Canad. Math. Bull. Vol. 67 (3), 2024, pp. 582–592
http://dx.doi.org/10.4153/S000843952300098X

Equal-Sum-Product problem II

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Abstract. In this paper, we present the results related to a problem posed by Andrzej Schinzel. Does the number $N_1(n)$ of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdot \ldots \cdot x_n, \ x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$$

tend to infinity with n? Let a be a positive integer. We give a lower bound on the number of integer solutions, $N_a(n)$, to the equation

$$x_1 + x_2 + \cdots + x_n = a x_1 x_2 \cdot \ldots \cdot x_n, \ x_1 \geq x_2 \geq \cdots \geq x_n \geq 1.$$

We show that if $N_2(n) = 1$, then the number $2n - 3$ is prime. The average behavior of $N_2(n)$ is studied. We prove that the set $\{n : N_2(n) \leq k, \ n \geq 2\}$ has zero natural density.

1 Introduction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of all natural numbers (i.e., positive integers). Equal-Sum-Product Problem is relatively easy to formulate but still unresolved (see [4]). Some early research focused on estimating the number of solutions, $N_1(n)$, to the equation

$$x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdot \ldots \cdot x_n, \ x_1 \geq x_2 \geq \cdots \geq x_n \geq 1, \tag{1.1}$$

which can be found in [3, 8]. Schinzel asked in papers [10, 11] if the number $N_1(n)$ tends toward infinity with n. This conjecture is yet to be proven. In [15], it was shown that the set $\{n : N_1(n) \leq k, \ n \in \mathbb{Z}, \ n \geq 2\}$ has zero natural density for all natural k. It is worth noting that the classical Diophantine equation $x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3$ was investigated by Markoff (1879), as mentioned in [1, 7]. Additionally, Hurwitz (see [5]) examined the family of equations $x_1^2 + x_2^2 + \cdots + x_n^2 = a x_1 x_2 \cdot \ldots \cdot x_n$, where $a, n \in \mathbb{N}$, $n \geq 3$. Let us now assume that $a, n \in \mathbb{N}$, $n \geq 2$. In this paper, we provide a lower bound for the number $N_a(n)$ of integer solutions $(x_1, x_2, \ldots, x_n)$ of the equation

$$x_1 + x_2 + \cdots + x_n = a x_1 x_2 \cdot \ldots \cdot x_n \tag{1.2}$$

such that $x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$. Some of the results presented can be generalized to the case of the equation

$$b(x_1 + x_2 + \cdots + x_n) = a x_1 x_2 \cdot \ldots \cdot x_n, \tag{1.3}$$

Received by the editors June 9, 2023; revised November 26, 2023; accepted November 29, 2023.
Published online on Cambridge Core September 13, 2023.
AMS subject classification: **11D72, 11D45.**
Keywords: Equal-sum-product, exceptional set, natural density.

where a, b are positive integers. In the case $a = 1,\ b = n$, the equation

$$n(x_1 + \cdots + x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

is called Erdós last equation (see [4, 12, 13]). Equation (1.3) is related to the problem of finding numbers divisible by the sum and product of their digits. It is worth noting that if equation (1.2) has solutions, then $a \le n$.

2 Basic results

In this section, we discuss the necessary basic results. First, we will show that the number of solutions $N_a(n)$ is finite for any fixed a and n.

Lemma 2.1 *Let n be a natural number. If $x_1, x_2, \ldots, x_n$ are any real numbers, then the following formula holds:*

$$\left(a\prod_{i=1}^{n-1} x_i - 1\right)(ax_n - 1) + a\sum_{s=1}^{n-2}\left(\left(\prod_{i=1}^{s} x_i - 1\right)(x_{s+1} - 1)\right) = a^2\prod_{i=1}^{n} x_i - a\sum_{i=1}^{n} x_i + a(n-2) + 1. \tag{2.1}$$

Proof Let us denote equation (2.1) as $T(n)$. We want to show by induction that $T(n)$ holds for every natural number n. The cases $n = 1$ and $n = 2$ are trivial: $(a-1)(ax_1 - 1) = a^2x_1 - ax_1 - a + 1$, $(ax_1 - 1)(ax_2 - 1) = a^2x_1x_2 - a(x_1 + x_2) + 1$. In both cases, equality is true. Therefore, the base step of the induction is satisfied, as $T(1)$ and $T(2)$ hold. Let us assume now that $n \ge 3$ and $T(n-1)$ holds, i.e., the following equality is true:

$$\left(a\prod_{i=1}^{n-2} x_i - 1\right)(ax_{n-1} - 1) + a\sum_{s=1}^{n-3}\left(\left(\prod_{i=1}^{s} x_i - 1\right)(x_{s+1} - 1)\right) = a^2\prod_{i=1}^{n-1} x_i - a\sum_{i=1}^{n-1} x_i + a(n-3) + 1. \tag{2.2}$$

