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Global dynamics for the stochastic nonlinear beam equations on the four-dimensional torus

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We study global-in-time dynamics of the stochastic nonlinear beam equations (SNLB) with an additive space-time white noise, posed on the four-dimensional torus. The roughness of the noise leads us to introducing a time-dependent renormalization, after which we show that SNLB is pathwise locally well-posed in all subcritical and most of the critical regimes. For the (renormalized) defocusing cubic SNLB, we establish pathwise global well-posedness below the energy space, by adapting a hybrid argument of Gubinelli-Koch-Oh-Tolomeo (2022) that combines the I-method with a Gronwall-type argument. Lastly, we show almost sure global well-posedness and invariance of the Gibbs measure for the stochastic damped nonlinear beam equations in the defocusing case.

Keywords: I-method; energy-critical; Gibbs measure; nonlinear beam equation; well-posedness; Wick renormalization

2020 Mathematics Subject Classification: 35L71; 35R60; 60H15

1. Introduction

We consider the stochastic nonlinear beam equation (SNLB) on $\mathbb{T}^4 = (\mathbb{R}/\mathbb{Z})^4$ with additive space-time white noise:

$$\begin{cases} \partial_t^2 u + \Delta^2 u \pm u^k = \xi, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^4), \end{cases} \tag{1.1}$$

where $u : \mathbb{R}_+ \times \mathbb{T}^4 \to \mathbb{R}$, Δ^2 denotes the bi-harmonic operator, $k \geq 2$ is a natural number, ξ is a space-time white-noise on $\mathbb{R}_+ \times \mathbb{T}^4$, and $\mathcal{H}^s(\mathbb{T}^4) = H^s(\mathbb{T}^4) \times H^{s-2}(\mathbb{T}^4)$. We refer to the Eq. (1.1) with '+' as defocusing and with '−' as focusing.

The deterministic beam equation appears in the literature under various names, such as the fourth-order wave equation, the extensible beam/plate equation, and the Bretherton equation. In the one-dimensional setting, it was first derived by Bretherton in [8] to describe the weak interaction between dispersive waves and it has a variety of applications in physics and mechanics; see [50] and references therein. We also refer to the non-local model derived by Woinowsky-Krieger [57] to describe the vibration of a clamped extensible beam.

Our main goal is to establish low regularity well-posedness of (1.1) on $\mathbb{T}^4$ with space-time white-noise, which is of analytical interest due to the roughness of the noise. This study on $\mathbb{T}^d$ for $d = 1, 2, 3$ was pursued in [38, 53, 54]. We also mention the results in [9, 10, 13] for a non-local version of (1.1) with multiplicative noises. For the study of the deterministic nonlinear beam equation, we refer the interested readers to [31, 48, 49] and references therein.

The main difficulty in studying (1.1) on $\mathbb{T}^4$ comes from the roughness of the noise ξ. To illustrate this, we first consider the mild formulation of (1.1):

$$u(t) = S(t)(u_0, u_1) \mp \int_0^t \frac{\sin((t-t')\Delta)}{\Delta} u^k(t')\, dt' + \Psi(t), \tag{1.2}$$

where $S(t)$ denotes the linear propagator

$$S(t)(u_0, u_1) = \cos(t\Delta)u_0 + \frac{\sin(t\Delta)}{\Delta} u_1, \tag{1.3}$$

with the understanding that $\frac{\sin(t\cdot 0)}{0} = t$, and Ψ is the stochastic convolution which solves the linear stochastic beam equation on $\mathbb{T}^4$:

$$\partial_t^2 \Psi + \Delta^2 \Psi = \xi. \tag{1.4}$$

More precisely, Ψ is given by

$$\Psi(t) = \int_0^t \frac{\sin((t-t')\Delta)}{\Delta} dW(t'), \tag{1.5}$$

where W denotes a cylindrical Wiener process on $L^2(\mathbb{T}^4)$:

$$W(t,x) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}^4} \beta_n(t) e_n(x), \tag{1.6}$$

with[1] $e_n(x) = e^{2\pi i n\cdot x}$, and $\{\beta_n\}_{n\in\mathbb{Z}^4}$ a family of mutually independent complex-valued Brownian motions conditioned to $\beta_{-n} = \overline{\beta_n}$, $n \in \mathbb{Z}^4$, with variance $\operatorname{Var}(\beta_n(t)) = t$. One can show that W lies almost surely in[2] $C^\alpha(\mathbb{R}_+; H^{-2-\varepsilon}(\mathbb{T}^4))$ for any $\alpha < \frac{1}{2}$ and $\varepsilon > 0$. Therefore, due to the two degrees of spatial smoothing of the linear beam equation, it follows that $\Psi(t) \in H^{-\varepsilon}(\mathbb{T}^4) \setminus L^2(\mathbb{T}^4)$ almost surely, for any $\varepsilon > 0$, thus it is merely a distribution. Consequently, we expect the solution u to (1.2) to also only be a distribution and thus the product u^k is classically ill-defined. To overcome this difficulty, we closely follow the work of Gubinelli-Koch-Oh [22] for wave equations (see also [47]), and construct solutions $u = \Psi + v$ which solve a suitably renormalized version of (1.1).

We now detail this renormalization procedure. We first smooth the noise ξ in (1.1) via Fourier truncation and consider the truncated stochastic convolution Ψ_N given by

$$\Psi_N(t,x) = \pi_N \Psi(t,x) = \sum_{\substack{n\in\mathbb{Z}^4 \\ |n|\le N}} e_n(x) \int_0^t \frac{\sin((t-t')|n|^2)}{|n|^2} d\beta_n(t'),$$

where π_N denotes the frequency truncation onto $\{|n| \le N\}$. Then, for each fixed $x \in \mathbb{T}^4$ and $t \ge 0$, it follows from the Ito isometry that the random variable $\Psi_N(t,x)$ is a mean-zero real-valued Gaussian random variable with variance

$$\sigma_N(t) = \mathbb{E}\big[\Psi_N^2(t,x)\big] \sim t \log N, \tag{1.7}$$

which is independent of $x \in \mathbb{T}^4$.

Let u_N be the solution to SNLB (1.1) with the regularized noise $\pi_N \xi$, which satisfies the mild formulation (1.2) with the truncated stochastic convolution Ψ_N. Motivated by (1.2), we introduce the first order expansion [6, 17, 36]:

$$u_N = \Psi_N + v_N, \tag{1.8}$$

where the remainder v_N solves the following nonlinear beam equation:

$$\partial_t^2 v_N + \Delta^2 v_N \pm \sum_{\ell=0}^k \binom{k}{\ell} \Psi_N^\ell v_N^{k-\ell} = 0. \tag{1.9}$$

[1] Here and after, we drop the harmless factor of 2π.

[2] In general, we have $W \in C^{\frac{1}{2}-}(\mathbb{R}_+; W^{-\frac{d}{2}-,\infty})$ for $d \ge 1$, which follows by Kolmogorov's continuity criterion and [23, Lemma 2.6]. Here $W^{s,r}(\mathbb{T}^4)$ denotes the usual L^r-based Sobolev spaces defined via the norm in (2.1).

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Unfortunately, due to (1.7), the monomials Ψ_N^ℓ in (1.9) do not have good limiting behavior as $N \to \infty$. Instead, we define the *Wick-ordered* power $\mathcal{W}_\sigma(\Psi_N^\ell)$ as

$$\mathcal{W}_\sigma(\Psi_N^\ell(t,x)) \stackrel{\text{def}}{=} H_\ell(\Psi_N(t,x);\sigma_N(t)), \tag{1.10}$$

where $H_\ell(x;\sigma)$ is the Hermite polynomial of degree ℓ, which can be shown to converge to a limit $\mathcal{W}_\sigma(\Psi^\ell)$ in $L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^4)))$, for any $1 \le p < \infty$ and $\varepsilon > 0$ as $N \to \infty$; see § 2.2. We then consider the Wick renormalized version of (1.9)

$$\partial_t^2 v_N + \Delta^2 v_N \pm \sum_{\ell=0}^{k} \binom{k}{\ell} \mathcal{W}_\sigma(\Psi_N^\ell) v_N^{k-\ell} = 0, \tag{1.11}$$

which converges, as $N \to \infty$, to the following equation:

$$\partial_t^2 v + \Delta^2 v \pm \sum_{\ell=0}^{k} \binom{k}{\ell} \mathcal{W}_\sigma(\Psi^\ell) v^{k-\ell} = 0. \tag{1.12}$$

Lastly, from (1.8) and (2.3) below, we can define the Wick-ordered nonlinearity $\mathcal{W}_\sigma(u_N^k)$ as

$$\mathcal{W}_\sigma(u_N^k(t,x)) \stackrel{\text{def}}{=} H_k(u_N(t,x);\sigma_N(t)) = \sum_{\ell=0}^{k} \binom{k}{\ell} \mathcal{W}_\sigma(\Psi_N^\ell(t,x)) v_N^{k-\ell}(t,x).$$

Consequently, if v_N solves (1.11), then $u_N = \Psi_N + v_N$ satisfies the following truncated Wick renormalized SNLB:

$$\partial_t^2 u_N + \Delta^2 u_N \pm \mathcal{W}_\sigma(u_N^k) = \pi_N \xi. \tag{1.13}$$

Similarly, with $u = \Psi + v$ for some suitable v, we define the Wick-ordered nonlinearity as

$$\mathcal{W}_\sigma(u^k) \stackrel{\text{def}}{=} \sum_{\ell=0}^{k} \binom{k}{\ell} \mathcal{W}_\sigma(\Psi^\ell) v^{k-\ell}, \tag{1.14}$$

and so if v solves (1.12), then $u = \Psi + v$ solves the following Wick renormalized SNLB:

$$\partial_t^2 u + \Delta^2 u \pm \mathcal{W}_\sigma(u^k) = \xi. \tag{1.15}$$

Before stating our first main result on local well-posedness of (1.15), let us discuss the scaling critical regularity associated to the deterministic nonlinear beam equation (NLB):

$$\partial_t^2 u + \Delta^2 u \pm u^k = 0. \tag{1.16}$$

On $\mathbb{R}^4$, (1.16) enjoys the following scaling symmetry: if u is a solution to (1.16) then $u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda^{\frac{4}{k-1}} u(\lambda^2 t, \lambda x)$ is also a solution to (1.16). This induces the scaling

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critical Sobolev index $s_{\text{scaling}} = 2 - \frac{4}{k-1}$, i.e., the homogeneous Sobolev $\dot{H}^s(\mathbb{R}^4)$-norm with $s = s_{\text{scaling}}$ is invariant under the scaling. Moreover, for a given integer $k \geq 2$, we define s_{crit} by

$$s_{\text{crit}} \overset{\text{def}}{=} \max(s_{\text{scaling}}, 0) = \max\left(2 - \frac{4}{k-1}, 0\right), \tag{1.17}$$

where the restriction $s_{\text{crit}} \geq 0$ appears in making sense of the powers of u. Although the scaling symmetry does not extend to $\mathbb{T}^4$, the numerology still plays an important role in predicting local well-posedness issues. In particular, our aim is to show that the SNLB (1.1) is locally well-posed in the scaling (sub)critical Sobolev spaces $\mathcal{H}^s(\mathbb{T}^4)$ with $s \geq s_{\text{crit}}$. In fact, we show pathwise local well-posedness of (1.15) in the subcritical regime for $s > s_{\text{crit}}$ and all order nonlinearities $k \geq 2$, and also in the critical case ($s = s_{\text{crit}}$) for $k \geq 4$.

THEOREM 1.1 *Given an integer $k \geq 2$, let s_{crit} be as in (1.17). Then, the Wick renormalized SNLB (1.15) is pathwise locally well-posed in $\mathcal{H}^s(\mathbb{T}^4)$ for*

$$\text{(i) } k \geq 4: \; s \geq s_{\text{crit}} \qquad \textit{or} \qquad \text{(ii) } k = 2, 3: \; s > s_{\text{crit}}.$$

More precisely, given any $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^4)$, there exists an almost surely positive stopping time $T = T(\omega, u_0, u_1) > 0$ such that there exists a unique solution u to (1.15) on $[0, T]$ with $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ and

$$u \in \Psi + C([0,T]; H^{s'}(\mathbb{T}^4)) \subset C([0,T]; H^{-\varepsilon}(\mathbb{T}^4))$$

for any $\varepsilon > 0$, where $s' = \min(s, 2 - \varepsilon)$.

The solution u in Theorem 1.1 is understood as $u = \Psi + v$ where we construct $(v, \partial_t v) \in C([0,T]; \mathcal{H}^{s'}(\mathbb{T}^4))$ with v solving the following Duhamel formulation:

$$v(t) = S(t)(u_0, u_1) \mp \int_0^t \frac{\sin((t-t')\Delta)}{\Delta} \mathcal{W}_\sigma(u^k(t'))dt', \tag{1.18}$$

for $\mathcal{W}_\sigma(u^k)$ and $S(t)$ as in (1.14) and (1.3), respectively. The main ingredient in proving Theorem 1.1 in the (almost) critical regime are the Strichartz estimates for the beam equation. In the Euclidean setting, by exploiting the formal decomposition

$$\partial_t^2 + \Delta^2 = (i\partial_t + \Delta)(-i\partial_t + \Delta),$$

which sheds light on the relation between the beam equation and the Schrödinger equation, and the analysis of oscillatory integrals, Pausader [48, 49] established Strichartz estimates for the beam equation. However, in contrast to the wave equation, the lack of finite speed of propagation poses difficulties in transferring these estimates from the Euclidean to the periodic setting. Instead, we exploit the connection between the operator $S(t)$ in (1.3) appearing in (1.18) and the free Schrödinger operators $e^{\pm it\Delta}$ via the periodic Schrödinger Strichartz estimates in [7, 27] from the ℓ^2-decoupling theory. See § 3 for details.

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Remark 1.2.

(i) In Theorem 1.1, we cannot reach the critical regularity $s = s_{\text{crit}} = 0$ for the quadratic and cubic SNLB (1.15), $k = 2, 3$. This restriction comes from the sharp Strichartz estimates for Schrödinger (see Lemma 3.1), where the endpoint $p = 3$ is not included, which is needed for our argument in the critical setting for $k = 2, 3$. Strichartz estimates for $p = 3$ are known to only hold with a derivative loss [4, 7], which prevents us from taking $s = 0$. Thus our result is sharp with respect to the method. It may be possible to reach the critical regularity in these cases by using the U^p- V^p spaces introduced in [28].

(ii) The proof of Theorem 1.1 can be easily adapted to show local well-posedness of the truncated Wick-ordered SNLB (1.13), uniformly in N. In fact, it follows that for $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^4)$, there exists an almost surely positive stopping time $T = T(\omega, u_0, u_1) > 0$ independent of N and a unique solution $u_N \in \Psi_N + C([0, T]; H^{s'}(\mathbb{T}^4))$ to (1.13). Moreover, we can show that u_N converges to the corresponding solution u to (1.15). We note that although this seems to depend on regularizing by π_N, one can consider a different regularization procedure, such as mollification. Indeed, one can show that the Wick-ordered monomials $\mathcal{W}_\sigma(\Psi^k)$ are independent of the choice of mollifier, and thus so is the renormalized nonlinearity (1.14). See [22, Remark 1.2] for further discussion.

