A REPRESENTATION THEOREM FOR RELATIVELY COMPLEMENTED DISTRIBUTIVE LATTICES

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In this note, we are concerned with the following generalization of a well-known theorem of M. H. Stone; see (**2**, 8.2).

THEOREM 1. *Let L be a relatively complemented distributive lattice.*

(I) If L has no least element, then L is isomorphic to the lattice of non-empty compact-open subsets of an anti-Hausdorff, nearly-Hausdorff, T_1-space with a base of open sets consisting of compact-open sets.

(II) (**3**, Theorem **1**) *If L has a least element, then L is isomorphic to the lattice of all compact-open subsets of a locally compact totally disconnected space.*

Moreover, the spaces of (I) *and* (II) *are compact if and only if L has a greatest element.*

The space in question is the space of prime ideals of L with the hull-kernel topology.

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Definitions. A space is *anti-Hausdorff* (superconnected in the paper by J. de Groot; see (**1**)) if and only if every two non-empty open subsets have non-empty intersection. A space is *nearly-Hausdorff* if and only if every closed propert subset is Hausdorff.

The proof. Let $\mathscr{P}$ be the set of prime ideals of L. For $a \in L$, let

$$\mathscr{P}(a) = \{P \in \mathscr{P}: a \notin P\}.$$

Clearly, $\mathscr{P}(a \wedge b) = \mathscr{P}(a) \cap \mathscr{P}(b)$ and $\mathscr{P}(a \vee b) = \mathscr{P}(a) \cup \mathscr{P}(b)$. Topologize $\mathscr{P}$ by letting the set of all $\mathscr{P}(a)$ $(a \in L)$ be a base for the open sets.

(1) *If I is an ideal of L and $a \notin I$, there is a prime ideal of L containing I but not a.* For, let K be a maximal element of the family of ideals containing I but not a. Suppose $x \wedge y \in K$ (for some $x, y \in L$). Let K' be the ideal generated by K and x, and let K'' be the ideal generated by K and y. If a belongs to both K' and K'', then $a \leqslant k \vee x$ and $a \leqslant k' \vee y$ for some $k, k' \in K$; hence,

$$a \leqslant (k \wedge k') \vee (k \wedge y) \vee (x \wedge k') \vee (x \wedge y) \in K.$$

Thus either $a \notin K'$ or $a \notin K''$, in which case, by maximality, either $K = K'$ or $K = K''$, i.e., either x or y belongs to K.

For each $a \in L$, $\mathscr{P}(a)$ is compact. For, let $\{\mathscr{P}(a) - \mathscr{P}(b): b \in B\}$ be a family of basic closed subsets of $\mathscr{P}(a)$ with the finite intersection property.

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Let I be the ideal generated by B; $a \notin I$. By (1), there is a prime ideal P containing I but not a. P belongs to the intersection,

$$\bigcap \{\mathscr{P}(a) - \mathscr{P}(b): b \in B\}.$$

If $S \subset \mathscr{P}$ *is a non-empty compact-open set, then* $S = \mathscr{P}(a)$ *for some* $a \in L$. For then S is a finite union $\mathscr{P}(a_1) \cup \ldots \cup \mathscr{P}(a_n)$, therefore, $S = \mathscr{P}(a_1 \vee \ldots \vee a_n)$.

(2) *If P and Q are distinct prime ideals of L, then neither contains the other, i.e., the prime ideals are exactly the maximal ideals.* For, suppose $Q \subset P$. Let $b \in P - Q$, $c \notin P$, and $d \in Q$. Let e be the complement of b in $[b \wedge c \wedge d, b \vee c]$. Then $e \vee b = b \vee c \notin P$, therefore, $e \notin P$. Also, $e \wedge b = b \wedge c \wedge d \in Q$, therefore, $e \in Q$, contradicting $Q \subset P$.

For each $a \in L$, $\mathscr{P} - \mathscr{P}(a)$ is a Hausdorff space. For, by (2), if $P \neq Q$, there are $b \in P - Q$, and $b' \in Q - P$. Let c be the complement of b in $[a \wedge b \wedge b', a \vee b \vee b']$. Now, $b \vee c = b \vee b' \vee a \notin P$, so $c \notin P$. Finally, if neither b nor c belongs to some $R \in \mathscr{P}$, then $b \wedge b' \wedge a = b \wedge c \notin R$, therefore $a \notin R$, i.e., $(\mathscr{P}(b) \cap \mathscr{P}(c)) \cap (\mathscr{P} - \mathscr{P}(a)) = \emptyset$.

$\mathscr{P}$ is a T_1-space. For $\{P\} = \bigcap \{\mathscr{P} - \mathscr{P}(a): a \in P\}$.

The mapping $a \rightarrow \mathscr{P}(a)$ is 1-1. For if $c \ngeq b$ in L, the principal ideal $I(c)$ generated by c does not contain b. By (1), $I(c)$ is contained in a prime ideal not containing b.