In the inductive step, we will be using the equivalent form of equation (2.2):

$$a\sum_{s=1}^{n-3}\left(\left(\prod_{i=1}^{s} x_i - 1\right)(x_{s+1} - 1)\right) = -\left(a\prod_{i=1}^{n-2} x_i - 1\right)(ax_{n-1} - 1) + a^2\prod_{i=1}^{n-1} x_i - a\sum_{i=1}^{n-1} x_i + a(n-3) + 1. \tag{2.3}$$

To prove the inductive step, i.e., to show that $T(n-1)$ implies $T(n)$ for $n \ge 3$, we will use the following algebraic identities that can be verified directly:

$$\left(a\prod_{i=1}^{n-1}x_i-1\right)(ax_n-1)=a^2\prod_{i=1}^{n}x_i-ax_n+1-a\prod_{i=1}^{n-1}x_i, \tag{2.4}$$

$$a\sum_{s=1}^{n-2}\left(\left(\prod_{i=1}^{s}x_i-1\right)(x_{s+1}-1)\right)= a\left(\prod_{i=1}^{n-2}x_i-1\right)(x_{n-1}-1)+a\sum_{s=1}^{n-3}\left(\left(\prod_{i=1}^{s}x_i-1\right)(x_{s+1}-1)\right). \tag{2.5}$$

Let us proceed to the proof of the inductive step. We want to show $T(n)$ assuming $T(n-1)$. Let us start by transforming the left side of $T(n)$ using equations (2.4) and (2.5)

$$\left(a\prod_{i=1}^{n-1}x_i-1\right)(ax_n-1)+a\sum_{s=1}^{n-2}\left(\left(\prod_{i=1}^{s}x_i-1\right)(x_{s+1}-1)\right)= a^2\prod_{i=1}^{n}x_i-ax_n+1-a\prod_{i=1}^{n-1}x_i+ +a\left(\prod_{i=1}^{n-2}x_i-1\right)(x_{n-1}-1)+a\sum_{s=1}^{n-3}\left(\left(\prod_{i=1}^{s}x_i-1\right)(x_{s+1}-1)\right). \tag{2.6}$$

Calculating directly, we notice that the following equality holds true

$$-a\prod_{i=1}^{n-1}x_i+a\left(\prod_{i=1}^{n-2}x_i-1\right)(x_{n-1}-1)= -a\prod_{i=1}^{n-1}x_i+a\prod_{i=1}^{n-1}x_i-ax_{n-1}-a\prod_{i=1}^{n-2}x_i+a=a-ax_{n-1}-a\prod_{i=1}^{n-2}x_i. \tag{2.7}$$

From equations (2.6) and (2.7), and then using the inductive assumption (2.3), we obtain

$$\begin{aligned}&\left(a\prod_{i=1}^{n-1}x_i-1\right)(ax_n-1)+a\sum_{s=1}^{n-2}\left(\left(\prod_{i=1}^{s}x_i-1\right)(x_{s+1}-1)\right)\\ &=a^2\prod_{i=1}^{n}x_i-ax_n+1+a-ax_{n-1}-a\prod_{i=1}^{n-2}x_i+a\sum_{s=1}^{n-3}\left(\left(\prod_{i=1}^{s}x_i-1\right)(x_{s+1}-1)\right)\overset{(2.3)}{=}\\ &a^2\prod_{i=1}^{n}x_i-ax_n+1+a-ax_{n-1}-a\prod_{i=1}^{n-2}x_i-\left(a\prod_{i=1}^{n-2}x_i-1\right)(ax_{n-1}-1)+\\ &+a^2\prod_{i=1}^{n-1}x_i-a\sum_{i=1}^{n-1}x_i+a(n-3)+1=a^2\prod_{i=1}^{n}x_i-a\sum_{i=1}^{n}x_i+a(n-2)+1.\end{aligned}$$

Thus, assuming $T(n-1)$, we have shown that $T(n)$ holds, completing the inductive step and concluding the proof of the lemma. ∎