Our next goal is to extend the solutions constructed in Theorem 1.1 globally-in-time. We restrict our attention to the defocusing case ('+' sign in (1.1)) and odd-ordered nonlinearities, as the energies corresponding to the deterministic NLB equation are sign definite in this setting. First, we construct pathwise global-in-time solutions for the cubic defocusing Wick-ordered (1.15) by adapting the hybrid method of Gubinelli-Koch-Oh-Tolomeo [24] to the beam equation. Then, we use Bourgain's invariant measure argument to show almost sure global well-posedness and invariance of the Gibbs measure for the defocusing damped Wick renormalized SNLB with odd-power nonlinearities.

We first consider the cubic Wick renormalized SNLB (1.15) in the defocusing case, with $k = 3$ and '+' sign. In Theorem 1.1, for $s > 0$ we constructed a solution $u = \Psi + v$ where the remainder v solves

$$\partial_t^2 v + \Delta^2 v + \mathcal{W}_\sigma(u^3) = 0, \tag{1.19}$$

and $\mathcal{W}_\sigma(u^3)$ is given in (1.14). A consequence of the (deterministic) contraction argument used to show Theorem 1.1 is the following (almost sure) blow-up alternative: either the solution v exists globally in time or there exists some finite time $T_* = T_*(\omega) > 0$ such that

$$\lim_{t \nearrow T_*} \|\vec{v}(t)\|_{\mathcal{H}^{s'}} = \infty, \tag{1.20}$$

where $\vec{v} = (v, \partial_t v)$ and $s' = \min(s, 2 - \varepsilon)$ for any small $\varepsilon > 0$.

To globalize solutions, we must control the growth of the norm in (1.20). In the parabolic setting, there are various results where deterministic arguments have

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been adapted to the stochastic setting to directly control the growth of norms of solutions; see [21, 37, 39, 40]. Unfortunately, for (1.15), due to the lack of a strong smoothing effect, such arguments do not apply. Instead, even in the deterministic setting, we must consider conservation laws. For the deterministic cubic nonlinear beam equation (NLB):

$$\partial_t^2 v + \Delta^2 v + v^3 = 0,$$

the associated energy

$$E(\vec{v}) = \frac{1}{2}\int_{\mathbb{T}^4} (\Delta v)^2 dx + \frac{1}{2}\int_{\mathbb{T}^4} (\partial_t v)^2 dx + \frac{1}{4}\int_{\mathbb{T}^4} v^4 dx, \tag{1.21}$$

gives control over the $\mathcal{H}^2(\mathbb{T}^4)$-norm of $\vec{v}$, as this quantity is conserved for sufficiently regular solutions. Unfortunately, when adding noise to the equation and considering a solution v to (1.19), two problems arise: (i) the energy $E(\vec{v})$ is not conserved under the dynamics of (1.19), and (ii) since $\vec{v} \in \mathcal{H}^{s'}(\mathbb{T}^4) \setminus \mathcal{H}^2(\mathbb{T}^4)$ for $s' = \min(s, 2-\varepsilon)$ for any $\varepsilon > 0$, the energy $E(\vec{v})$ is actually infinite.

In the context of the two-dimensional cubic stochastic nonlinear wave equation, Gubinelli-Koch-Oh-Tolomeo [24] introduced a new hybrid method to overcome these difficulties, by combining the I-method of Colliander-Keel-Staffilani-Takaoka-Tao [15, 16] and the Gronwall-type globalization argument by Burq-Tzvetkov [12]. See also [19, 55] for other instances of this method. To establish our next main result, we adapt this argument to show pathwise global well-posedness of (1.19).

THEOREM 1.3 *Let $s > \frac{7}{4}$. Then, the defocusing cubic Wick renormalized SNLB (1.19) is globally well-posed in $\mathcal{H}^s(\mathbb{T}^4)$. More precisely, given any $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^4)$, the solution v to the equation (1.19) exists globally in time and $(v, \partial_t v) \in C(\mathbb{R}_+; \mathcal{H}^{s'}(\mathbb{T}^4))$, almost surely, for $s' = \min(s, 2-\varepsilon)$ for any small $\varepsilon > 0$.*

We briefly detail the ideas of the proof of Theorem 1.3. For simplicity, let $\frac{7}{4} < s < 2$ so that $s' = s$. In view of the blow-up alternative (1.20), our main goal is to control the $H^s(\mathbb{T}^4)$-norm of the solution v to (1.19), where the conservation of $E(\vec{v})$ is not useful. Instead, the I-method is based on studying the growth of a modified energy obtained from $E(\vec{v})$ which controls the $H^s(\mathbb{T}^4)$-norm of v. In particular, for $N \in \mathbb{N}$, we consider $E(I\vec{v})$ where $I = I_N$ denotes the I-operator, a Fourier operator with a smooth, radially symmetric, non-increasing multiplier m_N given by

$$m_N(\xi) = \begin{cases} 1, & \text{if } |\xi| \le N, \\ \left(\frac{N}{|\xi|}\right)^{2-s}, & \text{if } |\xi| \ge 2N. \end{cases} \tag{1.22}$$

Note that $If \in H^2(\mathbb{T}^4)$ if and only if $f \in H^s(\mathbb{T}^4)$; see (4.1)-(4.2).

To study the growth of $E(I\vec{v})$, we consider the following I-SNLB:

$$\partial_t^2 Iv + \Delta^2 Iv + I\mathcal{W}_\sigma(u^3) = 0,$$

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where $\mathcal{W}_\sigma(u^3)$ is as in (1.14). Unfortunately, the modified energy $E(I\vec{v})$ is not conserved under the flow of I-SNLB, and by direct computation we obtain

$$\begin{aligned} E(I\vec{v})(t_2) - E(I\vec{v})(t_1) &= \int_{t_1}^{t_2}\int_{\mathbb{T}^4} (\partial_t Iv)\big\{ - I(v^3) + (Iv)^3\big\} dxdt' \\ &\quad - 3\int_{t_1}^{t_2}\int_{\mathbb{T}^4} (\partial_t Iv)\big\{ I(v^2\Psi) + I(v\mathcal{W}_\sigma(\Psi^2))\big\} dxdt' \\ &\quad - \int_{t_1}^{t_2}\int_{\mathbb{T}^4} (\partial_t Iv) I(\mathcal{W}_\sigma(\Psi^3)) dxdt', \end{aligned} \tag{1.23}$$

for $0 \leq t_1 < t_2$. The first term, due to the I-operator, requires a certain (deterministic) commutator estimate; see Lemma 4.1. The difficulty in the remaining contributions comes from the roughness of Ψ, which is handled by exploiting a finer regularity property of $I\Psi$ combined with commutator estimates and a Gronwall-type argument. Finally, due to the growth of the modified energy $E(I_N\vec{v})$, we iterate the argument above over time-intervals of fixed size, but with an increasing sequence N_k of parameters for the I-operator, extending the solution to (1.19) globally-in-time. See § 4 for details.

REMARK 1.4.

(i) There is a gap between the global well-posedness result for the Wick-ordered cubic SNLB (1.19) in Theorem 1.3 for $s > \frac{7}{4}$ and the local well-posedness threshold $s > 0$ from Theorem 1.1. The technical assumption of $s > \frac{7}{4}$ comes from controlling the growth of the energy (Proposition 4.5) and that of the chosen sequence of parameters N_k in a way that allows for an iterative argument (see (4.38) and (4.40)). We do not believe this restriction to be sharp, and it may be possible to improve it by refining the I-method part of the argument. However, we do not pursue this issue in this paper.

(ii) At this point, we do not know how to extend pathwise global well-posedness of the Wick ordered defocusing SNLB (1.15) to a (super-)quintic nonlinearity. As mentioned earlier, the method of proof for Theorem 1.3 is partially based on the Gronwall-type globalization argument by Burq-Tzvetkov [12], which only applies to the cubic case. Indeed, the main restriction comes from the term

$$\int_{\mathbb{T}^4} (\partial_t v)(v^2\Psi) dx,$$

appearing on the second contribution on the right-hand side of (1.23), where we dropped the I-operator for simplicity. In order to estimate this contribution by a power of the energy $E(\vec{v})$ in (1.21), by Cauchy-Schwarz inequality, we must place $\partial_t v$ in $L^2(\mathbb{T}^4)$, which implies that $v^2\Psi$ is also in $L^2(\mathbb{T}^4)$. Consequently, we obtain the L^4-norm of v, which is also controlled by the energy $E(\vec{v})$. However, one can see that the same argument fails for the analogous term for higher order nonlinearities; see, for example, the case-by-case analysis in [33, Section 5]. To deal with a (super-)quintic nonlinearity, one needs to exploit some other ideas such as those in [30, 43], but we choose not to pursue this issue in this paper.

(iii) A standard application of the I-method results in a polynomial growth bound (in time) on the Sobolev norm of a solution. See, for example, [16, Section 6]. The hybrid argument used for Theorem 1.3 yields a double exponential growth bound on the $\mathcal{H}^s$-norm of the solution; see Remark 4.6 below. It may be possible to improve this double exponential bound, but we expect that one can obtain at best a polynomial growth bound for SNLB (1.19) due to the polynomial growth (in time) of the stochastic convolution Ψ. One can compare this situation with the damped case in the next subsection, where the invariant measure argument yields a logarithmic growth bound; see remark 1.7(i) below.

Lastly, we restrict our attention to the following (defocusing) stochastic damped nonlinear beam equation (SdNLB):

$$\partial_t^2 u + \partial_t u + (1 - \Delta)^2 u + u^k = \sqrt{2}\xi, \tag{1.24}$$

for $k \in 2\mathbb{N} + 1$. By modifying the proof of Theorem 1.3, we can show global well-posedness for the damped dynamics (1.24) when $k = 3$, after renormalization, but we do not know how to extend this deterministic argument to higher nonlinearities. Instead, we consider a probabilistic approach and establish almost sure global well-posedness of (1.24) and invariance of the Gibbs measure $\vec{\rho}$ via Bourgain's invariant measure argument [5, 6], where $\vec{\rho}$ is formally given by

$$\text{“}d\vec{\rho}(u, \partial_t u) = Z^{-1} e^{-E(u, \partial_t u)} du\, d(\partial_t u)\text{”}. \tag{1.25}$$

Here, $E(u, \partial_t u)$ denotes the energy (or Hamiltonian) of the deterministic undamped defocusing NLB (1.16):

$$E(u, \partial_t u) = \frac{1}{2}\int_{\mathbb{T}^4} [(1 - \Delta)u]^2 dx + \frac{1}{2}\int_{\mathbb{T}^4} (\partial_t u)^2 dx + \frac{1}{k+1}\int_{\mathbb{T}^4} u^{k+1} dx. \tag{1.26}$$

We can understand SdNLB (1.24) as a superposition of the defocusing NLB dynamics (1.16) and the Ornstein-Uhlenbeck dynamics (for the component $\partial_t u$):

$$\partial_t(\partial_t u) = -\partial_t u + \sqrt{2} dW.$$

The latter leaves the Gibbs measure $\vec{\rho}$ invariant, which is also expected to hold under the dynamics of NLB (1.16) due to its Hamiltonian structure; see [45] and [58, Chapter 3]. Therefore, we expect $\vec{\rho}$ to also be invariant under SdNLB (1.24).

Moreover, this invariance is also inferred from the stochastic quantization viewpoint. In fact, (1.24) is the so-called canonical stochastic quantization equation of the Φ_4^{k+1}-model; see [51]. We thus refer to (1.24) as the hyperbolic Φ_4^{k+1}-model, which is of importance in constructive quantum field theory. The invariance of the Gibbs measure is also related to other applications in physics such as the study of equilibrium states, couplings of fields, and scattering of particles; see [1–3, 18, 20, 25, 52] and references therein. See also [22, 24, 35, 44, 54] for further results on wave-like Φ_d^{k+1}-models.

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Our first step is to rigorously construct the measure $\vec{\rho}$, since the expression in (1.25) is only formal. We want to define $\vec{\rho}$ as a weighted Gaussian measure of the form

$$“d\vec{\rho}(u, \partial_t u) = Z^{-1} e^{-\frac{1}{k+1}\int_{\mathbb{T}^4} u^{k+1} dx} d\vec{\mu}_2(u, \partial_t u)”, \tag{1.27}$$

where $\vec{\mu}_2 = \mu_2 \otimes \mu_0$ and μ_s denotes a Gaussian measure on periodic distributions given by

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n \in \mathbb{Z}^4} e^{-\frac{1}{2}\langle n \rangle^{2s} |\widehat{u}(n)|^2} d\widehat{u}(n), \tag{1.28}$$

for $s \in \mathbb{R}$. Note that μ_0 corresponds to the white noise measure. More precisely, $\vec{\mu}_2$ is defined as the induced probability measure under the map $\omega \in \Omega \longmapsto (X^1(\omega), X^2(\omega))$, where $X^1(\omega)$ and $X^2(\omega)$ are given by

$$X^1(\omega) = \sum_{n \in \mathbb{Z}^4} \frac{g_n(\omega)}{\langle n \rangle^2} e_n \qquad \text{and} \qquad X^2(\omega) = \sum_{n \in \mathbb{Z}^4} h_n(\omega) e_n. \tag{1.29}$$

Here, $\{g_n, h_n\}_{n \in \mathbb{Z}^4}$ denotes a family of independent standard complex-valued Gaussian random variables conditioned so that $\overline{g_n} = g_{-n}$ and $\overline{h_n} = h_{-n}$, $n \in \mathbb{Z}^4$. The main difficulty in making sense of (1.27) comes from the rough support of the base Gaussian measure $\vec{\mu}_2$, namely $\mathcal{H}^{-\varepsilon}(\mathbb{T}^4) \setminus \mathcal{H}^0(\mathbb{T}^4)$ for any $\varepsilon > 0$; see [11, Lemma B.1]. Since the typical element in the support of $\vec{\mu}_2$ is merely a distribution, the term $\int_{\mathbb{T}^4} u^{k+1}\, dx$ in (1.27) is ill-defined and a renormalization is needed in rigorously constructing $\vec{\rho}$.

