ON THE WIELANDT SUBGROUP OF INFINITE SOLUBLE GROUPS

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Dedicated to Professor Dr. H. Wielandt on his 80th birthday

(Received 28 September, 1988)

1. Introduction. The Wielandt subgroup $\omega(G)$ of a group G is defined to be the intersection of the normalizers of all the subnormal subgroups of G. If G is a group satisfying the minimal condition on subnormal subgroups then Wielandt [**10**] showed that $\omega(G)$ contains every minimal normal subgroup of G, and so contains the socle of G, and, later, Robinson [**6**] and Roseblade [**9**] proved that $\omega(G)$ has finite index in G.

In this paper we consider the somewhat dual situation. As, clearly, $\omega(G)$ contains the centre $Z(G)$, it will be sensible to ask how far $\omega(G)$ is away from $Z(G)$. If G is nilpotent, then every subgroup of G is subnormal and hence $\omega(G)$ is the norm of G, introduced by Baer in [**1**]. In this case Baer proved that either $\omega(G)=Z(G)$ or $\omega(G)$ is periodic.

Here we are interested in the Wielandt subgroup of a finitely generated soluble-by-finite group, particularly in its connections with the FC-centre of the group, which is the set of all elements with finitely many conjugates.

THEOREM A. *Let G be a finitely generated soluble-by-finite group with finite Prüfer rank. Then the Wielandt subgroup $\omega(G)$ is contained in the* FC-*centre of G.*

Cossey [**3**] proved that, if G is a polycyclic group, then $G/C_G(\omega(G))$ is finite. As an immediate consequence of Theorem A, we have the following slight generalization of this result.

COROLLARY. *If G is a polycyclic-by-finite group, then $G/C_G(\omega(G))$ is finite.*

In fact in this case $\omega(G)$ is finitely generated and by Theorem A each of its generators has finitely many conjugates in G.

Another obvious consequence of Theorem A is that in a soluble-by-finite min-by-max group G the Wielandt subgroup is always contained in the second FC-centre. Note that the locally dihedral 2-group G is a Černikov group all of whose subnormal subgroups are normal, i.e. $G=\omega(G)$, but G is not an FC-group.

In general, $\omega(G)$ need not be contained in any term of the upper FC-central series of the soluble group G, even if the group has finite Prüfer rank or is finitely generated, as can be seen from Examples 1 and 2 below.

Also $\omega(G)/Z(G)$ need not in general be finite. Indeed Cossey [**3**] has constructed a polycyclic and nilpotent-by-finite group of Fitting length three which has trivial centre and an infinite cyclic Wielandt subgroup. This example shows that the following theorem, which generalizes a result of Cossey [**3**], is best possible.

THEOREM B. *If G is a polycyclic group which is either (a) metanilpotent or (b) abelian-by-finite, then $\omega(G)/Z(G)$ is finite.*

† The first author wishes to thank the Mathematics Department of the University of Napoli for its excellent and warm hospitality.

Here the hypothesis on G to be soluble cannot be dispensed with, as we will construct a finitely generated abelian-by-finite group with trivial centre and infinite Wielandt subgroup (see Example 3).

Our notation is standard and can be found in [**8**]. In particular:

A group G is a T-*group* if all of its subnormal subgroups are normal.

An automorphism of a group G is a *power automorphism* if it maps every subgroup of G onto itself.

A soluble-by-finite group is *minimax* if it has a series of finite length whose factors are either infinite cyclic or of type p^∞, for some prime p, or finite. The *spectrum* of an abelian minimax group G is the set of all primes p such that G has a quotient of type p^∞.

2. Proof of the Theorems. Our first lemma, which is probably already known, concerns the structure of soluble-by-finite T-groups.

LEMMA 1. *Let G be a soluble-by-finite* T*-group. Then G is finite-by-soluble.*

Proof. Let G_0 be the soluble radical of G, and put $C = C_G(G_0)$. Since G induces a group of power automorphisms on each factor of the derived series of G_0, it follows that the commutator subgroup of the factor group G/C is nilpotent and hence G/C is soluble. Moreover $C \cap G_0 \leq Z(C)$, so that $C/Z(C)$ is finite and C' is finite. Therefore G is finite-by-soluble.

