi_{p+1}(H_{p+1}(X); Y, Y^p) \arrow[r, "\bar\epsilon_1"] \arrow[d, "\cong"] & \pi_{p+1}(H_{p+1}(X), Y/Y^p) \arrow[d, "\cong"] \\
\mathrm{Hom}\ (H_{p+1}(X), \pi_{p+1}(Y, Y^p)) \arrow[r] \arrow[d, "h_1"] & \mathrm{Hom}\ (H_{p+1}(X), \pi_{p+1}(Y/Y^p)) \arrow[d, "h_2"] \\
\mathrm{Hom}\ (H_{p+1}(X), H_{p+1}(Y, Y^p)) \arrow[r, "\cong"] & \mathrm{Hom}\ (H_{p+1}(X), H_{p+1}(Y/Y^p))
\end{tikzcd}$$

and since $H_r(Y, Y^p) = 0$ if $r \leqq p$, and $Y$ and $Y^p$ are simply connected, $h_1$ is an isomorphism. Also $H_r(Y/Y^p) = 0$ for $r \leqq p$ and $Y/Y^p \simeq Y \cup CY^p/CY^p \simeq Y \cup CY^p$ since $Y^p$ is a subcomplex of $Y$ and $CY^p$ is contractible. Now $\pi_1(Y \cup CY^p) = 0$ since $\pi_1(Y) = 0 = \pi_1(CY^p) = \pi_1(Y \cap CY^p) = \pi_1(Y^p)$ and so by van Kampen's theorem, $Y/Y^p$ is simply connected and so $h_2$ is an isomorphism. By naturality, so is $\bar\epsilon_1$.

Assume for the moment that $f_*{}^c j(Y) = (f/f^p)_\# j(X)$. Then

$$\bar\epsilon_1(f, f^p)_\#\xi(X) = (f/f^p)_\# j(X) = f_*{}^c j(Y) = f_*{}^c\epsilon_1\xi(Y) = \bar\epsilon_1 f_*{}^c\xi(Y)$$

and so $(f, f^p)_\#\xi(X) = f_*{}^c\xi(Y)$. Therefore,

$$f^p{}_\# k'(X) = f^p{}_\#\partial\xi(X) = \partial(f, f^p)_\#\xi(X) = \partial f_*{}^c\xi(Y) = f_*{}^c\partial\xi(X) = f_*{}^c k'(Y)$$

as required.

Finally, $f_*{}^c j(Y)$ and $(f/f^p)_\# j(X)$ are represented by

$K'(H_{p+1}(X/X^p), p + 1) \to X/X^p \to Y/Y^p$ and

$K'(H_{p+1}(X), p + 1) \to K'(H_{p+1}(Y), p + 1) \to Y/Y^p$

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respectively, inducing $(f/f^p)_*$ and $f_*$ in homology. Since

$$\begin{tikzcd} H_{p+1}(X) \arrow[r] \arrow[d] & H_{p+1}(Y) \arrow[d] \\ H_{p+1}(X/X^p) \arrow[r] & H_{p+1}(Y/Y^p) \end{tikzcd}$$

commutes, applying [**15**, 3.1], we get the desired equality.

It follows from Theorem 4 that *the* homology decomposition of a rational space is well-defined (up to homotopy type).

We proceed to show that if $X$ is a rational space, then $\Sigma\Omega X$ is (up to homotopy) a wedge of rational spheres.

PROPOSITION 5. *If $X$ is a rational space, so is $\Sigma\Omega X$.*

*Proof.* Since $X$ has the homotopy of a $CW$ complex, so does $\Omega X$[**12**], and hence $\Sigma\Omega X$ has the homotopy of a $CW$ complex; further, $\Omega X$ is connected and hence by [**14**, 8.5.3], $\Sigma\Omega X$ is simply connected. It follows from [**1**] that $\Omega X$ has finite type if $X$ has finite type and so we need only see that $H_n(\Sigma\Omega X, \mathbf{Z})$ is a vector space over $\mathbf{Q}$. Now, we have the usual fibration $\Omega X \hookrightarrow PX \rightarrow X$ and $X$ is simply connected. Hence by [**16**, 1.8], $\Omega X$ has rational homology and we are done using the suspension homomorphism.