Similarly to the local theory for SNLB (1.1), where we introduced a renormalization based on the logarithmically diverging variance of Ψ in (1.7), here the same difficulty appears due to the roughness of the support of $\vec{\mu}_2$. In fact, for $N \in \mathbb{N}$, the typical element X^1 in the support of μ_2 satisfies

$$\alpha_N \stackrel{\text{def}}{=} \mathbb{E}\big[\big(\pi_N X^1(x)\big)^2\big] = \sum_{\substack{n \in \mathbb{Z}^4 \\ |n| \leq N}} \frac{1}{\langle n \rangle^4} \sim \log N, \tag{1.30}$$

which is independent of both $t \in \mathbb{R}_+$ and $x \in \mathbb{T}^4$. We then define the Wick renormalized truncated potential energy

$$R_N(u) = \frac{1}{k+1} \int_{\mathbb{T}^4} \mathcal{W}_\alpha((\pi_N u)^{k+1}) dx, \tag{1.31}$$

where the Wick-ordered power $\mathcal{W}_\alpha((\pi_N u)^{k+1})$ is defined by

$$\mathcal{W}_\alpha((\pi_N u)^{k+1}(t,x)) \stackrel{\text{def}}{=} H_{k+1}(\pi_N u(t,x); \alpha_N). \tag{1.32}$$

One can show that $\{R_N\}_{N\in\mathbb{N}}$ forms a Cauchy sequence in $L^p(\mu_2)$ for any finite $p \geq 1$, from which we conclude that there exists a limiting random variable $R(u)$ given by

$$\lim_{N\to\infty} R_N(u) \overset{\text{def}}{=} R(u) = \frac{1}{k+1}\int_{\mathbb{T}^4} \mathcal{W}_\alpha(u^{k+1}(x))dx. \tag{1.33}$$

See [46, Proposition 1.1] and [32, Proposition 3.4] for details. We then construct the Gibbs measure $\vec{\rho}$ as the limit of the following truncated Gibbs measures

$$d\vec{\rho}_N(u, \partial_t u) = Z_N^{-1} e^{-R_N(u)} d\vec{\mu}_2(u, \partial_t u). \tag{1.34}$$

PROPOSITION 1.5. *Given any $1 \leq p < \infty$, we have*

$$\lim_{N\to\infty} e^{-R_N(u)} = e^{-R(u)} \qquad \text{in } L^p(\mu_2). \tag{1.35}$$

Consequently, the truncated Gibbs measure $\vec{\rho}_N$ in (1.34) converges, in the sense of (1.35), to a limiting Gibbs measure $\vec{\rho}$ given by

$$d\vec{\rho}(u, \partial_t u) = Z^{-1} e^{-R(u)} d\vec{\mu}_2(u, \partial_t u). \tag{1.36}$$

We now sketch the proof of Proposition 1.5. From an application of Nelson's estimate, we obtain uniform in N integrability of the truncated density; for any $1 \leq p < \infty$,

$$\sup_{N\in\mathbb{N}} \big\|e^{-R_N(u)}\big\|_{L^p(\mu_2)} < \infty. \tag{1.37}$$

See, for example, [46, Proposition 1.2] and [32, Proposition 3.6]. Combining the uniform bound (1.37) with a convergence in measure deduced from (1.33), we obtain (1.35); see, for example, [56, Remark 3.8] and [32, (3.32)]. This allows us to construct the Gibbs measure $\vec{\rho}$ in (1.36), which is mutually absolutely continuous with respect to the base Gaussian measure $\vec{\mu}_2$.

We can now consider the dynamical problem for the Φ_4^{k+1}-model (1.24). In particular, we consider the following truncated Wick renormalized SdNLB

$$\partial_t^2 u_N + \partial_t u_N + (1-\Delta)^2 u_N + \pi_N\big(\mathcal{W}_\alpha((\pi_N u)^k)\big) = \sqrt{2}\xi, \tag{1.38}$$

and show almost sure global well-posedness and invariance of the Gibbs measure $\vec{\rho}$ for the limiting equation:

$$\partial_t^2 u + \partial_t u + (1-\Delta)^2 u + \mathcal{W}_\alpha(u^k) = \sqrt{2}\xi. \tag{1.39}$$

THEOREM 1.6 *Let $k \in 2\mathbb{N}+1$. The Wick renormalized SdNLB (1.39) is almost surely globally well-posed with respect to the Gibbs measure $\vec{\rho}$ in (1.36) and the Gibbs measure $\vec{\rho}$ is invariant under the dynamics. More precisely, there exists a non-trivial stochastic process $(u, \partial_t u) \in C(\mathbb{R}_+; \mathcal{H}^{-\varepsilon}(\mathbb{T}^4))$ for any $\varepsilon > 0$ such that, given any $T > 0$, the solution $(u_N, \partial_t u_N)$ to the renormalized truncated SdNLB (1.38) with random initial data $(u_N, \partial_t u_N)|_{t=0}$ distributed according to the truncated Gibbs measure $\vec{\rho}_N$ in (1.34), converges in probability to some stochastic process $(u, \partial_t u)$ in*

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$C([0,T];\mathcal{H}^{-\varepsilon}(\mathbb{T}^4))$. Moreover, the law of $(u(t),\partial_t u(t))$ is given by the renormalized Gibbs measure $\vec{\rho}$ in (1.36) for any $t\geq 0$.

By using Bourgain's invariant measure argument, due to the convergence of $\vec{\rho}_N$ to $\vec{\rho}$, Theorem 1.6 follows once we construct the limiting process $(u,\partial_t u)$ locally-in-time with a good approximation property for the solution u_N to (1.38) and establish invariance of the truncated measures $\vec{\rho}_N$ under (1.38). The former follows from adapting the proof of Theorem 1.1 to the damped models (1.38)-(1.39), while the latter exploits the Hamiltonian structure of the truncated system (1.38). See § 5 for details.

Remark 1.7.

(i) Let $(u,\partial_t u)$ be the limiting process constructed in Theorem 1.3. Then, as a consequence of Bourgain's invariant measure argument, one can obtain the following logarithmic growth bound (in time):

$$\|(u(t),\partial_t u(t))\|_{\mathcal{H}^{-\varepsilon}}\leq C(\omega)\big(\log(1+t)\big)^{\frac{k}{2}},$$

for any $t\geq 0$. For details, see [44].

(ii) The local well-posedness in Theorem 1.1 can be easily adapted to the Wick renormalized SNLB with damped massive linear part $\partial_t^2 u+\partial_t u+(1-\Delta)^2 u$, which we detail in § 5. We choose to consider the massive linear part $(1-\Delta)^2$ instead of Δ^2 to avoid a problem at the zero-th frequency when constructing the Gibbs measure $\vec{\rho}$, as in [24, 46].

(iii) The Gaussian measure μ_2 is the log-correlated Gaussian free field on $\mathbb{T}^4$ studied in [45], and thus the SdNLB dynamics (1.24) are associated with this log-correlated Gibbs measure. Our construction of $\vec{\rho}$ in (1.36) is valid for $k\in 2\mathbb{N}+1$ and with a plus sign in front of the potential energy in (1.26). However, in the case of a focusing quartic interaction (i.e., with a minus sign in front of the potential energy and $k=3$ in (1.26)), the authors in [45] obtained a non-normalizability result for the corresponding measure and established its exact divergence rate; see [45, Theorem 1.4].

2. Preliminaries

In this section, we introduce notations and recall basic lemmas. For $a,b>0$, we use $a\lesssim b$ to denote that there exists a constant $C>0$ such that $a\leq Cb$. We write $a\sim b$ if $a\lesssim b$ and $b\lesssim a$. When writing the norm of a space-time function, we usually use short-hand notation, such as $L^q_I L^r_x = L^q(I;L^r(\mathbb{T}^4))$ for a given time interval $I\subset\mathbb{R}_+$. We will also use the notation $L^q_T L^r_x = L^q([0,T];L^r(\mathbb{T}^4))$ for $T>0$.

2.1. Deterministic tools

We first introduce some function spaces. For $s\in\mathbb{R}$, we define the L^2-based Sobolev space $H^s(\mathbb{T}^4)$ via the norm:

$$\|f\|_{H^s}=\big\|\langle n\rangle^s\widehat{f}(n)\big\|_{\ell^2_n},$$

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where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ and $\widehat{f}$ denotes the spatial Fourier transform of f. For $1 \le p \le \infty$, we define the L^p-based Sobolev space $W^{s,p}(\mathbb{T}^4)$ via the norm:

$$\|f\|_{W^{s,p}} = \big\| \mathcal{F}^{-1}\big(\langle n \rangle^s \widehat{f}(n)\big)\big\|_{L^p}, \tag{2.1}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Note that $H^s(\mathbb{T}^4) = W^{s,2}(\mathbb{T}^4)$.

We now introduce notation for Littlewood-Paley projections. Let $\phi : \mathbb{R} \to [0,1]$ be a smooth bump function with supp$\phi \subset [-\frac{8}{5}, \frac{8}{5}]$ and $\phi \equiv 1$ on $[-\frac{5}{4}, \frac{5}{4}]$. For $\xi \in \mathbb{R}^4$, we define

$$\varphi_1(\xi) = \phi(|\xi|) \qquad \text{and} \qquad \varphi_N(\xi) = \phi\Big(\frac{|\xi|}{N}\Big) - \phi\Big(\frac{2|\xi|}{N}\Big),$$

for $N \ge 2$ a dyadic number. For a dyadic number $N \ge 1$, we define the Littlewood-Paley projector $\mathbf{P}_N$ as the Fourier multiplier operator with the symbol φ_N. Then,

$$f = \sum_{N \ge 1 \ \text{dyadic}} \mathbf{P}_N f.$$

We also write

$$\mathbf{P}_{\le N} f = \sum_{1 \le M \le N \ \text{dyadic}} \mathbf{P}_M f.$$

Next, we recall the following Christ–Kiselev lemma. For a proof, see [14, 26].

LEMMA 2.1. *Let X, Y be Banach spaces and $K(s,t) : X \to Y$ be an operator-valued kernel from X to Y. Suppose that we have the estimate*

$$\Big\| \int_{-\infty}^{t_0} K(s,t) f(s)\, ds \Big\|_{L^q([t_0,\infty);Y)} \lesssim \|f\|_{L^p(\mathbb{R};X)},$$

for some $1 \le p < q \le \infty$, all $t_0 \in \mathbb{R}$, and all $f \in L^p\big((-\infty, t_0); X\big)$. Then, we have

$$\Big\| \int_{-\infty}^{t} K(s,t) f(s)\, ds \Big\|_{L^q(\mathbb{R};Y)} \lesssim \|f\|_{L^p(\mathbb{R};X)}.$$

Note that the assumption in the above lemma is satisfied in particular if we have

$$\Big\| \int_{\mathbb{R}} K(s,t) f(s)\, ds \Big\|_{L^q(\mathbb{R};Y)} \lesssim \|f\|_{L^p(\mathbb{R};X)}.$$

Lastly, we recall the following product estimates. See for example [22, Lemma 3.4].

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LEMMA 2.2. *Let $0 \leq s \leq 1$.*
(i) Suppose that $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. Then, we have

$$\|\langle\nabla\rangle^{-s}(fg)\|_{L^r(\mathbb{T}^4)} \lesssim \|f\|_{L^{p_1}(\mathbb{T}^4)}\|\langle\nabla\rangle^{s}(g)\|_{L^{q_1}(\mathbb{T}^4)} + \|\langle\nabla\rangle^{-s}(f)\|_{L^{p_2}(\mathbb{T}^4)}\|g\|\|\langle\nabla\rangle^{-s}(fg)\|_{L^{q_2}(\mathbb{T}^4)}.$$

(ii) Suppose that $1 < p, q, r < \infty$ satisfy the scaling condition $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{s}{4}$. Then, we have

$$\|\langle\nabla\rangle^{-s}(fg)\|_{L^r(\mathbb{T}^4)} \lesssim \|\langle\nabla\rangle^{-s}f\|_{L^p(\mathbb{T}^4)}\|\langle\nabla\rangle^{s}g\|_{L^q(\mathbb{T}^4)}.$$

2.2. Tools from stochastic analysis

In the following, we first review some basic facts on Hermite polynomials. See, for example, [29, 41]. We define the kth Hermite polynomial $H_k(x;\sigma)$ with variance $\sigma > 0$ via the following generating function:

$$e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma), \tag{2.2}$$

for $t, x \in \mathbb{R}$. We list the first few Hermite polynomials for readers' convenience:

$$H_0(x;\sigma) = 1, \qquad H_1(x;\sigma) = x, \qquad H_2(x;\sigma) = x^2 - \sigma,$$
$$H_3(x;\sigma) = x^3 - 3\sigma x, \qquad H_4(x;\sigma) = x^4 - 6\sigma x^2 + 3\sigma^2.$$

From (2.2), we obtain the following identities for any $k \in \mathbb{N}$ and $x, y \in \mathbb{R}$:

$$H_k(x+y;\sigma) = \sum_{\ell=0}^{k} \binom{k}{\ell} x^{k-\ell} H_\ell(y;\sigma). \tag{2.3}$$

We now recall the regularities of the stochastic convolutions and their Wick-powers introduced in § 1. Let Ψ be the stochastic convolution defined in (1.5) and Ψ^{d} be the stochastic convolution associated with SdNLB (1.24), namely the solution to the linear stochastic damped beam equation:

$$\begin{cases} \partial_t^2 \Psi^{\mathrm{d}} + \partial_t \Psi^{\mathrm{d}} + (1-\Delta)^2 \Psi^{\mathrm{d}} = \sqrt{2}\xi \\ (\Psi^{\mathrm{d}}, \partial_t \Psi^{\mathrm{d}})|_{t=0} = (u_0^\omega, u_1^\omega), \end{cases} \tag{2.4}$$

with initial data with law $\mathcal{L}(u_0^\omega, u_1^\omega) = \vec{\mu}_2$. See § 5 for further details on Ψ^{d}. Then, using standard stochastic analysis with the Wiener chaos estimate, we have the following regularity and convergence result. For an analogous proof, we refer the readers to [34, Lemma 2.1]. See also [22, 24].

LEMMA 2.3. *Let $Z = \Psi$ or Ψ^{d}, $\ell \in \mathbb{N}$, $T > 0$, and $1 \leq p < \infty$. For $\mathcal{W}(Z_N^\ell) = \mathcal{W}((\pi_N Z)^\ell)$ denoting the truncated Wick power defined in (1.10) or (1.32), respectively. Then, $\{\mathcal{W}(Z_N^\ell)\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p\big(\Omega; C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^4))\big)$.*

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Moreover, denoting the limit by $\mathcal{W}(Z^\ell)$, we have $\mathcal{W}(Z^\ell) \in C\big([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^4)\big)$ almost surely, with the following tail estimate for any $1 \le q < \infty$, $T \ge 1$, and $\lambda > 0$:

$$P\Big(\|\mathcal{W}(Z^\ell)\|_{L^q_T W^{-\varepsilon,\infty}_x} > \lambda\Big) \le C \exp\bigg(- c \frac{\lambda^{\frac{2}{\ell}}}{T^{1+\frac{2}{q\ell}}}\bigg).$$

When $q = \infty$, we also have the following tail estimate:

$$P\Big(\|\mathcal{W}(Z^\ell)\|_{L^\infty([j,j+1]; W^{-\varepsilon,\infty}_x)} > \lambda\Big) \le C \exp\bigg(- c \frac{\lambda^{\frac{2}{\ell}}}{j+1}\bigg) \tag{2.5}$$

for any $j \in \mathbb{Z}_{\ge 0}$ and $\lambda > 0$.

In order to prove Theorem 1.3, we need the following finer regularity property of Ψ. For a proof, see [24, Lemma 2.4].

Lemma 2.4. *Let Ψ be as in (1.5) and fix $0 < s < 2$. Then, given any $x \in \mathbb{T}^4$ and $t \in \mathbb{R}_+$, $I\Psi(t,x)$ is a mean-zero Gaussian random variable with variance bounded by $C_0 t \log N$, where the constant C_0 is independent of $x \in \mathbb{T}^4$ and $t \in \mathbb{R}_+$.*

3. Local well-posedness of SNLB

In this section, we show pathwise local well-posedness of the Wick renormalized SNLB (1.15) in Theorem 1.1. In § 3.1, we show homogeneous and inhomogeneous Strichartz estimates for the linear beam operators appearing in the mild formulation (1.18). We then apply these in § 3.2 to show Theorem 1.1 via a contraction mapping argument.