The proof of Theorem A rests heavily on the following lemma.

LEMMA 2. *Let G be a soluble-by-finite minimax group whose maximum periodic normal subgroup is finite. If F is the Fitting subgroup of G and A is the Wielandt subgroup of G, then $G/C_G(A \cap F)$ is finite.*

Proof. From the structure of minimax groups it follows that F is nilpotent with finite torsion subgroup T and G/F is a finitely generated abelian-by-finite group (see [**8**, Part 2, p. 169]). Hence there exists a torsion-free abelian normal subgroup H/F of G/F such that G/H is finite. Let p be a prime which does not belong to the spectrum of the abelian minimax group $Z(F/T)$, and for each positive integer t put $F_t = F/F^{p^t}T$. Then F_t is a finite p-group and the Frattini factor group $F_t/\Phi(F_t)$ has order at most p^r, where r is the Prüfer rank of F.

Let y be any element of $H \backslash F$ and put $Y = \langle y \rangle$. The factor group $Y/C_Y(F_t/\Phi(F_t))$ is isomorphic with a subgroup of the general linear group $GL(r, p)$ and hence, if m is the maximum p'-divisor of $|GL(r, p)|$, the element $z = y^m$ acts as a p-automorphism on $F_t/\Phi(F_t)$ and so also on F_t. Since $F \leq \langle z, F \rangle \leq H$, the subgroup $\langle z, F \rangle$ is subnormal in G. Write $K_t = C_{\langle z \rangle}(F_t)$. Then $\langle K_t, F^{p^t}T \rangle$ is normal in $\langle z, F \rangle$ and the quotient $\langle z, F \rangle / \langle K_t, F^{p^t}T \rangle$ is a finite p-group. Hence the subgroup $S_t = \langle z, F^{p^t}T \rangle$ is subnormal in $\langle z, F \rangle$ and so also in G. Therefore A normalizes S_t, and for each element x of $A \cap F$ we have that

$$[x, z] \in S_t \cap F = F^{p^t}T(\langle z \rangle \cap F) = F^{p^t}T,$$

since $\langle z \rangle \cap F = 1$. Thus $[x, z] \in \bigcap_{t \geq 1} F^{p^t}T = T$ (see [**8**, Part 2, p. 170]) and so y^m centralizes $(A \cap F)T/T$.

The Wielandt subgroup A acts as a group of power automorphisms on the non-periodic nilpotent group F. If F is not abelian, it has no non-trivial power automorphisms (see [**2**]), and so A centralizes F. This proves that in any case $A \cap F$ is contained in the centre of F. Therefore H^m centralizes $(A \cap F)T/T$ and so also $(A \cap F)/(A \cap T)$. Hence $H/C_H((A \cap F)/(A \cap T))$ has finite exponent and so is finite. It follows that also $G/C_G((A \cap F)/(A \cap T))$ is finite. Since $A \cap T$ is finite, the subgroup $C = C_G((A \cap F)/(A \cap T)) \cap C_G(A \cap T)$ has finite index in G. Moreover the group $C/C_G(A \cap F)$ has finite exponent and so is finite. Therefore $G/C_G(A \cap F)$ is finite.

Proof of Theorem A. Suppose first that G is soluble. By a result of Robinson [**7**] G is a minimax group, and so it contains a characteristic subgroup R, which is the direct product of finitely many quasicyclic subgroups, such that the Fitting subgroup F/R of G/R is nilpotent with finite torsion subgroup and G/F is polycyclic (see [**8**, Part 2, p. 169]).

Let A be the Wielandt subgroup of G. Since AR/R is contained in the Wielandt subgroup of G/R, it follows from Lemma 2 that $G/C_G((AR \cap F)/R)$ is finite. Therefore $E = C_G((A \cap F)/(A \cap R))$ has finite index in G and so it is finitely generated. Let $\{x_1, \ldots, x_t\}$ be a finite set of generators of E. If a is an element of $A \cap F$, we have that $a^{x_i} = au_i$, where $u_i \in A \cap R$ $(i = 1, \ldots, t)$, and there exists a finite G-invariant subgroup S of R containing $u_1, \ldots, u_t$. Then $[a, E]$ is contained in S and so a has finitely many conjugates in G. Therefore $A \cap F$ is contained in the FC-centre of G.