LEMMA 6. *Let $X$ be a rational space. There is a simply connected $CW$ complex $Y$ with integral homology of finite type, and a map $f:Y \rightarrow X$ inducing an isomorphism or rational homology.*

*Proof.* The proof is suggested by the construction in Theorem 2.2 of [**16**]. We may assume that $X$ is a $CW$ complex. If dim $X = 2$,

$$X \simeq \bigvee_\alpha S_0{}^2 \quad (\text{where } S_0{}^n \text{ denotes } K'(\mathbf{Q}, n))$$

and we may take $Y = \bigvee_\alpha S^2$ and $f:Y \rightarrow X$ to be the "localization" map (induced by the inclusion $\mathbf{Z} \rightarrow \mathbf{Q}$) on each summand.

Suppose now that the statement is true for rational spaces of dimension $\leqq n$ and dim $X = n + 1$. Then $X$ is the cofibre of $\bigvee_\beta S_0{}^n \rightarrow X^n$ and by the inductive construction we have a cellular map

$$Y^n \xrightarrow{f'} X^n.$$

We seek to fill in the diagram

$$\begin{tikzcd} \bigvee_\beta S^n \arrow[r, dashed, "j"] \arrow[d, "f''"] & Y^n \arrow[d, "f'"] \\ \bigvee_\beta S_0{}^n \arrow[r, "i"] & X^n \end{tikzcd}$$

where $f''$ is the obvious map. Now by the inductive hypothesis and [**16**, Theorem 2.1], we have isomorphisms

$$\pi_n(Y^n) \otimes \mathbf{Q} \xrightarrow{\cong} \pi_n(Y_0{}^n) \xrightarrow[\cong]{(f_0')_\#} \pi_n(X^n)$$

where $f_0'$ denotes the localization of $f'$—see [**15**]), and so if $i_\beta$ denotes the restriction of $i$ to a rational sphere $[i_\beta] = (f_0')_\#([j_\beta] \otimes 1)$ for some $j_\beta : S^n \to Y^n$. It follows that if $j: \bigvee S^n \to Y^n$ is the obvious map, then $f'j \simeq if''$ and hence if $Y = Y^n \cup_j C(\bigvee_\beta S^n)$, that we can extend $f''$ to a map $f: Y \to X$. It is clear that $Y$ has finite type, and the fact that $f$ induces isomorphisms of rational homology is proved in [**16**]. If $X$ is infinite, take $Y = \bigcup_{n=2}^\infty Y^n$ where $Y^n$ is constructed for $X^n$ and $X = \bigcup_{n=2}^\infty X^n$.

THEOREM 7. *If $X$ is a rational space, $\Sigma\Omega X$ has the homotopy type of a wedge of rational spheres.*

*Proof.* We induct on *the* length of a homology decomposition of $\Sigma\Omega X$, which is well-defined by Theorem 4 and Proposition 5. Since the first stage is a wedge of 2-spheres, we may assume that the result is true for the $r$th stage $Z^r$, where $r \geqq 3$. Applying Lemma 6 and [**6**, Proposition 2.5] to $Z^{r+1}$, we conclude that next $k'$ invariant is zero. Thus all $k'$ invariants of $\Sigma\Omega X$ are zero and this yields the result.

COROLLARY 8. *If the cohomology suspension $\sigma^*: H^*(X) \to H^*(\Omega X)$ is injective and $X$ is rational, then $X$ has the homotopy type of a wedge of rational spheres.*

*Proof.* Let $p_1: \Sigma\Omega X \to X$ be the evaluation map. Then $p_1{}^*$ coincides with $\sigma^*$ up to a sign, and $p_1$ induces maps $p_1{}^n (n \geqq 2)$ of the homology sections by Proposition 5 and Theorem 3, and

$$(p_1{}^n)_\# k_{n+1}'(\Sigma\Omega X) = (\tilde{p}_1)_*{}^c k_{n+1}'(X).$$

Since $p_1{}^*$ is injective, so is $(\tilde{p}_1)_*{}^c$ and hence by Theorem 7

$$(\tilde{p}_1)_*{}^c k_{n+1}'(X) = (p_1{}^n)_\#(0) = 0, \quad \text{implying } k_{n+1}'(X) = 0.$$

May [**11**] has shown that ker $\sigma^*$ consists of all matric Massey products, and Berstein (unpublished) was able to show that

THEOREM 9. *If $X$ is a rational space, $\sigma^*: H(X, \mathbf{Z}) \to H^*(\Omega X, \mathbf{Z})$ is injective if and only if all Massey products in $H^*(X, \mathbf{Z})$ vanish.*

Thus Corollary 8 characterizes rational spaces in which all Massey products vanish as wedges of rational spheres. In particular, we have the dual of [**7**, 4.4]:

COROLLARY 10. *If $X$ is an $(n-1)$-connected rational space and* dim $X \leqq 3n - 2$, *$X$ has the homotopy of a wedge of rational spheres if all cup products vanish.*

*Proof.* The hypotheses imply that ker $\sigma^*$ is injective.