3.1. Strichartz estimates

To obtain Strichartz estimates for the beam equation, we need the following sharp Strichartz estimates for the linear Schrödinger propagator $e^{\pm it\Delta}$ due to Bourgain-Demeter [7] and Killip-Vişan [27].

Lemma 3.1. *Let $3 < p \le \infty$ and $N \ge 1$ be a dyadic integer. Then, we have*

$$\big\|e^{\pm it\Delta}\mathbf{P}_{\le N} f\|_{L^p_{t,x}([0,1]\times\mathbb{T}^4)} \lesssim N^{2-\frac{6}{p}} \|f\|_{L^2_x(\mathbb{T}^4)}.$$

For $1 \le q, r < \infty$, we define the index $s_{q,r}$ as follows

$$s_{q,r} \overset{\text{def}}{=} 2 - \tfrac{2}{q} - \tfrac{4}{r}. \tag{3.1}$$

We then obtain the following estimate.

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LEMMA 3.2. *For* $3 < q \le r < \infty$ *and* $s_{q,r}$ *in* (3.1), *we have*

$$\|e^{\pm it\Delta} f\|_{L_t^q([0,1];L_x^r(\mathbb{T}^4))} \lesssim \|f\|_{H^{s_{q,r}}(\mathbb{T}^4)}.$$

Proof. Let $I = [0,1]$. We start by writing $f = \sum_N \mathbf{P}_N f$ from Littlewood-Paley decomposition. From Bernstein's inequality and Lemma 3.1, we have

$$\|e^{\pm it\Delta}\mathbf{P}_N f\|_{L_I^q L_x^r} \lesssim N^{\frac{4}{q}-\frac{4}{r}} \|e^{\pm it\Delta}\mathbf{P}_{\le 2N}(\mathbf{P}_N f)\|_{L_I^q L_x^q} \lesssim N^{2-\frac{2}{q}-\frac{4}{r}} \|\mathbf{P}_N f\|_{L_x^2}.$$

Using the Littlewood-Paley theorem, Minkowski's inequality, and the above estimate, we obtain

$$\begin{aligned}
\|e^{\pm it\Delta} f\|_{L_I^q L_x^r} &\lesssim \Big(\sum_{N\ge 1 \text{ dyadic}} \|e^{\pm it\Delta}\mathbf{P}_N f\|_{L_I^q L_x^r}^2 \Big)^{1/2} \\
&\lesssim \Big(\sum_{N\ge 1 \text{ dyadic}} N^{2s_{q,r}} \|\mathbf{P}_N f\|_{L_x^2}^2 \Big)^{1/2} \\
&\sim \|f\|_{H^{s_{q,r}}}.
\end{aligned}$$

as desired. □

From the definition of the linear beam operator $S(t)$ in (1.3), the fact that $e^{\pm it\Delta}$ are isometries in $H^s(\mathbb{T}^4)$ for any $s \in \mathbb{R}$, and Lemma 3.2, we obtain the following homogeneous Strichartz estimates for the linear beam operator.

LEMMA 3.3. *Let $S(t)$ be the linear operator in (1.3),* $0 < T \le 1$, $3 < q \le r < \infty$, *and* $s \ge s_{q,r}$. *Then, we have*

$$\|S(t)(u_0,u_1)\|_{L_T^\infty H_x^s} + \|S(t)(u_0,u_1)\|_{L_T^q L_x^r} \lesssim \|(u_0,u_1)\|_{\mathcal{H}^s}.$$

We now establish the following inhomogeneous Strichartz estimate, using a TT^*-argument.

LEMMA 3.4. *For* $3 < q \le r < \infty$ *and* $s_{q,r}$ *as in* (3.1), *we have*

$$\Big\| \int_0^t \frac{\sin((t-t')\Delta)}{\Delta} F(t') dt' \Big\|_{L_t^q([0,1];L_x^r(\mathbb{T}^4))} \lesssim \|F\|_{L_t^1 H_x^{s_{q,r}-2}([0,1]\times\mathbb{T}^4)}.$$

Proof. Note that the zero-th frequency of F can be estimated easily, and so we can assume that F has mean zero below. Let $I = [0,1]$. First note that

$$\begin{aligned}
&\Big\| \int_0^t \frac{\sin((t-t')\Delta)}{\Delta} F(t')\, dt' \Big\|_{L_I^q L_x^r} \\
&\qquad \lesssim \Big\| \int_0^t \frac{e^{i(t-t')\Delta}}{\Delta} F(t')\, dt' \Big\|_{L_I^q L_x^r} + \Big\| \int_0^t \frac{e^{-i(t-t')\Delta}}{\Delta} F(t')\, dt' \Big\|_{L_I^q L_x^r}.
\end{aligned}$$

Thus, we focus on estimating the first term, as the estimate for the second term follows from an analogous strategy. The operator T defined by $Tu_0 = e^{it\Delta}u_0$ is a

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bounded operator $T: H^{sq,r} \to L^q_I L^r_x$ from Lemma 3.2. Note that we have

$$\begin{aligned}\langle Tu_0, G\rangle_{t,x} &= \int_0^1 \int_{\mathbb{T}^4} e^{it\Delta} u_0(x) \overline{G(t,x)}\, dx\, dt \\ &= \sum_{n\in\mathbb{Z}^4} \widehat{u}_0(n) \overline{\int_0^1 e^{-it|n|^2} \widehat{G}(t,n)\, dt} = \langle u_0, T^*G\rangle_x,\end{aligned}$$

where the dual operator T^* is given by

$$T^*G = \int_0^1 e^{-it\Delta} G(t,\cdot)\, dt,$$

which in turn is bounded from $L^{q'}_I L^{r'}_x$ to $H^{-sq,r}_x$. From the trivial boundness of $T: H^s_x \to L^\infty_I H^s_x$ for any $s \in \mathbb{R}$, we conclude that $T^*: L^1_I H^{sq,r}_x \to H^{sq,r}_x$ is also bounded. Consequently, we have that $TT^*: L^1_I H^{sq,r}_x \to L^q_t L^r_x$ and

$$TT^*G = \int_0^1 e^{i(t-t')\Delta} G(t',\cdot)\, dt'.$$

From the Christ–Kiselev lemma (Lemma 2.1), we get that

$$\left\| \int_0^t e^{i(t-t')\Delta} G(t',\cdot)\, dt' \right\|_{L^q_I L^r_x} \lesssim \|G\|_{L^1_I H^{sq,r}_x},$$

and by choosing $G = \frac{1}{\Delta} F$, we obtain the intended result. □

3.2. Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1 by constructing a solution $u = \Psi + v$ where Ψ denotes the stochastic convolution solving (1.4) and the remainder v solves (1.12). In particular, we consider the following mild formulation for v:

$$v(t) = S(t)(u_0,u_1) \mp \int_0^t \frac{\sin((t-t')\Delta)}{\Delta} \sum_{\ell=0}^k \binom{k}{\ell} \Xi_\ell\, v^{k-\ell}(t') dt' \tag{3.2}$$

for given initial data (u_0,u_1) and a source $(\Xi_0, \Xi_1, \ldots, \Xi_k)$ with the understanding that $\Xi_0 \equiv 1$, where $S(t)$ is the linear propagator as defined in (1.3). Given $s, \varepsilon \in \mathbb{R}$, we define the space $\mathcal{X}^{s,\varepsilon}(\mathbb{T}^4) = \mathcal{H}^s(\mathbb{T}^4) \times \big(C([0,1]; W^{-\varepsilon,\infty}(\mathbb{T}^4))\big)^{\otimes k}$ with the following norm for $\boldsymbol{\Xi} = (u_0, u_1, \Xi_1, \ldots, \Xi_k) \in \mathcal{X}^{s,\varepsilon}(\mathbb{T}^4)$:

$$\|\boldsymbol{\Xi}\|_{\mathcal{X}^{s,\varepsilon}} = \|(u_0,u_1)\|_{\mathcal{H}^s} + \sum_{j=1}^k \|\Xi_j\|_{C([0,1];W^{-\varepsilon,\infty})}.$$

Moreover, we introduce our solution space $X^{s,q,r}(T)$ for $s \in \mathbb{R}$ and $1 \le q, r \le \infty$:

$$X^{s,q,r}(T) \overset{\text{def}}{=} C([0,T]; H^s(\mathbb{T}^4)) \cap L^q([0,T]; L^r(\mathbb{T}^4)).$$

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The local well-posedness in Theorem 1.1 follows from local well-posedness of (3.2) and Lemma 2.3, which states that the random enhanced data set $(u_0, u_1, \Psi, \mathcal{W}_\sigma(\Psi^2), \ldots, \mathcal{W}_\sigma(\Psi^k))$ almost surely belongs to $\mathcal{X}^{s,\varepsilon}(\mathbb{T}^4)$ for any $\varepsilon > 0$. We then show the following deterministic result for (3.2).

Proposition 3.5. *Given an integer $k \geq 2$, let s_{crit} be as defined in (1.17). Then, the mild formulation (3.2) is locally well-posed in $\mathcal{X}^{s,\varepsilon}(\mathbb{T}^4)$ for*

$$\text{(i) } k \geq 4: \ \ s \geq s_{\text{crit}} \qquad \textit{or} \qquad \text{(ii) } k = 2, 3: \ \ s > s_{\text{crit}},$$

and $\varepsilon > 0$ sufficiently small. More precisely, given an enhanced data set

$$\Xi = (u_0, u_1, \Xi_1, \ldots, \Xi_k) \in \mathcal{X}^{s,\varepsilon}(\mathbb{T}^4),$$

there exist $T = T(\Xi) \in (0, 1]$ and a unique solution v to the mild formulation (3.2) in the class $X^{s',q,r}(T)$ for $s' = \min(s, 2-\varepsilon)$ and for some appropriate $1 \leq q, r \leq \infty$.

Proof. We define the map Γ by

$$\begin{aligned}
\Gamma[v](t) &\stackrel{\text{def}}{=} S(t)(u_0, u_1) \mp \int_0^t \frac{\sin((t-t')\Delta)}{\Delta} \sum_{\ell=0}^{k} \binom{k}{\ell} (\Xi_\ell v^{k-\ell})(t') dt' \\
&\stackrel{\text{def}}{=} S(t)(u_0, u_1) \mp \sum_{\ell=0}^{k} \binom{k}{\ell} \mathcal{I}\big(\Xi_\ell v^{k-\ell}\big)(t),
\end{aligned} \tag{3.3}$$

and consider the following three cases.

<u>Case 1:</u> $k \geq 4$ <u>and</u> $s > s_{\text{crit}}$.

Let $\varepsilon > 0$ sufficiently small and $(q, r) = (k-1+\theta, 2k-2)$ for $\theta > 0$ such that $s' \geq s_{q,r} > s_{\text{crit}}$ for $s_{q,r}$ in (3.1). For $\ell = 0$, by Lemma 3.4, Sobolev's inequality and Hölder's inequality, we obtain

$$\begin{aligned}
\big\|\mathcal{I}\big(v^k\big)\big\|_{X^{s',q,r}(T)} &\lesssim \|v^k\|_{L^1_T H^{s'-2}_x} \lesssim \|v^k\|_{L^1_T L_x^{\frac{4}{4-s'}}} \lesssim \|v\|_{L^\infty_T L_x^{\frac{4}{2-s'}}} \|v^{k-1}\|_{L^1_T L^2_x} \\
&\lesssim \|v\|_{L^\infty_T H^{s'}_x} \|v\|^{k-1}_{L^{k-1}_T L^{2k-2}_x} \lesssim T^\eta \|v\|^k_{X^{s',q,r}(T)}
\end{aligned} \tag{3.4}$$

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for some $\eta > 0$. For $1 \le \ell \le k-1$, proceeding as before, with Lemma 2.2 (ii) and Lemma 2.2 (i) repetitively, we obtain

$$\begin{aligned}
\big\|\mathcal{I}\big(\Xi_\ell v^{k-\ell}\big)\big\|_{X^{s',q,r}(T)} &= \big\|\langle\nabla\rangle^{-\varepsilon}\big(\Xi_\ell v^{k-\ell}\big)\big\|_{L^1_T H^{s'-2+\varepsilon}_x} \\
&\lesssim \big\|\langle\nabla\rangle^{-\varepsilon}\big(\Xi_\ell v^{k-\ell}\big)\big\|_{L^1_T L_x^{\frac{4}{4-s'-\varepsilon}}} \\
&\lesssim \|\langle\nabla\rangle^{-\varepsilon}\Xi_\ell\|_{L^\infty_T L_x^{\frac{4}{\varepsilon}}} \big\|\langle\nabla\rangle^{\varepsilon} v^{k-\ell}\big\|_{L^1_T L_x^{\frac{4}{4-s'-\varepsilon}}} \\
&\lesssim \|\Xi_\ell\|_{L^\infty_T W_x^{-\varepsilon,\infty}} \|\langle\nabla\rangle^{\varepsilon} v\|_{L^\infty_T L_x^{\frac{4}{2-s'+\varepsilon}}} \|v\|^{k-\ell-1}_{L_T^{k-\ell-1} L_x^{\frac{2(k-\ell-1)}{1-\varepsilon}}} \\
&\lesssim \|\Xi_\ell\|_{L^\infty_T W_x^{-\varepsilon,\infty}} \|v\|_{L^\infty_T H^{s'}_x} T^\eta \|v\|^{k-\ell-1}_{L^q_T L^r_x} \\
&\lesssim T^\eta \|\Xi_\ell\|_{L^\infty_T W_x^{-\varepsilon,\infty}} \|v\|^{k-\ell}_{X^{s',q,r}(T)}
\end{aligned} \tag{3.5}$$

for some $\eta > 0$ and $\varepsilon > 0$ sufficiently small. Lastly, for $\ell = k$, by Lemma 3.4, since $s' < 2$, we have

$$\big\|\mathcal{I}(\Xi_k)\big\|_{X^{s',q,r}(T)} \lesssim \|\Xi_k\|_{L^1_T H^{s'-2}_x} \lesssim T\|\Xi_k\|_{L^\infty_T W_x^{-\varepsilon,\infty}}. \tag{3.6}$$

By Lemma 3.3, (3.3), (3.4), (3.5), and (3.6), we have

$$\|\Gamma[v]\|_{X^{s',q,r}(T)} \lesssim \|\Xi\|_{\mathcal{X}^{s,\varepsilon}} + T^\eta\big[\|\Xi\|^k_{\mathcal{X}^{s,\varepsilon}} + \|v\|^k_{X^{s',q,r}(T)}\big].$$

A straightforward modification of the above steps yields the following difference estimate:

$$\begin{aligned}
\|\Gamma[v_1] - \Gamma[v_2]\|_{X^{s',q,r}(T)} \lesssim T^\eta\big[&\|\Xi\|^k_{\mathcal{X}^{s,\varepsilon}} + \|v_1 - v_2\|_{X^{s',q,r}(T)} \\
&\big(\|v_1\|^{k-1}_{X^{s',q,r}(T)} + \|v_2\|^{k-1}_{X^{s',q,r}(T)}\big)\big].
\end{aligned}$$

Then, by $T = T(\|\Xi\|_{\mathcal{X}^{s,\varepsilon}}) > 0$ sufficiently small, the local well-posedness of (3.2) on $[0, T]$ follows from a contraction mapping argument.