Theorem 2.2 *Let $a, k \in \mathbb{N}$, $b \in \mathbb{N} \cup \{0\}$. For any integer $n \geq 2$, the system of Diophantine equations*

$$\left\{ \begin{array}{rcl} x_{1,1} + x_{1,2} + \cdots + x_{1,n} & = & a x_{2,1} \cdot x_{2,2} \cdot \ldots \cdot x_{2,n} + b\,, \\ x_{2,1} + x_{2,2} + \cdots + x_{2,n} & = & a x_{3,1} \cdot x_{3,2} \cdot \ldots \cdot x_{3,n} + b\,, \\ & \cdots & \\ x_{k-1,1} + x_{k-1,2} + \cdots + x_{k-1,n} & = & a x_{k,1} \cdot x_{k,2} \cdot \ldots \cdot x_{k,n} + b\,, \\ x_{k,1} + x_{k,2} + \cdots + x_{k,n} & = & a x_{1,1} \cdot x_{1,2} \cdot \ldots \cdot x_{1,n} + b \end{array} \right. \tag{2.8}$$

has only finite number of solutions $x_{i,j}$ which are natural numbers.

Proof By adding sides of equations of the system of equations (2.8), we obtain

$$\sum_{i=1}^{k} \sum_{j=1}^{n} x_{i,j} = \sum_{i=1}^{k} a \prod_{j=1}^{n} x_{i,j} + kb.$$

Hence,

$$\sum_{i=1}^{k} \left(a^2 \prod_{j=1}^{n} x_{i,j} - a \sum_{j=1}^{n} x_{i,j} + a(n-2) + 1 \right) = k(a(n-2)+1) - kab.$$

By (2.1), we have

$$\sum_{i=1}^{k} \left(\left(a \prod_{j=1}^{n-1} x_{i,j} - 1 \right) (a x_{i,n} - 1) + a \sum_{s=1}^{n-2} \left(\prod_{j=1}^{s} x_{i,j} - 1 \right) (x_{i,s+1} - 1) \right) = \\ k(a(n-2)+1) - kab. \tag{2.9}$$

For given a, k, b, n, the number of solutions of equation (2.9) in positive integers is bounded above. Hence, the system of equations (2.8) has only a finite number of solutions in positive integers $x_{i,j}$. ■

Taking $k = 1$, an immediate consequence of Theorem 2.2 is the following result.

Corollary 2.3 *For given $a \in \mathbb{N}$, $b \in \mathbb{N} \cup \{0\}$ and any integer $n \geq 2$, the number of solutions of the equation*

$$x_1 + x_2 + \cdots + x_n = a x_1 \cdot x_2 \cdot \ldots \cdot x_n + b \tag{2.10}$$

in positive integers $x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$ is finite. In particular, in the case $b = 0$, the number of solutions $N_a(n)$ is finite.

Remark 2.4 Theorem 2.2 is true for all $a, b \in \mathbb{Q}$, $a \geq 1$.

Remark 2.5 In the case of $b = 0$, we can provide a different proof of Corollary 2.3. Let $z_i = x_1 x_2 \cdot \ldots \cdot x_{i-1} x_{i+1} \cdot \ldots \cdot x_n = \frac{1}{x_i} \prod_{j=1}^{n} x_j \in \mathbb{N}$ for $i \in \{1, 2, \ldots, n\}$. Notice that from the inequality $x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$, we get the inequality $1 \leq z_1 \leq z_2 \leq \cdots \leq z_n$. Then, equation (2.10) takes the form

$$\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n} = a \geq 1. \tag{2.11}$$

Equation (2.11) has finitely many solutions in positive integers, as we can find upper bounds on z_i. The bounds we will find are not optimal, but they are sufficient for our purposes. If $n \geq 2$, then $ax_1x_2 \cdot \ldots \cdot x_n = x_1 + x_2 + \cdots + x_n \geq x_1 + x_2 \geq x_1 + 1 > x_1$, and hence $ax_2 \cdot \ldots \cdot x_n \geq 2$. From here, we can deduce

$$(n-1)x_2 \geq x_2 + \cdots + x_n = x_1(ax_2 \cdot \ldots \cdot x_n - 1) \geq x_1.$$

Therefore, $nx_2 > x_1$ and $nz_1 > z_2$. We also have for $k \in \{2, 3, \ldots, n-1\}$, that

$$nz_1z_2 \cdot \ldots \cdot z_k \geq z_1z_2 \geq \prod_{i=1}^{n} x_i \geq z_{k+1}.$$