Clearly it may be assumed that F/R is non-periodic. But obviously A induces a group of power automorphisms on F/R. Thus either A centralizes F/R or else F/R is abelian, $A/C_A(F/R)$ has order two and the non-trivial automorphism induced by A on F/R is the inversion (see [**2**]). In the former case $A = A \cap F$ and A is contained in the FC-centre of G.

It remains to show, by obtaining a contradiction, that the latter case cannot hold. In this situation G/R is a finitely generated abelian-by-polycyclic group, and so it satisfies the maximal condition on normal subgroups (see [**8**, Part 1, p. 161]). In particular $(A \cap F)R/R$ is the normal closure of a finite subset of G/R, and so it is finitely generated, since every element of $A \cap F$ has finitely many conjugates in G. Since $A/(A \cap F)$ is finite, it follows that the group AR/R is a finitely generated T-group. Thus AR/R is either abelian or finite (see [**5**, Theorem 3.3.1]). If AR/R is abelian, then $A \leq F$ and $A = A \cap F$ is contained in the FC-centre of G. Assume finally that AR/R is finite. Hence $F/C_F(AR/R)$ is finite and A acts trivially on $C_F(AR/R)$. But A has an element acting by inversion on F/R and $C_F(AR/R)/R$ is a non-periodic subgroup, which is the required contradiction.

In the general case it follows from Lemma 1 that the T-group A contains a finite G-invariant subgroup B such that A/B is soluble. Then A/B is contained in the Wielandt subgroup of the soluble radical of G/B, and the first part of the proof shows that A/B is contained in the FC-centre of G/B. Since B is finite, A is contained in the FC-centre of G.

The following two examples show that the two finiteness conditions in Theorem A are both necessary.

Example 1. Let G be the split extension of the additive group A of rational numbers by $\langle \alpha \rangle$, where α is an automorphism of infinite order of A. Then every subnormal subgroup of G is either contained in A or contains A, and so A is the Wielandt subgroup of G. On the other hand, it is clear that the FC-centre of G is trivial.

Example 2 (P. Hall [**4**]). Let A be the additive group of a rational vector space V of

https://doi.org/10.1017/S0017089500009149 Published online by Cambridge University Press

dimension $\aleph_0$ with basis $\{a_i \mid i \in \mathbb{Z}\}$. Then $A = \bigoplus_{i \in \mathbb{Z}} A_i$, where A_i is the subspace generated by a_i. Let ξ and τ be the linear transformations of V defined by

$$a_i^{\xi} = a_{i+1}, \qquad a_i^{\tau} = p_i a_i \qquad (i \in \mathbb{Z}),$$

where the map $i \mapsto p_i$ is a bijection between $\mathbb{Z}$ and the set of all prime numbers. Hence $G = \langle \xi, \tau \rangle \ltimes A$ is a 3-generator soluble group with derived length three and trivial FC-centre. Since $\langle \xi, \tau \rangle$ is isomorphic to the wreath product of two infinite cyclic groups, it is easy to see that each subnormal subgroup H of G which is not contained in A contains an element x acting on every A_i as the multiplication by a rational number $r_i \neq 1, -1$. Then $[A_i, H] = A_i$ and so $[A, H] = A$. Since H is subnormal in G, it follows that $A \leq H$. Therefore A is the Wielandt subgroup of G.

In the proof of Theorem B the following result due to Cossey [3] will be used.