Case 2: $k = 2, 3$ and $s > s_{\text{crit}} = 0$.

In this case, we take $(q, r) = (3 + \theta, 3 + \theta)$ for $0 < \theta \le \frac{3s'}{2-s'}$ which guarantees that $s' \ge s_{q,r}$. For $\ell = 0$, proceeding as in (3.4), we have

$$\begin{aligned}
\big\|\mathcal{I}(v^k)\big\|_{X^{s',q,r}(T)} &\lesssim \|v^k\|_{L^1_T H^{s'-2}_x} \lesssim \|v^k\|_{L^1_T L_x^{\frac{4}{4-s'}}} \\
&\lesssim \|v\|_{L^k_T L_x^{\frac{4k}{4-s'}}} \lesssim T^\eta \|v\|_{X^{s',q,r}(T)},
\end{aligned} \tag{3.7}$$

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for some $\eta > 0$, since $\frac{4k}{4-s'} \leq \frac{6}{2-s'}$ for $k = 2, 3$. For $1 \leq \ell \leq k-1$, noticing that $\frac{2(k-\ell-1)}{1-\varepsilon} < 3 + \theta$ for $k \leq 3$ and $\varepsilon > 0$ sufficiently small, we proceed as in (3.5) to obtain

$$\big\| \mathcal{I}\big(\Xi_\ell v^{k-\ell}\big) \big\|_{X^{s',q,r}(T)} = T^\eta \|\Xi_\ell\|_{L_T^\infty W_x^{-\varepsilon,\infty}} \|v\|_{X^{s',q,r}(T)}^{k-\ell} \tag{3.8}$$

for some $\eta > 0$. By Lemma 3.3, (3.3), (3.7), (3.8), and (3.6), we have

$$\|\Gamma[v]\|_{X^{s',q,r}(T)} \lesssim \|\Xi\|_{\mathcal{X}^{s,\varepsilon}} + T^\eta \|\Xi\|_{\mathcal{X}^{s,\varepsilon}}^k + T^\eta \|v\|_{X^{s',q,r}(T)}^k.$$

Similar steps yield a difference estimate and we conclude the argument as in Case 1.
Case 3: $k \geq 4$ and $s = s_{\text{crit}}$.
In this case, we take $(q, r) = (k, \frac{2k(k-1)}{k+1})$ so that $s' = s = s_{\text{crit}} = s_{q,r}$ and $q, r > 3$. By proceeding as in Case 1, the estimates (3.5) and (3.6) hold, but we can only show (3.4) without the gain of T^η on the right-hand side. Thus, we have

$$\begin{aligned} \|\Gamma[v]\|_{L_T^\infty H_x^{s'}} \leq C\|(u_0,u_1)\|_{\mathcal{H}^s} + CT^\eta \|\Xi\|_{\mathcal{X}^{s,\varepsilon}} \|v\|_{L_T^\infty H_x^{s'}} \sum_{\ell=0}^{k-2} \|v\|_{L_T^q L_x^r}^\ell \\ + CT\|\Xi\|_{\mathcal{X}^{s,\varepsilon}} + C\|v\|_{L_T^q L_x^r}^k, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \|\Gamma[v]\|_{L_T^q L_x^r} \leq C\|S(t)(u_0,u_1)\|_{L_T^q L_x^r} + CT^\eta \|\Xi\|_{\mathcal{X}^{s,\varepsilon}} \|v\|_{L_T^\infty H_x^{s'}} \sum_{\ell=0}^{k-2} \|v\|_{L_T^q L_x^r}^\ell \\ + CT\|\Xi\|_{\mathcal{X}^{s,\varepsilon}} + C\|v\|_{L_T^q L_x^r}^k, \end{aligned} \tag{3.10}$$

for some $C > 0$ and $\eta > 0$. We now define the set $B_{a,b,T}$ as

$$B_{a,b,T} \overset{\text{def}}{=} \big\{ v \in X^{s',q,r}(T) : \|v\|_{L_T^\infty H_x^{s'}} \leq a \text{ and } \|v\|_{L_T^q L_x^r} \leq b \big\}.$$

Suppose that $\|(u_0,u_1)\|_{\mathcal{H}^s} \leq A$ for some $A > 0$. We let $a = 4CA$ and $0 < b \leq 1$ small enough such that

$$Cb^k \leq \min(\tfrac{a}{4}, \tfrac{b}{4}). \tag{3.11}$$

By dominated convergence theorem, we can let $T = T(u_0, u_1) > 0$ be small enough so that

$$\|S(t)(u_0,u_1)\|_{L_T^q L_x^r} \leq \tfrac{b}{4C}. \tag{3.12}$$

Choosing T smaller, if necessary, we also assume that

$$CT^\eta (k-1)\|\Xi\|_{\mathcal{X}^{s,\varepsilon}} \leq \min(\tfrac{1}{4a}, \tfrac{1}{4b}) \qquad \text{and} \qquad CT\|\Xi\|_{\mathcal{X}^{s,\varepsilon}} \leq \min(\tfrac{a}{4}, \tfrac{b}{4}) \tag{3.13}$$

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Combining (3.9), (3.10), (3.11), (3.12), and (3.13), we know that for $v \in B_{a,b,T}$, we have

$$\|\Gamma[v]\|_{L^\infty_T H^{s'}_x} \le a \qquad \text{and} \qquad \|\Gamma[v]\|_{L^q_T L^r_x} \le b,$$

so that Γ maps $B_{a,b,T}$ to $B_{a,b,T}$. By further shrinking b and T if necessary, we can use similar steps to obtain

$$\|\Gamma[v_1] - \Gamma[v_2]\|_{X^{s',q,r}(T)} \le \tfrac{1}{2}\|v_1 - v_2\|_{X^{s',q,r}(T)},$$

so that Γ is a contraction map on $B_{a,b,T}$. We can then conclude the proof of local well-posedness of (3.2). □

Remark 3.6. Note that in Case 3 above, to extend the argument to cover the critical regularity $s = 0$ for $k = 2, 3$ (even without the noise terms), we would need to find suitable q, r such that $s_{q,r} = 0$ with $s_{q,r}$ in (3.1). However, we can easily see that this requires that $q > 3$ which implies that $r < 3$ and vice-versa, thus the Strichartz estimates in Lemmas 3.3–3.4 do not apply. Moreover, since these are derived from the sharp Strichartz estimates in Lemma 3.1 which are known to fail at the endpoint $p = 3$ [4], the argument above is insufficient to reach critical regularity for quadratic and cubic nonlinearities.

4. Pathwise global well-posedness of the cubic SNLB

In this section, we show pathwise global well-posedness of the Wick-ordered cubic SNLB (1.19) via the hybrid argument in [24]. We restrict our attention to $0 < s < 2$, since the result for $s \ge 2$ follows from the same argument. In § 4.1, we first show some preliminary estimates involving the I-operator, and establish commutator estimates to control (1.23). We then prove Theorem 1.3 in § 4.2.

4.1. Commutator estimates and other preliminaries

We recall the definition of the I-operator with Fourier multiplier m_N in (1.22). In the following, we fix $N \in \mathbb{N}$. From the definition of the I-operator and the Littlewood-Paley theorem, we have that

$$\|f\|_{H^s} \lesssim \|If\|_{H^2} \lesssim N^{2-s}\|f\|_{H^s}, \tag{4.1}$$

$$\|If\|_{W^{s_0+s_1,p}} \lesssim N^{s_1}\|f\|_{W^{s_0,p}}, \tag{4.2}$$

for any $s_0 \in \mathbb{R}$, $0 \le s_1 \le 2 - s$, and $1 < p < \infty$. For simplicity, we will use the notations

$$f_{\lesssim N} \overset{\text{def}}{=} \pi_{\frac{N}{3}} f \qquad \text{and} \qquad f_{\gtrsim N} \overset{\text{def}}{=} \pi^{\perp}_{\frac{N}{3}} f \overset{\text{def}}{=} f - f_{\lesssim N}, \tag{4.3}$$

where π_N denotes the projection onto frequencies $\{|n| \le N\}$.

We first go over some basic commutator estimates in the following lemmas.

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Lemma 4.1. *Let* $\frac{4}{3} \le s < 2$*. Then, for* $k = 1, 2, 3$*, we have*

$$\|(If)^k - I(f^k)\|_{L^2} \lesssim N^{-2+k(2-s)}\|If\|_{H^2}^k.$$

Proof. By the definition of the I-operator and (4.3), we have $I(f_{\lesssim N}^k) = f_{\lesssim N}^k$ for $k = 1, 2, 3$. Thus, we obtain

$$\begin{aligned}
(If)^k - I(f^k) &= \big(I(f_{\lesssim N} + f_{\gtrsim N})\big)^k - I\big((f_{\lesssim N} + f_{\gtrsim N})^k\big) \\
&= \big(f_{\lesssim N} + I(f_{\gtrsim N})\big)^k - I\big((f_{\lesssim N} + f_{\gtrsim N})^k\big) \\
&= f_{\lesssim N}^k - I\big(f_{\lesssim N}^k\big) + \sum_{j=0}^{k-1}\binom{k}{j}\Big(f_{\lesssim N}^j (If_{\gtrsim N})^{k-j} - I\big(f_{\lesssim N}^j f_{\gtrsim N}^{k-j}\big)\Big) \\
&= \sum_{j=0}^{k-1}\binom{k}{j}\Big(f_{\lesssim N}^j (If_{\gtrsim N})^{k-j} - I\big(f_{\lesssim N}^j f_{\gtrsim N}^{k-j}\big)\Big). \qquad (4.4)
\end{aligned}$$

We first consider the case when $1 \le j \le k-1$. We let $1 < q < \infty$ sufficiently large and $\delta > 0$ small such that $\frac{1}{2} = \frac{j}{q} + \frac{1}{2+\delta}$. Then, by Hölder's and Sobolev's inequalities, we have

$$\begin{aligned}
\|f_{\lesssim N}^j (If_{\gtrsim N})^{k-j}\|_{L^2} &\le \|f_{\lesssim N}\|_{L^q}^j \|If_{\gtrsim N}\|_{L^{(2+\delta)(k-j)}}^{k-j} \\
&\lesssim \|f_{\lesssim N}\|_{H^2}^j \|If_{\gtrsim N}\|_{H^{2-\frac{4}{(2+\delta)(k-j)}}}^{k-j} \qquad (4.5) \\
&\lesssim N^{-\frac{4}{2+\delta}}\|If\|_{H^2}^k.
\end{aligned}$$

Similarly, using the boundedness of the multiplier m_N and (4.1), we have

$$\begin{aligned}
\big\|I\big(f_{\lesssim N}^j f_{\gtrsim N}^{k-j}\big)\big\|_{L^2} &\lesssim \|f_{\lesssim N}^j f_{\gtrsim N}^{k-j}\|_{L^2} \\
&\le \|f_{\lesssim N}\|_{L^q}^j \|f_{\gtrsim N}\|_{L^{(2+\delta)(k-j)}}^{k-j} \\
&\lesssim \|If_{\lesssim N}\|_{H^2}^j \|f_{\gtrsim N}\|_{H^{2-\frac{4}{(2+\delta)(k-j)}}}^{k-j} \qquad (4.6) \\
&\lesssim N^{(k-j)(2-s)-\frac{4}{2+\delta}}\|If_{\lesssim N}\|_{H^2}^j \|f_{\gtrsim N}\|_{H^s}^{k-j} \\
&\lesssim N^{-2+k(2-s)}\|If\|_{H^2}^k.
\end{aligned}$$

Here, we have used the fact that $2 - \frac{4}{(2+\delta)(k-j)} \le s$, which is guaranteed by $\frac{4}{3} \le s < 2$. When $j = 0$, similar estimates to (4.5) and (4.6) hold with $q = \infty$ and $\delta = 0$. Therefore, the desired estimate follows from (4.4), (4.5), and (4.6). □

Lemma 4.2. *Let* $0 < s < 2$ *and* $0 < \gamma < 1$*. Given* $\delta = \delta(s) > 0$ *sufficiently small, there exist small* $\gamma_0 = \gamma_0(\delta) > 0$ *and large* $p = p(\delta) \gg 1$ *such that*

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$$\|(If)(Ig) - I(fg)\|_{L^2} \lesssim N^{-\frac{1-\gamma}{2}+\delta} \|f\|_{H^{2-\gamma}} \|g\|_{W^{-\gamma_0,p}}$$

for any sufficiently large $N \gg 1$.

Proof. By writing $f = f_{\lesssim N^{\frac{1}{2}}} + f_{\gtrsim N^{\frac{1}{2}}}$ and $g = g_{\lesssim N} + g_{\gtrsim N}$, we have

$$\begin{aligned}(If)(Ig) - I(fg) &= \Big\{(If_{\lesssim N^{\frac{1}{2}}})(Ig_{\lesssim N}) - I(f_{\lesssim N^{\frac{1}{2}}} g_{\lesssim N})\Big\} \\ &\quad + \Big\{(If_{\lesssim N^{\frac{1}{2}}})(Ig_{\gtrsim N}) - I(f_{\lesssim N^{\frac{1}{2}}} g_{\gtrsim N})\Big\} \\ &\quad + (If_{\gtrsim N^{\frac{1}{2}}})(Ig) - I(f_{\gtrsim N^{\frac{1}{2}}} g) \\ &=: B_1 + B_2 + B_3 + B_4.\end{aligned}$$

Since the Fourier support of $f_{\lesssim N^{\frac{1}{2}}} g_{\lesssim N}$ is contained in $\{|n| \leq \frac{2}{3}N\}$, then $B_1 \equiv 0$.

For B_2, note that for $(n_1, n_2) \in \Lambda_n \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{Z}^4 \times \mathbb{Z}^4 : \ n = x + y, |x| \leq \frac{N^{\frac{1}{2}}}{3}, |y| > \frac{N}{3}\}$, by considering the sub-regions $|n_2| \geq 3N$ and $|n_2| < 3N$, from the mean value theorem and the definition in (1.22), we get

$$|m_N(n_1 + n_2) - m_N(n_2)| \lesssim N^{2-s} |n_2|^{-3+s} |n_1|.$$

From the above, the fact that $m_N(n_1) \equiv 1$ on Λ_n, and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|B_2\|_{L^2} &= \Bigg\| \sum_{(n_1,n_2)\in\Lambda_n} \big(m(n_2) - m(n_1+n_2)\big) \widehat{f}(n_1) \widehat{g}(n_2) \Bigg\|_{\ell^2_n} \\ &\lesssim N^{2-s} \Bigg\| \sum_{(n_1,n_2)\in\Lambda_n} \frac{1}{\langle n_1\rangle^{1-\gamma} \langle n_2\rangle^{3-s-\delta}} \langle n_1\rangle^{2-\gamma} |\widehat{f}(n_1)| \frac{|\widehat{g}(n_2)|}{\langle n_2\rangle^{\delta}} \Bigg\|_{\ell^2_n} \\ &\lesssim N^{-\frac{1}{2}+\delta} \|f\|_{H^{2-\gamma}} \|g\|_{H^{-\delta}}.\end{aligned}$$

As for B_3, by Hölder's inequality, Sobolev's embedding, and (4.2), we have

$$\begin{aligned}\|B_3\|_{L^2} &\leq \|If_{\gtrsim N^{\frac{1}{2}}}\|_{L^2} \|Ig\|_{L^\infty} \lesssim N^{-1+\frac{\gamma}{2}} \|f\|_{H^{2-\gamma}} \|Ig\|_{W^{5\delta,\delta^{-1}}} \\ &\lesssim N^{-1+\frac{\gamma}{2}+6\delta} \|f\|_{H^{2-\gamma}} \|g\|_{W^{-\delta,\delta^{-1}}}\end{aligned}$$

for $\delta = \delta(s) > 0$ sufficiently small.