Thus, for all $k \in \{1, 2, \ldots, n-1\}$, we have $z_{k+1} \leq nz_1 \cdot z_2 \cdot \ldots \cdot z_k$. Now we can proceed with the inductive proof of the upper bound: $z_i \leq a^{-1}n^{2^{i-1}}$, where $i \in \{1, 2, \ldots, n\}$. Base step, as the z_i are increasing, we can use equation (2.11) to obtain an inequality:

$$\tfrac{n}{z_1} \geq \tfrac{1}{z_1} + \tfrac{1}{z_2} + \cdots + \tfrac{1}{z_n} = a \geq 1, \text{ hence } z_1 \leq a^{-1}n.$$

If we now make the assumption that $z_i \leq a^{-1}n^{2^{i-1}}$ for all $i \in \{1, 2, \ldots, k\}$, where $k < n$, then $z_{k+1} \leq nz_1z_2 \cdot \ldots \cdot z_k \leq n\frac{n^{2^0+2^1+2^2+\cdots+2^{k-1}}}{a} = \frac{n^{2^k}}{a}$; this establishes the inductive step.

The proof of Theorem 2.2 can be modified in the specific case of a, n to create an efficient algorithm for finding solutions to equation (2.10).

Kurlandchik and Nowicki [6, Theorem 3] had earlier shown that $N_1(n)$ is finite for any $n \geq 2$.

Schinzel's question can be generalized. For given $a \in \mathbb{N}$, does the number $N_a(n)$ tend to infinity with n? We can show with the elementary method the following theorems.

Theorem 2.6 *If $a, n \in \mathbb{N}$, then* $\limsup_{n\to\infty} N_a(n) = \infty$.

Proof We shall consider two cases. Let $a \in \{1, 2\}$. If $t \in \{0, 1, \ldots, \lfloor \frac{s}{2} \rfloor\}$, where s is a nonnegative integer, then

$$\tfrac{1}{a}((a+1)^{s-t}+1) + \tfrac{1}{a}((a+1)^{t}+1) + \underbrace{1+1+\cdots+1}_{\frac{1}{a}((a+1)^s-1)\text{ times}} =$$
$$a \cdot \tfrac{1}{a}((a+1)^{s-t}+1) \cdot \tfrac{1}{a}((a+1)^{t}+1) \cdot \underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{\frac{1}{a}((a+1)^s-1)\text{ times}}.$$

We have $s - t \geq t$ and $\frac{1}{a}((a+1)^i + 1) \in \mathbb{N}$, where i is a nonnegative integer. Hence, $N(\frac{1}{a}((a+1)^s + 2a - 1)) \geq \lfloor \frac{s}{2} \rfloor + 1$. Therefore, $\limsup_{n\to\infty} N_a(n) = \infty$.

Let $a \geq 3$. If $t \in \{1, \dots, \lfloor \frac{s+1}{2} \rfloor\}$, where $s \in \mathbb{N}$, then

$$\frac{1}{a}((a-1)^{2s-2t+1}+1) + \frac{1}{a}((a-1)^{2t-1}+1) + \underbrace{1+1+\cdots+1}_{\frac{1}{a}((a-1)^{2s}-1) \text{ times}} =$$
$$a \cdot \frac{1}{a}((a-1)^{2s-2t+1}+1) \cdot \frac{1}{a}((a-1)^{2t-1}+1) \cdot \underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{\frac{1}{a}((a-1)^{2s}-1) \text{ times}} .$$

We have $2s - 2t + 1 \geq 2t - 1$ and $\frac{1}{a}((a-1)^{2i-1}+1), \frac{1}{a}((a-1)^{2i}-1) \in \mathbb{N}$, where $i \in \mathbb{N}$. Hence, $N(\frac{1}{a}((a-1)^{2s}+2a-1)) \geq \lfloor \frac{s+1}{2} \rfloor$.

Therefore, $\limsup_{n\to\infty} N_a(n) = \infty$. ∎

Remark 2.7 Let $a \geq 3$. Depending on the choice of $a \leq n$, equation (1.2) may not have solutions. The simplest example is $a = 3$ and $n = 4$. In this case, equation (1.2) is equivalent to

$$(3x_1x_2x_3 - 1)(3x_4 - 1) + 3(x_1x_2 - 1)(x_3 - 1) + 3(x_1 - 1)(x_2 - 1) = 7,$$

but the corresponding equation has no integer solutions $x_1 \geq x_2 \geq x_3 \geq x_4 \geq 1$. This gives $N_3(4) = 0$.