*Example* 11. Suppose $n \geqq 2$ and let $\iota_1, \iota_2: S^n \hookrightarrow S^n \vee S^n$ denote the obvious maps. Let $[\alpha] \in \pi_{3n-2}(S^n \vee S^n)$ denote the basic Whitehead product $[\iota_1, [\iota_1, \iota_2]]$. It is shown in [**4**] that $\alpha$ has infinite order and hence the localization (see [**16**])

$$\alpha_{(0)}: S^{3n-2}{}_{(0)} \to (S^n \vee S^n)_{(0)}$$

of $\alpha$ is non-trivial. Now $(S^n \vee S^n)_{(0)}$ is just $S^n{}_{(0)} \vee S^n{}_{(0)}$. Let $C_1$ denote the rational space $(S^n{}_{(0)} \vee S^n{}_{(0)}) \cup_{\alpha(0)} CS^{3n-2}{}_{(0)}$. Then $C_1$ is $(n-1)$-connected by the Hurewicz Theorem, dim $C_1 = 3n - 1$ and all cup products vanish in $\tilde{H}^*(C_1, \mathbf{Z})$. But $C_1$ is not a wedge of rational spheres, and thus Corollary 10 is a "best possible" result. Notice also that $C_1$ has a non-zero Massey product [**10**], and so the cohomology suspension map is not injective.

In a later paper we will prove Corollary 10 by entirely different methods, and generalize this corollary as follows:

THEOREM 12. *If $X$ is an $(n-1)$-connected rational space and* dim $X \leqq (k+2)n - 2$, cat $X \leqq k$ *if all cup products of length $k+1$ vanish.*

We will also give further examples to show that this result cannot be improved unless the dimensionality condition is weakened.

COROLLARY 13 (Berstein [**3**]). *Let $X$ be a simply connected CW complex of finite type. Then if* cat $X \leqq 1$, *$X$ has the homotopy type of a wedge of rational spheres modulo the class of finite groups.*

*Proof.* If cat $X \leqq 1$, $p_1$ has a section by [**8**, p. 209] and hence $p_1{}^*$ is injective. The result follows immediately.

*Remarks.* 1. Corollary 8 arose from an equivalent conjecture and was in fact first proved by Berstein as follows: If $p_1{}^*$ is injective, $p_{1*}$ is surjective and so

$$H_*(\Sigma\Omega X) \cong H_*(X) \oplus A$$

for some summand $A$. Write $H_*(X) = H_*(Z)$, $Z$ a wedge of spheres. Since $\Sigma\Omega X$ is a wedge of spheres by Proposition 5 and Lemma 6, we can realize the map $H_*(Z) \hookrightarrow H_*(\Sigma\Omega X)$ as a map $r: Z \to \Sigma\Omega X$. Then $p_1 r: Z \to X$ induces the identity map in homology, and since $X$ is simply connected, $r$ is a homotopy equivalence.

2. Notice that the proof of Corollary 8 above combined with May's result on the kernel of the cohomology suspension map yields a quick proof of the well-known (but tedious to prove) fact that: All (matric) Massey products vanish in a space of category one—see, e.g., [**13**].

3. The author has recently proved the dual of Theorem 9: Let $X$ be a rational space and let $\Sigma_n: \pi_n(X) \to \pi_{n+1}(\Sigma X)$ be the Freudenthal suspension homomorphism. Then

$$\Sigma_n \otimes 1_{\mathbf{Q}}: \pi_n(X) \otimes \mathbf{Q} \to \pi_{n+1}(\Sigma X) \otimes \mathbf{Q}$$

is injective if and only if all (rational) higher Whitehead products in $\pi_n(X)$

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(in the sense of [**2**]) vanish. It follows that a rational space is a product of $K(\mathbf{Q}, n)$'s if and only if all rational higher Whitehead products vanish, giving a quick proof of [**7**, 4.4]. (See [**18**] for details.)

4. Berstein's original proof of Corollary 13 used the Hilton-Milnor Theorem.

*Added in proof.* In Remark 3 above we also need $\pi_i(X) \otimes \mathbf{Q} = 0$ unless $n \leqq i \leqq 4n - 3, n \geqq 2$.

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