Lastly, by (4.2) and Lemma 2.2 (ii), we have

$$\begin{aligned}\|B_4\|_{L^2} &\lesssim N^{4\delta} \|f_{\gtrsim N^{\frac{1}{2}}} g\|_{H^{-4\delta}} \lesssim N^{4\delta} \|f_{\gtrsim N^{\frac{1}{2}}}\|_{H^{4\delta}} \|g\|_{W^{-4\delta,\delta^{-1}}} \\ &\lesssim N^{-1+\frac{\gamma}{2}+6\delta} \|f\|_{H^{2-\gamma}} \|g\|_{W^{-4\delta,\delta^{-1}}}\end{aligned}$$

for $\delta = \delta(s) > 0$ sufficiently small. □

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We now show the following commutator estimate using Lemma 4.1 and Lemma 4.2.

LEMMA 4.3. *Let* $\frac{3}{2} < s < 2$ *and* $k = 1, 2$. *Given* $\delta = \delta(s) > 0$ *sufficiently small, there exist small* $\gamma_0 = \gamma_0(\delta) > 0$ *and* $p = p(\delta) \gg 1$ *such that*

$$\|I(f^k g) - (If)^k Ig\|_{L^2} \lesssim N^{-\frac{1-k(2-s)}{2}+\delta} \|If\|_{H^2}^k \|g\|_{W^{-\gamma_0,p}}$$

for sufficiently large $N \gg 1$.

Proof. Using triangle inequality, we have

$$\begin{aligned}\|I(f^k g) - (If)^k Ig\|_{L^2} &\leq \|I(f^k g) - I(f^k)Ig\|_{L^2} + \big\|\big(I(f^k) - (If)^k\big)Ig\big\|_{L^2} \\ &=: D_1 + D_2.\end{aligned}$$

For D_1, by Sobolev's inequality and Lemma 2.2(i), we have

$$\|f^k\|_{H^{2-k(2-s)}} \lesssim \|f^k\|_{W^{s,\frac{4}{2+(k-1)(2-s)}}} \lesssim \|f\|_{H^s} \|f\|_{L^{\frac{4}{2-s}}}^{k-1} \lesssim \|f\|_{H^s}^k. \tag{4.7}$$

Thus, by Lemma 4.2 with $\gamma = k(2-s)$, (4.7), and (4.1), there exists $\delta > 0$ sufficiently small such that

$$\|D_1\|_{L^2} \lesssim N^{-\frac{1-k(2-s)}{2}+\delta} \|f\|_{H^s}^k \|g\|_{W^{-\gamma_0,p}} \lesssim N^{-\frac{1-k(2-s)}{2}+\delta} \|If\|_{H^2}^k \|g\|_{W^{-\gamma_0,p}}$$

for some small $\gamma_0 = \gamma_0(\delta) > 0$ and large $p = p(\delta) \gg 1$.

Lastly, by Hölder's inequality, emma 4.1, Sobolev embedding, and (4.2), we have

$$\begin{aligned}\|D_2\|_{L^2} &\leq \|I(f^k) - (If)^k\|_{L^2} \|Ig\|_{L^\infty} \\ &\lesssim N^{-2+k(2-s)} \|If\|_{H^2}^k \|Ig\|_{W^{5\delta,\delta^{-1}}} \\ &\lesssim N^{-2+k(2-s)+6\delta} \|If\|_{H^2}^k \|g\|_{W^{-\delta,\delta^{-1}}}\end{aligned}$$

given that $\delta = \delta(s) > 0$ is sufficiently small. □

We conclude this subsection by showing the following estimates, which will be useful in estimating the second and third terms in (1.23).

LEMMA 4.4. *(i) Let* $0 < s < 2$ *and* $k = 0, 1$. *Then, for any* $0 \leq \lambda \leq 2 - s$, *we have*

$$\left|\int_{\mathbb{T}^4} (\partial_t Iv(t))(Iv(t))^k Iw(t)\, dx\right| \lesssim N^{\lambda}\big(1 + \big[E(I\vec{v})(t)\big]^{\frac{3}{4}}\big)\|w(t)\|_{W_x^{-\lambda,4}}$$

for any $t \geq 0$, *where* E *is the energy defined in* (1.21).

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(ii) We have

$$\begin{aligned}&\bigg|\int_{t_1}^{t_2}\int_{\mathbb{T}^4}(\partial_t Iv)(Iv)^2 Iw\,dxdt\bigg|\\&\lesssim \|Iw\|_{L^{\eta^{-1}}_{[t_1,t_2]}L^{\eta^{-1}}_x}\int_{t_1}^{t_2}\bigg(E^{\frac{1+\eta}{1-2\eta}}(I\vec{v})(t)+\frac{\eta}{(t-t_1)^{\frac{1}{2}}}\bigg)dt,\end{aligned}$$

uniformly in $0<\eta<\frac{1}{8}$ *and* $t_2\ge t_1\ge 0$.

Proof. (i) By Hölder's inequality, (1.21), and (4.2), we have

$$\begin{aligned}\bigg|\int_{\mathbb{T}^4}(\partial_t Iv(t))(Iv(t))^k Iw(t)\,dx\bigg| &\lesssim \|\partial_t Iv(t)\|_{L^2}\|Iv(t)\|_{L^4}^k\|Iw(t)\|_{L^4}\\&\lesssim N^\lambda\big[E(I\vec{v})(t)\big]^{\frac{1}{2}+\frac{k}{4}}\|w(t)\|_{W^{-\lambda,4}}.\end{aligned}$$

(ii) From Hölder's inequality, Sobolev inequality, and (1.23), we have

$$\begin{aligned}\bigg|\int_{t_1}^{t_2}\int_{\mathbb{T}^4}(\partial_t Iv)(Iv)^2 Iw\,dxdt\bigg| &\le \int_{t_1}^{t_2}\|\partial_t Iv\|_{L^2_x}\|Iv\|_{L^4_x}\|Iv\|_{L_x^{\frac{4}{1-4\eta}}}\|Iw\|_{L_x^{\eta^{-1}}}dt\\&\le\int_{t_1}^{t_2}\big[E(I\vec{v})(t)\big]^{\frac{3}{4}}\|Iv\|_{L_x^{\frac{4}{1-4\eta}}}\|Iw\|_{L_x^{\eta^{-1}}}\,dt\end{aligned}\qquad(4.8)$$

for $\eta>0$. By Sobolev inequality, interpolation, and (1.21), we get

$$\|Iv\|_{L_x^{\frac{4}{1-4\eta}}}\lesssim\|Iv\|_{W_x^{8\eta,\frac{4}{1+4\eta}}}\lesssim\|Iv\|_{H^2_x}^{4\eta}\|Iv\|_{L^4-x}^{1-4\eta}\le\big[E(I\vec{v})\big]^{\frac{1+4\eta}{4}},\qquad(4.9)$$

uniformly in $0<\eta<\frac{1}{8}$. From (4.8), (4.9), and Hölder's inequality, we get

$$\begin{aligned}\bigg|\int_{t_1}^{t_2}\int_{\mathbb{T}^4}(\partial_t Iv)(Iv)^2 Iw\,dxdt\bigg|\lesssim\bigg(\int_{t_1}^{t_2}\big[E(I\vec{v})\big]^{\frac{1+\eta}{1-\eta}}\,dt\bigg)^{1-\eta}\\ \|Iw\|_{L^{\eta^{-1}}_{[t_1,t_2]}L^{\eta^{-1}}_x}.\end{aligned}\qquad(4.10)$$

To estimate the first factor in (4.10), let

$$p=\frac{1-\eta}{1-2\eta},\qquad q=\frac{1}{1-2\eta},\qquad p'=\frac{1-\eta}{\eta},\qquad q'=\frac{1}{2\eta},$$

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where p', q' are the Hölder conjugates of p, q, respectively. By Hölder's and Young's inequalities, we have

$$\begin{aligned}\bigg(\int_{t_1}^{t_2} f(t)dt\bigg)^{1-\eta} &\leq \bigg(\int_{t_1}^{t_2} |f(t)|^p dt\bigg)^{\frac{1-\eta}{p}} (t_2-t_1)^{\frac{1-\eta}{p'}} \\ &\leq \frac{1}{q}\bigg(\int_{t_1}^{t_2} |f(t)|^p dt\bigg)^{\frac{q(1-\eta)}{p}} + \frac{1}{q'}(t_2-t_1)^{\frac{q'(1-\eta)}{p'}} \\ &= (1-2\eta)\int_{t_1}^{t_2} |f(t)|^{\frac{1-\eta}{1-2\eta}}\, dt + 2\eta(t_2-t_1)^{\frac{1}{2}}.\end{aligned}$$

Thus, we obtain

$$\bigg(\int_{t_1}^{t_2} E^{\frac{1+\eta}{1-\eta}}(I\vec{v})(t)dt\bigg)^{1-\eta} \lesssim \int_{t_1}^{t_2} \bigg(E^{\frac{1+\eta}{1-2\eta}}(I\vec{v})(t) + \frac{\eta}{(t-t_1)^{\frac{1}{2}}}\bigg)dt.$$

Combining the above estimates gives the intended estimate. □

4.2. Proof of Theorem 1.3

In this subsection, we construct a solution to the Wick renormalized cubic SNLB (1.19) on the time interval $[0,T]$ for any given $T \gg 1$. The argument is based on that in [24].

We first fix $\frac{3}{2} < s < 2$, $N \gg 1$ sufficiently large, and $T > 0$, and establish an estimate for the growth of the modified energy $E(t) = E(I_N\vec{v})(t)$ on the time interval $[0,T]$. Note that by (1.21) and Hölder's inequality, we have

$$\|Iv\|_{H_x^2}^2 = \|Iv\|_{L_x^2}^2 + \|\Delta(Iv)\|_{L_x^2}^2 \leq 2E^{\frac{1}{2}}(t) + 2E(t). \tag{4.11}$$

Then, by (1.23), Cauchy-Schwarz inequality, Lemmas 4.1, 4.3–4.4, and (4.11), we have

$$\begin{aligned}E(t_2)-E(t_1) \lesssim \int_{t_1}^{t_2} & N^{-2+3(2-s)}\big(1+E^2(t)\big)dt \\ &+ \sum_{k=1}^{2}\int_{t_1}^{t_2} N^{-\frac{1-k(2-s)}{2}+\delta}\big(1+E^{\frac{k+1}{2}}(t)\big)\|\mathcal{W}_\sigma(\Psi^{3-k}(t))\|_{W_x^{-\gamma_0,p}}dt \\ &+ \sum_{k=2}^{3}\int_{t_1}^{t_2} N^{\lambda}\big(1+E^{\frac{3}{4}}(t)\big)\|\mathcal{W}_\sigma(\Psi^k(t))\|_{W_x^{-\lambda,4}}dt \\ &+ \bigg\{\int_{t_1}^{t_2}\bigg(E^{\frac{1+\eta}{1-2\eta}}(t) + \frac{\eta}{(t-t_1)^{\frac{1}{2}}}\bigg)dt\bigg\}\|I\Psi\|_{L_{[t_1,t_2]}^{\eta^{-1}}L_x^{\eta^{-1}}}\end{aligned} \tag{4.12}$$

for any $t_2 \geq t_1 \geq 0$, where $\gamma_0 = \gamma_0(\delta) > 0$ is sufficiently small, $p = p(\delta) \gg 1$ sufficiently large, $0 \leq \lambda \leq 2-s$, and $0 < \eta < \frac{1}{8}$.

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Before proceeding to the iterative argument, we introduce some notations. Given $j \in \mathbb{Z}_{\geq 0}$, we define $V_j = V_j(\omega)$ by

$$V_j = \max_{k=1,2} \|\mathcal{W}_\sigma(\Psi^{3-k})\|_{L^\infty_{[j,j+1]} W_x^{-\gamma_0,p}} + \max_{k=0,1} \|\mathcal{W}_\sigma(\Psi^{3-k})\|_{L^\infty_{[j,j+1]} W_x^{-\lambda,4}}$$

and define $V = V(\omega)$ by

$$e^{V^{\frac{1}{3}}} = \sum_{j=0}^{\infty} e^{-jK} e^{V_j^{\frac{1}{3}}} \tag{4.13}$$

for some $K > 0$ large enough. Note that by applying (2.5) in Lemma 2.3 and letting $K > 0$ be sufficiently large, we have

$$\mathbb{E}\big[e^{V^{\frac{1}{3}}}\big] = \sum_{j=0}^{\infty} e^{-jK} \mathbb{E}\big[e^{V_j^{\frac{1}{3}}}\big] \leq \sum_{j=0}^{\infty} e^{-jK} e^{c(j+1)} < \infty,$$

so that V is almost surely finite. Also, for $T > 0$, we define $M_T = M_T(\omega)$ as follows

$$M_T = \max_{k=1,2} \|\mathcal{W}_\sigma(\Psi^{3-k})\|_{L^\infty_T W_x^{-\gamma_0,p}} + \max_{k=0,1} \|\mathcal{W}(\Psi^{3-k})\|_{L^\infty_T W_x^{-\lambda,4}}. \tag{4.14}$$

From (4.13) we have that $V_j^{\frac{1}{3}} \leq V^{\frac{1}{3}} + jK$, and therefore

$$M_T = \max_{j \leq T} V_j \lesssim V + K^3 T^3. \tag{4.15}$$

Furthermore, we define $R = R(\omega)$ by

$$R = 1 + \sum_{N=1}^{\infty} \sum_{j=1}^{\infty} e^{-jK \log N} \int_0^j \int_{\mathbb{T}^4} e^{|I_N \Psi(t,x)|} dx dt. \tag{4.16}$$

Then, by using Lemma 2.4 and taking $K > 0$ possibly larger, we have

$$\begin{aligned} \mathbb{E}[R] &= 1 + \sum_{N=1}^{\infty} \sum_{j=1}^{\infty} e^{-jK \log N} \int_0^j \int_{\mathbb{T}^4} \mathbb{E}\Big[e^{|I_N \Psi(t,x)|}\Big] dx dt \\ &\lesssim \sum_{N=1}^{\infty} \sum_{j=1}^{\infty} e^{-jK \log N} j e^{cj \log N} < \infty. \end{aligned}$$

Therefore, $1 \leq R(\omega) < \infty$ almost surely.

In the following, we fix $\omega \in \Omega$, where Ω is the full probability set where for all $\omega \in \Omega$ we have $V(\omega), R(\omega) < \infty$, and prove pathwise well-posedness of (1.19) on Ω. We first need the following crucial result.