Definition. A topological space X is an *L-space* if and only if X is T_1, anti-Hausdorff, nearly-Hausdorff, and the compact-open subsets of X form a base for the open sets of X.

The next theorem shows that the lattice of non-empty compact-open subsets of an L-space characterizes it as an L-space.

THEOREM 2. *If X is an L-space, then X is homeomorphic to the space of prime ideals of the lattice L of non-empty compact-open subsets of X.*

Proof. The lattice L is relatively complemented and distributive. If L had a least element, e, then every non-empty open subset of X would contain e, contradicting the statement X is T_1.

Let $\mathscr{P}$ be the prime ideal space of L with the hull-kernel topology. For $x \in X$, let $\phi(x)$ be the set of all non-empty compact-open subsets of X not containing x. This defines a function $\phi: X \rightarrow \mathscr{P}$.

If $\mathscr{P}(a) = \{P \in \mathscr{P}: a \notin P\}$ is a basic open set of $\mathscr{P}$, then

$$\phi^{-1}(\mathscr{P}(a)) = \{x \in X: a \notin \phi(x)\} = a,$$

which is open in X. Hence ϕ is continuous. If x and y are distinct elements of X, x lies in a compact-open set Y not containing y; then Y belongs to $\phi(y)$ but not to $\phi(x)$. Hence ϕ is 1-1.

Let a be a compact-open subset of X. If $P \in \phi[a]$, then $a \notin P$; hence $\phi[a] \subset \mathscr{P}(a)$. Let $P \in \mathscr{P}(a)$. Let P' be the set of complements of elements of P. P' is closed under finite intersections; if $S \in P'$, then $X - S \in P$, and $X - S$ does not contain a, so $S \cap a \neq \emptyset$; also S is closed. Hence there is

$$x \in \bigcap \{a \cap S: S \in P'\}.$$

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Clearly, $\phi(x)$ contains P; but $\phi(x)$ is a prime ideal, therefore by (2), $\phi(x) = P$. Since $x \in a$, $P \in \phi[a]$. Hence ϕ is an open map. Since $\mathscr{P} = \bigcup \mathscr{P}(a)$, ϕ is onto.

Further remarks. Every open subset of an L-space is an L-space. No closed proper subset of an L-space with more than one point is an L-space.

Let X be an L-space and Y a non-empty open subset of X. Let $\mathscr{L}(X)$ and $\mathscr{L}(Y)$ be the lattices of non-empty compact-open subsets of X and Y, respectively. Then the inclusion map is an isomorphism of $\mathscr{L}(Y)$ onto an ideal of $\mathscr{L}(X)$, and this ideal is prime if and only if Y is the complement of a singleton subset of X.

If I is an ideal of $\mathscr{L}(X)$, then I is $\mathscr{L}(Y)$ for some non-empty open subset Y of X. (*Proof.* Every ideal is the intersection of prime ideals. Every prime ideal is the set of non-empty compact-open subsets not containing a fixed point; hence every ideal is the set of non-empty compact-open subsets not containing a subset S of X, hence not containing the closure of S. Let $Y = X - \text{cl}\, S$.)

The principal ideals correspond to compact-open subsets. Hence, unless $\mathscr{L}(X)$ has a greatest element, no principal ideal is prime.

The L-space X can give information about the ideals of $\mathscr{L}(X)$ only because every proper ideal of $\mathscr{L}(X)$ is an intersection of prime ideals. Thus it is not reasonable to expect a characterization of the sub-(relatively complemented)-lattices of $\mathscr{L}(X)$ in terms of subsets of X.

Let X be an L-space and Z a closed subset of X. Then Z is a Boolean space (in the sense of **(3)**: a locally compact totally disconnected space) and $a \to a \cap Z$ is a homomorphism of $\mathscr{L}(X)$ onto the lattice of compact-open subsets of Z.

Let M be a relatively complemented distributive lattice and $\phi: \mathscr{L}(X) \to M$ an epimorphism. Then ϕ induces a homeomorphism of the prime ideal space, $\mathscr{P}(M)$, of M onto a subset Z of X; if M has a least element, then Z is closed and $X - Z$ represents the ideal $\ker \phi$. (*Proof.* Define $\theta: \mathscr{P}(M) \to X$ by $\theta(P) = \phi^{-1}(P)$, identifying X with the prime ideal space of $\mathscr{L}(X)$ by Theorem 2. Then θ^{-1} takes the basic open set determined by $a \in \mathscr{L}(X)$ to the basic open set determined by $\phi(a)$. Also θ is open: θ takes the basic open set determined by $b \in M$ to the trace on $\text{im}\, \theta$ of the basic open set determined by any element of $\phi^{-1}(b)$. Now, if M has a least element, the image of θ consists of all (prime ideals) $x \in X$ containing $\ker \phi$, i.e., all prime ideals corresponding to points $X - U$, where U represents the kernel of ϕ.)

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