Lemma 3. *Let G be a nilpotent-by-abelian polycyclic group. Then $\omega(G)/Z(G)$ is finite.*

Proof of Theorem B. (a) By Theorem 3.3.1 of [5] it can be assumed that $\omega(G)$ is infinite abelian, so that $\omega(G)$ is contained in the Fitting subgroup F of G. Let $\{x_1, \ldots, x_t\}$ be a finite set of generators of G and put $H_i = \langle x_i, F \rangle$. Since G/F is nilpotent, H_i is subnormal in G, and so $\omega(G)$ is contained in the Wielandt subgroup of H_i. As H_i is nilpotent-by-abelian, it follows from Lemma 3 that $\omega(H_i)/Z(H_i)$ is finite; thus there exists a positive integer e_i such that $\omega(G)^{e_i} \leq Z(H_i) \leq C_G(x_i)$. If $e = e_1 \cdots e_t$, it follows that $\omega(G)^e \leq \bigcap_{i=1}^{t} C_G(x_i) = Z(G)$. Therefore $\omega(G)/Z(G)$ has finite exponent and so is finite.

(b) The proof will be by induction on the Hirsch length of G. Suppose first that G has no non-trivial finite normal subgroups, so that in particular the Fitting subgroup F of G is torsion-free. Clearly we may suppose that G is not nilpotent, so that G/F contains an abelian non-trivial normal subgroup H/F.

Let N be an abelian normal subgroup of G such that G/N is finite and let p be a prime number which does not divide the order of the finite group $H/C_H(N)$. Then, for each positive integer i, we have that

$$N/N^{p^i} = C_{N/N^{p^i}}(H) \times [N/N^{p^i}, H].$$

Hence $C_N(H) \cap [N, H] \leq \bigcap_{i>0} N^{p^i} = 1$.

If $[N, H] = 1$, then $N \leq Z(H)$, so that H is central-by-finite. Thus H' is finite, and so $H' = 1$ and H is abelian. This contradiction shows that $[N, H] \neq 1$. Suppose now that $C_N(H) = 1$. Then $Z(H) \cap N = 1$ and so $Z(H)$ is finite. But H is nilpotent-by-abelian and hence from Lemma 3 it follows that $\omega(H)$ is finite. Therefore $\omega(G)$ is finite (and even trivial in this case). Therefore we may assume that the normal subgroups $C_N(H)$ and $[N, H]$ are both non-trivial. It follows by induction that there exists a positive integer e such that $\omega(G)^e$ centralizes both $G/C_N(H)$ and $G/[N, H]$. Then $\omega(G)^e \leq Z(G)$, and $\omega(G)/Z(G)$ has finite exponent and so is finite.

In the general case, let T be the maximum finite normal subgroup of G. By the above the result is true for the factor group G/T. Since T is finite, there exists a positive integer e such that $\omega(G)^e$ centralizes G/T and T. Therefore $\omega(G)^e Z(G)/Z(G)$ has finite exponent, so that also $\omega(G)/Z(G)$ has finite exponent, and hence is finite.

The following example shows that the solubility hypothesis in Theorem B cannot in general be dropped.

EXAMPLE 3. Let $Q = A_5$ and let A be the free abelian group with basis $\{x_1, x_2, x_3, x_4, x_5\}$ endowed with the natural Q-module structure. Clearly $B = \langle x_1^{-1}x_2, x_2^{-1}x_3, x_3^{-1}x_4, x_4^{-1}x_5\rangle$ is a Q-submodule of A and $G = Q \ltimes B$ is a finitely generated abelian-by-finite group with trivial centre. Moreover $\omega(G)$ is infinite, since $\omega(G) = B$. This will follow directly from the fact that every proper subnormal subgroup of G is contained in B.

To see this first note that, if $\alpha = (12)(34)$, $\beta = (123)$ and $\gamma = (23)(45)$, then

$$([x_1^{-1}x_2, \alpha]^{-1}[x_1^{-1}x_2, \beta])^{\gamma} = x_1^{-1}x_2$$

and so $x_1^{-1}x_2$ belongs to $[B, Q]$. Similar computations show that all generators of B belong to $[B, Q]$, and hence $[B, Q] = B$. Now let S be any subnormal subgroup of G. Then, if S is not contained in B, we have $G = BS$, as Q is simple, and so $B = [B, Q] = [B, S] = [B,_k S]$ for all k. It follows that B is contained in S and so $S = G$.

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