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PROPOSITION 4.5. *Let* $\frac{3}{2} < s < 2$, $T \geq T_0 \gg 1$, *and* $N \in \mathbb{N}$ *with* $N > 10$. *Let* $V = V(\omega) < \infty$ *and* $R = R(\omega) < \infty$ *be as in (4.13) and (4.15). Then, there exist* $0 < \alpha \leq 2s - 3$ *and* $0 < \beta < \alpha$ *such that if*

$$E(t_0) \leq N^{\beta} \tag{4.17}$$

for some $0 \leq t_0 < T$, *then there exists small* $\tau = \tau(s, T, K, \omega) > 0$ *such that*

$$E(t) \leq N^{\alpha}$$

for any t *satisfying* $t_0 \leq t \leq \min(T, t_0 + \tau)$.

Proof. By replacing $E(t)$ by $E(t) + 1$, we can assume that $E(t) \geq 1$. Then, from (4.12) with (4.14), we have

$$\begin{aligned} E(t) - E(t_0) \lesssim & \int_{t_0}^{t} N^{-2+3(2-s)} E^2(t')dt' \\ & + M_T \sum_{k=1}^{2} \int_{t_0}^{t} N^{-\frac{1-k(2-s)}{2}+\delta} E^{\frac{k+1}{2}}(t')dt' \\ & + M_T \int_{t_0}^{t} N^{\lambda} E^{\frac{3}{4}}(t')dt' \\ & + \left\{ \int_{t_0}^{t} \left(E^{1+c\eta}(t') + \frac{\eta}{(t'-t_0)^{\frac{1}{2}}} \right) dt' \right\} \|I\Psi\|_{L^{\eta^{-1}}_{[t_0,t]} L^{\eta^{-1}}_x} \end{aligned} \tag{4.18}$$

for any $t \geq t_0$ and for $c = \frac{3}{1-2\eta} > 0$.

We assume that (4.17) holds for some $0 \leq t_0 < T$. By the continuity in time of $E(t)$ and (4.17) with $\alpha > \beta$, there exists $t_1 > t_0$ sufficiently close to t_0 such that

$$\max_{t_0 \leq \tau \leq t} E(\tau) \leq 100 N^{\alpha} \tag{4.19}$$

for any $t_0 \leq t \leq t_1$, where $\alpha > \beta$ is to be determined later. Note that at this point, t_1 depends on t_0. This issue will be dealt with later.

Let $\eta = \frac{1}{n}$ for some $n \in \mathbb{N}$. We note from (4.16) and $n! \leq n^n$ that

$$\begin{aligned} \|I_N\Psi\|^n_{L^n_{[t_0,t],x}} &= \int_{t_0}^{t}\int_{\mathbb{T}^4} |I_N\Psi(x,t)|^n dxdt \leq n! \int_0^T \int_{\mathbb{T}^4} e^{|I_N\Psi(x,t)|} dxdt \\ &\leq n! e^{KT\log N} R \leq n^n e^{KT\log N} R. \end{aligned}$$

We now choose

$$n \sim KT\log N + c\log(100N^{\alpha}) \sim KT\log N \gg 1,$$

where we may have to take $K \gg 1$ larger.

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Then, due to (4.19) and $\eta = n^{-1}$, we can estimate the last term on the right-hand side of (4.18) as

$$\begin{aligned}
&\bigg\{ \int_{t_0}^{t} \bigg(E^{1+c\eta}(t') + \frac{\eta}{(t'-t_0)^{\frac{1}{2}}} \bigg) dt' \bigg\} \|I\Psi\|_{L^{\eta^{-1}}_{[t_0,t]} L^{\eta^{-1}}_x} \\
&\leq \int_{t_0}^{t} \bigg(E(t') n e^{\frac{1}{n}(KT\log N + c\log(100N^{\alpha}))} R^{\frac{1}{n}} + \frac{e^{\frac{1}{n}KT\log N} R^{\frac{1}{n}}}{(t'-t_0)^{\frac{1}{2}}} \bigg) dt' \\
&\lesssim \int_{t_0}^{t} \bigg(\big[KRT\log N\big] E(t') + \frac{R}{(t'-t_0)^{\frac{1}{2}}} \bigg) dt',
\end{aligned} \tag{4.20}$$

where we used that $R = R(\omega) \geq 1$.

Next, we define F by

$$F(t) \overset{\text{def}}{=} \max_{t_0 \leq \tau \leq t} E(\tau) - E(t_0) + N^{\beta} \geq E(t). \tag{4.21}$$

Then, by (4.19), we have

$$N^{\beta} \leq F(t) \leq 200 N^{\alpha} \tag{4.22}$$

for $t_0 \leq t \leq t_1$. In particular, we have $\log F(t) \sim \log N$. Moreover, from (4.22), we have

$$\begin{cases}
N^{-2+3(2-s)} F^2(t) \lesssim N^{-\alpha} F^2(t) \leq 200 F(t), \\
N^{-\frac{1-2(2-s)}{2}+\delta} F^{\frac{3}{2}}(t) \lesssim N^{-\frac{\alpha}{2}} F^{\frac{3}{2}}(t) \leq \sqrt{200} F(t), \\
N^{-\frac{1-(2-s)}{2}+\delta} F(t) \leq F(t), \\
N^{\lambda} F^{\frac{3}{4}}(t) \lesssim N^{\lambda} F^{-\frac{1}{4}}(t) F(t) \leq F(t),
\end{cases} \tag{4.23}$$

provided that

$$\alpha \leq \min(3s-4, 2s-3), \quad \delta \leq \min\left(\tfrac{2s-3-\alpha}{2}, \tfrac{s-1}{2}\right), \quad \text{and} \quad \lambda \leq \tfrac{\beta}{4}, \tag{4.24}$$

which requires that

$$s > \max\left(\frac{4}{3}, \frac{3}{2}\right) = \frac{3}{2}.$$

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Hence, by (4.21), (4.18), (4.23), (4.20), (4.22), and (4.15), we obtain

$$\begin{aligned} F(t) - F(t_0) &\\ &= \max_{t_0 \le \tau \le t} E(\tau) - E(t_0) \\ &\lesssim (1 + M_T) \int_{t_0}^{t} F(t')dt' + \int_{t_0}^{t} \Big(KRTF(t') \log F(t') + \frac{R}{(t'-t_0)^{\frac{1}{2}}} \Big) dt' \\ &\lesssim (1 + V + K^3T^3) \int_{t_0}^{t} F(t')dt' + \int_{t_0}^{t} \Big(KRTF(t') \log F(t') + \frac{R}{(t'-t_0)^{\frac{1}{2}}} \Big) dt' \\ &\lesssim (1 + V + KRT) \int_{t_0}^{t} F(t')(\log F(t') + K^2T^2) dt' + 2R(t-t_0)^{\frac{1}{2}} \end{aligned} \tag{4.25}$$

for any $t_0 \le t \le t_1$ such that (4.22) holds. Denoting by $C_0 = C_0(s)$ the implicit constant in (4.25), we define G by

$$G(t) = F(t) - 2C_0R(t-t_0)^{\frac{1}{2}}. \tag{4.26}$$

Let us pick $t_*(s, R) > 0$ such that

$$2C_0R(t-t_0)^{\frac{1}{2}} \ll 1, \tag{4.27}$$

sufficiently small so that

$$F(t) \le 5^{\frac{\alpha-\beta}{2}} G(t). \tag{4.28}$$

Then, from (4.26) and (4.28), we get that $F(t) \sim G(t)$, which combined with (4.25) gives

$$G(t) - G(t_0) \le C(1 + V + KRT) \int_{t_0}^{t} G(t')(\log G(t') + K^2T^2) dt' \tag{4.29}$$

for any $t_0 \le t \le \min(t_1, t_0 + t_*(s, R))$ and some $C > 0$.

Now, note that the equation

$$\partial_t H(t) = \widetilde{C} H(t)(\log H(t) + K^2T^2)$$

has a solution $H(t) = \exp\big(\exp(\widetilde{C}t)(\log H(0) + K^2T^2) - K^2T^2\big)$. Then, by comparison, we deduce from (4.29) that

$$G(t) \le \exp\Big(e^{C(1+V+KRT)(t-t_0)} (\log G(t_0) + K^2T^2) - K^2T^2 \Big), \tag{4.30}$$

for some constant $C > 0$.

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Recall from (4.26) and (4.21) that $G(t_0) = N^\beta$. Then, if the condition

$$e^{C(1+V+KRT)(t-t_0)}(\beta \log N + K^2T^2) \leq \alpha \log N + K^2T^2 - \frac{\alpha-\beta}{2}\log 5, \tag{4.31}$$

holds for $t_0 \leq t \leq \min(t_1, t_0 + t_*(s,V,R,T,K))$ (where $t_*(s,V,R,T,K) > 0$ will be specified later), the bound (4.30) implies

$$G(t) \leq 5^{\frac{\beta-\alpha}{2}} N^\alpha, \tag{4.32}$$

for any $t_0 \leq t \leq \min(t_1, t_0 + t_*(s,V,R,T))$. Then, we conclude from (4.21), (4.26), (4.27), and (4.28) that

$$E(t) \leq F(t) \leq N^\alpha, \tag{4.33}$$

for any $t_0 \leq t \leq \min(t_1, t_0 + t_*(s,V,R,T,K))$. This in turn guarantees that the conditions (4.19) and (4.22) are met. Therefore, by a standard continuity argument, we conclude that the bounds (4.32) and (4.33) hold for any t with $t_0 \leq t \leq t_0 + t_*(s,V,R,T,K)$ such that the condition (4.31) holds.

Finally, let us consider the condition (4.31). Let $\alpha = \alpha(s) > \beta = \beta(s)$ be such that the conditions in (4.24) hold. Since $\alpha > \beta$, there exists $t_{**}(s,V,R,T,K)$ such that, for $0 \leq \tau \leq t_{**}$, we have

$$\alpha - e^{C(1+V+R+KRT)\tau}\beta \geq \frac{\alpha-\beta}{2} > 0. \tag{4.34}$$

Then, since $N > 10$, by choosing $0 \leq \tau \leq t_{**}$ sufficiently small such that

$$e^{C(1+V+R+KRT)\tau} - 1 \leq \frac{\frac{\alpha-\beta}{2}\log N - \frac{\alpha-\beta}{2}\log 5}{K^2T^2}, \tag{4.35}$$

we can guarantee that the condition (4.31) is satisfied for $t_0 \leq t \leq t_0 + \tau$, and hence so is (4.33). This concludes the proof of Proposition 4.5. □

We now present the proof of Theorem 1.3. Fix $\frac{7}{4} < s < 2$, $T \gg 1$, $\omega \in \Omega$ such that $V = V(\omega) < \infty$ and $R = R(\omega) < \infty$, and let the parameters α, β, τ be as in Proposition 4.5.

Fix $N_0 \gg 1$ which is to be determined later. Then, for $k \in \mathbb{Z}_{\geq 0}$, define an increasing sequence $\{N_k\}_{k\in\mathbb{Z}_{\geq 0}}$ by setting

$$N_k = N_0^{\sigma^k}, \tag{4.36}$$

for some $\sigma = \sigma(s) > 1$ sufficiently large satisfying

$$\alpha \ll \sigma(\beta - 2(2-s)) \quad \text{and} \quad 2\alpha \ll \sigma\beta, \tag{4.37}$$

which implies

$$N_{k+1}^{2(2-s)} N_k^\alpha + N_k^{2\alpha} \ll N_{k+1}^\beta. \tag{4.38}$$

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From (4.37) and the assumptions on α, β in Proposition 4.5, we have

$$2(2-s) < \beta < \alpha \leq 2s-3,$$

which imposes the further restriction on s:

$$s > \max\left(\frac{7}{4}, \frac{3}{2}\right) = \frac{7}{4}.$$

Suppose that for some $k \in \mathbb{Z}_{\geq 0}$ and $t \geq 0$, it holds that

$$E(I_{N_k}\vec{v})(t) \leq N_k^{\alpha}. \tag{4.39}$$

Then, by (1.21), (4.1), Sobolev inequality, (4.39), and (4.38), we have

$$\begin{aligned} E(I_{N_{k+1}}\vec{v})(t) &\lesssim N_{k+1}^{2(2-s)}\|\vec{v}\|_{\mathcal{H}^s}^2 + \|v\|_{H_x^1}^4 \\ &\lesssim N_{k+1}^{2(2-s)}\|I_{N_k}\vec{v}\|_{\mathcal{H}_x^2}^2 + \|I_{N_k}v\|_{H_x^2}^4 \\ &\lesssim N_{k+1}^{2(2-s)}E(I_{N_k}\vec{v})(t) + E(I_{N_k}\vec{v})^2(t) \\ &\lesssim N_{k+1}^{2(2-s)}N_k^{\alpha} + N_k^{2\alpha} \ll N_{k+1}^{\beta}. \end{aligned} \tag{4.40}$$

We are now ready to implement an iterative argument. Given $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^4)$, choose $N_0 = N_0(u_0, u_1, s) \gg 1$ such that

$$E(I_{N_0}\vec{v})(0) \leq N_0^{\beta}. \tag{4.41}$$

By applying Proposition 4.5, there exists $\tau = \tau(s, T, K, \omega) > 0$ such that

$$E(I_{N_0}\vec{v})(t) \leq N_0^{\alpha},$$

for any $0 \leq t \leq \tau$. By (4.39) and (4.40), this then implies

$$E(I_{N_1}\vec{v})(\tau) \leq N_1^{\beta}.$$

Applying Proposition 4.5 once again, we in turn obtain

$$E(I_{N_1}\vec{v})(t) \leq N_1^{\alpha},$$

for $0 \leq t \leq 2\tau$. By (4.39) and (4.40), this then implies

$$E(I_{N_2}\vec{v})(2\tau) \leq N_2^{\beta}.$$

By iterating this argument $\left[\frac{T}{\tau}\right]+1$ times, we obtain a solution v to the renormalized cubic SNLB (1.19) on the time interval $[0, T]$. Since the choice of $T \gg 1$ was arbitrary, this proves global well-posedness of (1.19).

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REMARK 4.6. From the argument above, we can also establish a growth bound on the Sobolev norm of the solution v to SNLB (1.19). Namely, for $T \gg 1$ and with the same choice of parameters, we have

$$\|\vec{v}(t)\|_{\mathcal{H}^s} \lesssim \big(1 + E(I_{N_k}\vec{v})(t)\big)^{\frac{1}{2}} \le N_k^{\frac{\alpha}{2}}$$

for any $0 \le t \le T$ such that $k\tau \le t \le (k+1)\tau$ for some $k \in \mathbb{Z}_{\ge 0}$. Then, by (4.36), we have

$$\|\vec{v}(t)\|_{\mathcal{H}^s} \lesssim \exp\big(\tfrac{\alpha}{2}\sigma^k \log N_0\big) \le \exp\Big(\tfrac{\alpha}{2}\log N_0 \cdot \exp\big(\tfrac{(\log\sigma)t}{\tau}\big)\Big) \tag{4.42}$$

for $0 \le t \le T$. Moreover, in view of (4.41), we choose $N_0 \in \mathbb{N}$ such that $1 + E(I_{N_0}\vec{v})(0) \sim N_0^{\beta}$, so that by (4.1) and the fact that $\beta > 2(2-s)$, we have

$$\log N_0 \sim \log\big(2 + \|\vec{v}(0)\|_{\mathcal{H}^s}\big). \tag{4.43}$$

In order to iteratively apply Proposition 4.5 $\frac{T}{\tau}$-many times to reach the target time T, we need to guarantee the condition (4.35). By taking

$$\tau \sim_{s,V,R,K} T^{-1}, \tag{4.44}$$

the condition (4.34) holds. Thus, in view of (4.36) with $k \sim \frac{T}{\tau}$, the condition (4.35) becomes

$$0 < C_0 \le \frac{\frac{\alpha-\beta}{2}\sigma^{T^2}\log N_0 - \frac{\alpha-\beta}{2}\log 5}{K^2T^2},$$

which holds true for any sufficiently large $T \gg 1$. Finally, from (4.42), (4.43), and (4.44), we conclude the following double exponential bound for any $t \ge 0$

$$\|\vec{v}(t)\|_{\mathcal{H}^s} \le C\exp\Big(c\log\big(2 + \|\vec{v}(0)\|_{\mathcal{H}^s}\big)\cdot e^{C(\omega)t^2}\Big).$$

5. Almost sure global well-posedness of the hyperbolic Φ_4^{k+1}-model

In this section, we prove Theorem 1.6, i.e. almost sure global well-posedness of the renormalized SdNLB (1.39) and invariance of the corresponding Gibbs measure (1.36). Due to the convergence of $\vec{\rho}_N$ to $\vec{\rho}$, the invariance of $\vec{\rho}_N$ under the truncated SdNLB dynamics (1.38), and Bourgain's invariant measure argument [5, 6], Theorem 1.6 follows once we construct the limiting process $(u, \partial_t u)$ locally-in-time with a good approximation property for the solution u_N to (1.38). Furthermore, since $\vec{\rho}$ is mutually absolutely continuous with respect to $\vec{\mu}_2$, it suffices to study the renormalized SdNLB (1.38) and (1.39) with the Gaussian random initial data (u_0^ω, u_1^ω) with $\mathcal{L}(u_0^\omega, u_1^\omega) = \vec{\mu}_2$.

We first detail how to adapt the proof of Theorem 1.1 to show local well-posedness of (1.38) and (1.39), uniformly in the truncation N, and then show invariance of the truncated Gibbs measure $\vec{\rho}_N$ in (1.34) under the dynamics of the truncated SdNLB (1.38).

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As in § 3, to construct solutions for SdNLB (1.38)-(1.39), we proceed with a first order expansion centered around the stochastic convolution Ψ^{d} which solves (2.4). By defining the operator $\mathcal{D}(t)$ as

$$\mathcal{D}(t) \overset{\text{def}}{=} e^{-\frac{t}{2}} \frac{\sin\big(t[[\nabla]]^2\big)}{[[\nabla]]^2} \qquad \text{with} \qquad [[\nabla]] \overset{\text{def}}{=} \Big(\langle\nabla\rangle^4 - \frac{1}{4}\Big)^{1/4},$$

the stochastic convolution Ψ^{d} which solves the stochastic damped linear beam equation in (2.4) can be expressed as

$$\Psi^{\mathrm{d}}(t) = \partial_t \mathcal{D}(t) u_0^\omega + \mathcal{D}(t)(u_0^\omega + u_1^\omega) + \sqrt{2}\int_0^t \mathcal{D}(t-t')dW(t'),$$

where W is a cylindrical Wiener process on $L^2(\mathbb{T}^4)$ as in (1.6). A direct but tedious computation shows that $\Psi_N^{\mathrm{d}} = \pi_N \Psi^{\mathrm{d}}$ is a mean-zero real-valued Gaussian random variable with variance

$$\mathbb{E}\big[\Psi_N^{\mathrm{d}}(t,x)^2\big] = \mathbb{E}\big[\big(\pi_N \Psi^{\mathrm{d}}(t,x)\big)^2\big] = \alpha_N$$

for any $t \in \mathbb{R}_+$, $x \in \mathbb{T}^4$, and $N \in \mathbb{N}$, where α_N is as in (1.30). Unlike $\sigma_N(t)$ in (1.7), the variance α_N is independent of time t. This is due to the fact that the massive Gaussian free field μ_2 is invariant under the dynamics of (2.4).

Let u_N be the solution to (1.38) with $\mathcal{L}\big((u_N, \partial_t u_N)|_{t=0}\big) = \vec{\mu}_2$. Then, we write u_N as

$$u_N = v_N + \Psi^{\mathrm{d}} = (v_N + \Psi_N^{\mathrm{d}}) + \pi_N^\perp \Psi^{\mathrm{d}}, \tag{5.1}$$

where $\pi_N^\perp = \mathrm{Id} - \pi_N$. Note that the dynamics of the truncated Wick-ordered SdNLB (1.38) decouple into the linear dynamics for the high frequency part given by $\pi_N^\perp \Psi^{\mathrm{d}}$ and the nonlinear dynamics for the low frequency part $\pi_N u_N$:

$$\partial_t^2 \pi_N u_N + \partial_t \pi_N u_N + (1-\Delta)^2 \pi_N u_N + \pi_N\big(\mathcal{W}_\alpha((\pi_N u)^k)\big) = \sqrt{2}\pi_N \xi. \tag{5.2}$$

Then, by (2.3), the remainder term $v_N = \pi_N u_N - \Psi_N^{\mathrm{d}}$ satisfies the following equation:

$$\begin{cases} \partial_t^2 v_N + \partial_t v_N + (1-\Delta)^2 v_N + \sum\limits_{\ell=0}^{k} \binom{k}{\ell} \pi_N\big(\mathcal{W}_\alpha((\Psi_N^{\mathrm{d}})^\ell) v_N^{k-\ell}\big) = 0, \\ (v_N, \partial_t v_N)|_{t=0} = (0,0), \end{cases} \tag{5.3}$$

where the Wick power $\mathcal{W}_\alpha((\Psi_N^{\mathrm{d}})^\ell) \overset{\text{def}}{=} H_\ell(\Psi_N^{\mathrm{d}}; \alpha_N)$

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converges to a limit, denoted by $\mathcal{W}_\alpha((\Psi^{\rm d})^\ell)$, in $C([0,T];W^{-\varepsilon,\infty}(\mathbb{T}^4))$ for any $\varepsilon>0$ and $T>0$, almost surely (and also in $L^p(\Omega)$ for any $p<\infty$); see emma 2.3. Thus, we formally obtain the limiting equation:

$$\begin{cases}\partial_t^2 v+\partial_t v+(1-\Delta)^2 v+\sum_{\ell=0}^{k}\binom{k}{\ell}\mathcal{W}_\alpha((\Psi^{\rm d})^\ell)v^{k-\ell}=0,\\ (v,\partial_t v)|_{t=0}=(0,0).\end{cases}\tag{5.4}$$

We now detail how to modify the proof of Theorem 1.1 to show local well-posedness of (5.3)–(5.4), uniformly in $N\in\mathbb{N}$. Note that v is a solution to (5.4) if and only if $w=e^{\frac{t}{2}}v$ satisfies the following equation:

$$\partial_t^2 w+(1-\Delta)^2 w-\frac{1}{4}w+e^{\frac{t}{2}}\sum_{\ell=0}^{k}\binom{k}{\ell}\mathcal{W}_\alpha((\Psi^{\rm d})^\ell)(e^{-\frac{t}{2}}w)^{k-\ell}=0.$$

The terms in the mild formulation corresponding to the w-equation can be treated as in Proposition 3.5, except for the one coming from $\frac{3}{4}w-2\Delta w$ term. However, this term can be viewed as a perturbation thanks to the two degrees of smoothing in the integral Duhamel operator, and the analogue of Proposition 3.5 follows. The same argument allows us to show local well-posedness of (5.3) where the time of existence depends only on the stochastic convolution $\Psi^{\rm d}$ and its Wick-powers, but not on $N\in\mathbb{N}$.

Now, it remains to show the invariance of the truncated Gibbs measure $\vec{\rho}_N$ under the truncated SdNLB dynamics (1.38) in the following proposition. In fact, the rest of the proof of Theorem 1.6 follows from a standard application of Bourgain's invariant measure argument, whose details we omit. See, for example, [44] for further details.

PROPOSITION 5.1. *Let $N\in\mathbb{N}$. Then, the truncated SdNLB Eq (1.38) is almost surely globally well-posed with respect to the random initial data distributed by the truncated Gibbs measure $\vec{\rho}_N$ in (1.34). Moreover, the truncated Gibbs measure $\vec{\rho}_N$ (1.34) is invariant under the dynamics of (1.38). More precisely, denoting by u_N the global solution to truncated SdNLB Eq (1.38), we have $\mathcal{L}(u_N(t),\partial_t u_N(t))=\vec{\rho}_N$ for any $t\in\mathbb{R}_+$.*

Proof. The idea of the proof has already appeared in [24, 34, 42, 44] and so we only sketch the key steps. Given $N\in\mathbb{N}$, we define $\vec{\mu}_{2,N}$ and $\vec{\mu}_{2,N}^\perp$ to be the marginal probability measures on $\pi_N\mathcal{H}^{-\varepsilon}(\mathbb{T}^4)$ and $\pi_N^\perp\mathcal{H}^{-\varepsilon}(\mathbb{T}^4)$, respectively. In other words, recalling X^1 and X^2 in (1.29), $\vec{\mu}_{2,N}$ and $\vec{\mu}_{2,N}^\perp$ are the induced probability measures under the maps $\omega\in\Omega\mapsto(\pi_N X^1(\omega),\pi_N X^2(\omega))$ and $\omega\in\Omega\mapsto(\pi_N^\perp X^1(\omega),\pi_N^\perp X^2(\omega))$, respectively. Then, with $\vec{\mu}_2=\vec{\mu}_{2,N}\otimes\vec{\mu}_{2,N}^\perp$ and (1.34), we can write

$$\vec{\rho}_N=\vec{\nu}_N\otimes\vec{\mu}_{2,N}^\perp,\tag{5.5}$$

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where the measure $\vec{\nu}_N$ is given by

$$d\vec{\nu}_N = Z_N^{-1} R_N(u) d\vec{\mu}_{2,N},$$

with the density R_N as in (1.31).

We recall the decomposition (5.1). Since the high frequency part $\pi_N^{\perp} u_N = \pi_N^{\perp} \Psi^{\mathrm{d}}$ satisfies

$$\partial_t^2 \pi_N^{\perp} \Psi^{\mathrm{d}} + \partial_t \pi_N^{\perp} \Psi^{\mathrm{d}} + (1-\Delta)^2 \pi_N^{\perp} \Psi^{\mathrm{d}} = \sqrt{2} \pi_N^{\perp} \xi, \tag{5.6}$$

the dynamics of $\pi_N^{\perp} \Psi^{\mathrm{d}}$ are linear and thus we can study the evolution of each frequency on the Fourier side to conclude that the Gaussian measure $\vec{\mu}_{2,N}^{\perp}$ is invariant under the dynamics of (5.6). In fact, a tedious but direct computation shows that

$$\mathbb{E}\big[|\widehat{\Psi}^{\mathrm{d}}(t,n)|^2\big] = \frac{1}{\langle n \rangle^4} \quad \text{and} \quad \mathbb{E}\big[|\widehat{\partial_t \Psi^{\mathrm{d}}}(t,n)|^2\big] = 1,$$

for any $t \in \mathbb{R}_+$ and $n \in \mathbb{Z}^4$, so that $\mathcal{L}(\Psi^{\mathrm{d}}(t), \partial_t \Psi^{\mathrm{d}}(t)) = \vec{\mu}_2$ for any $t \in \mathbb{R}_+$.

We now consider the low frequency part $\pi_N u_N$, which solves (5.2). Denoting $(u_{1,N}, u_{2,N}) = (\pi_N u_N, \partial_t \pi_N u_N)$, we can write (5.2) in the following Ito formulation:

$$\begin{aligned} d\begin{pmatrix} u_{1,N} \\ u_{2,N} \end{pmatrix} + \bigg\{ \begin{pmatrix} 0 & -1 \\ (1-\Delta)^2 & 0 \end{pmatrix} \begin{pmatrix} u_{1,N} \\ u_{2,N} \end{pmatrix} + \begin{pmatrix} 0 \\ \pi_N \mathcal{W}_\alpha(u_{1,N}^k) \end{pmatrix} \bigg\} dt \\ = \begin{pmatrix} 0 \\ -u_{2,N} dt + \sqrt{2} \pi_N dW \end{pmatrix}. \end{aligned} \tag{5.7}$$

This shows that the generator $\mathcal{L}^N$ of the Markov semigroup for (5.7) can be written as $\mathcal{L}^N = \mathcal{L}_1^N + \mathcal{L}_2^N$, where $\mathcal{L}_1^N$ corresponds to the (deterministic) NLB with truncated nonlinearity

$$d\begin{pmatrix} u_{1,N} \\ u_{2,N} \end{pmatrix} + \bigg\{ \begin{pmatrix} 0 & -1 \\ (1-\Delta)^2 & 0 \end{pmatrix} \begin{pmatrix} u_{1,N} \\ u_{2,N} \end{pmatrix} + \begin{pmatrix} 0 \\ \pi_N \mathcal{W}_\alpha(u_{1,N}^k) \end{pmatrix} \bigg\} dt = 0, \tag{5.8}$$

while $\mathcal{L}_2^N$ corresponds to the Ornstein-Uhlenbeck process:

$$du_{2,N} = -u_{2,N} dt + \sqrt{2} \pi_N dW. \tag{5.9}$$

The invariance of $\vec{\nu}_N$ under the dynamics of (5.8) follows from Liouville's theorem and the conservation of the Hamiltonian $E_N(u_{1,N}, u_{2,N})$ under the dynamics of (5.8), where

$$\begin{aligned} E_N(u_{1,N}, u_{2,N}) = \frac{1}{2} \int_{\mathbb{T}^4} |(1-\Delta) u_{1,N}|^2 dx + \frac{1}{2} \int_{\mathbb{T}^4} (u_{2,N})^2 dx \\ + \frac{1}{k+1} \int \mathcal{W}_\alpha(u_{1,N}^{k+1}) \, dx. \end{aligned}$$

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Hence, we have $(\mathcal{L}_1^N)^*\vec{\nu}_N = 0$, where $(\mathcal{L}_1^N)^*$ denotes the adjoint of $\mathcal{L}_1^N$. Regarding (5.9), we recall that the Ornstein-Uhlenbeck process preserves the standard Gaussian measure. Thus, $\vec{\nu}_N$ is also invariant under the dynamics of (5.9), since the measure $\vec{\nu}_N$ on the second component is the white noise μ_0 (see (1.28) with $s=0$ and projected onto the low frequencies $|n| \leq N$). Hence, we have $(\mathcal{L}_2^N)^*\vec{\nu}_N = 0$, and so

$$(\mathcal{L}^N)^*\vec{\nu}_N = (\mathcal{L}_1^N)^*\vec{\nu}_N + (\mathcal{L}_2^N)^*\vec{\nu}_N = 0.$$

This shows invariance of $\vec{\nu}_N$ under (5.7) and hence under (5.2).

Therefore, invariance of the truncated Gibbs measure $\vec{\rho}_N$ in (1.34) under the truncated SdNLB dynamics (1.38) follows from (5.5) and the invariance of $\vec{\nu}_N$ and $\vec{\mu}_{2,N}^{\perp}$ under (5.7) and (5.6) respectively. □

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