Remark 2.8 Due to the solutions $(\underbrace{2, 2, \dots, 1}_{4a-2 \text{ times}})$, $(\underbrace{m, 1, \dots, 1}_{ma-m+1 \text{ times}})$, where $m \in \mathbb{N}$ and certain technical computations based on the method from Remark 2.5, we can prove that:

(1) $N_a(a) = N_a(2a-1) = N_a(3a-2) = N_a(4a-3) = 1$, where $a \geq 2$,
(2) $N_2(6) = 2$, $N_a(4a-2) = 1$, where $a \geq 3$,
(3) $N_a(n) = 0$ if $n \in ((a, 2a-1) \cup (2a-1, 3a-2) \cup (3a-2, 4a-3)) \cap \mathbb{N}$,
(4) $N_a(ma - m + 1) \geq 1$, where $m \in \mathbb{N}$.

Points (1)–(3) partially explain the basic structure of the right side of Table 1.

It has been proven in [15] that in the case of $a = 1$, the following theorem holds.

Theorem 2.9 *If $n \in \mathbb{N}$, $n \geq 2$, then*

$$N_1(n) \geq \left\lfloor \frac{d(n-1)+1}{2} \right\rfloor + \left\lfloor \frac{d(2n-1)+1}{2} \right\rfloor - 1, \tag{2.12}$$

where $d(j)$ is the number of positive divisors of j. Moreover,

$$\begin{aligned} N_1(n) \geq &\left\lfloor \frac{d(n-1)+1}{2} \right\rfloor + \left\lfloor \frac{d(2n-1)+1}{2} \right\rfloor - 1 \\ &+ \left\lfloor \frac{d_2(3n+1)+1}{2} \right\rfloor + \left\lfloor \frac{d_3(4n+1)+1}{2} \right\rfloor + \left\lfloor \frac{d_3(4n+5)+1}{2} \right\rfloor \\ &- \delta(2|n+1) - \delta(3|n+1) - \delta(3|n+2) \\ &- \delta(5|n+2,\ n \geq 8) - \delta(7|n+3,\ n \geq 11) - \delta(11|n+4,\ n \geq 29), \end{aligned} \tag{2.13}$$

where $d_i(m)$ is the number of positive divisors of m which lie in the arithmetic progression $i(\text{mod } i+1)$. The function δ is the Dirac delta function.

Table 1: The table shows values of $N_a(n)$ for small natural numbers $a \le n \le 10$. The bold numbers are $N_a(n)$, such that $n \ge 4a - 1$.

$n\backslash a$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	1														
3	**1**	1	1													
4	**1**	1	0	1												
5	**3**	1	1	0	1											
6	**1**	2	0	0	0	1										
7	**2**	**1**	1	1	0	0	1									
8	**2**	**1**	0	0	0	0	0	1								
9	**2**	**2**	1	0	1	0	0	0	1							
10	**2**	**1**	1	1	0	0	0	0	0	1						
11	**3**	**1**	**1**	0	0	1	0	0	0	0	1					
12	**2**	**2**	**0**	0	0	0	0	0	0	0	0	1				
13	**4**	**2**	**1**	1	1	0	1	0	0	0	0	0	1			
14	**2**	**2**	**0**	1	0	0	0	0	0	0	0	0	0	1		
15	**2**	**2**	**2**	**0**	0	0	0	1	0	0	0	0	0	0	1	
16	**2**	**1**	**0**	**1**	0	1	0	0	0	0	0	0	0	0	0	1

Remark 2.10 In the case $a = 2$, equation (1.2) has at least one *typical* solution in the form $(n-1, \underbrace{1, 1, \ldots, 1}_{n-1 \text{ times}})$. Therefore, $N_2(n) \ge 1$ for all integers $n \ge 2$.

3 Main results

We give a lower bound on the number of solutions $N_a(n)$ of equation (1.2).

Theorem 3.1 *If $a, n \in \mathbb{N}$, $n \ge 2$, then*

$$N_a(n) \ge \left\lfloor \frac{d_{a-1}(a(n-2)+1)+1}{2} \right\rfloor + \left\lfloor \frac{d_{2a-1}(2a(n-1)+1)+1}{2} \right\rfloor - \delta(2a-1|n), \tag{3.1}$$

where $d_i(m)$ is the number of positive divisors of m which lie in the arithmetic progression $i \pmod{i+1}$. The function δ is the Dirac delta function.

Proof In the set $\mathbb{N}^n$, we have the following pairwise disjoint families of pairwise different $(x_1, x_2, \ldots, x_n)$ solutions of equation (1.2). Note that in each case x_i is an integer and $x_1 \ge x_2 \ge \cdots \ge x_n \ge 1$. We define

$$\begin{aligned} A_1(n) = \{(\tfrac{n-2+\frac{d+1}{a}}{d}, \tfrac{d+1}{a}, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ times}}) : \\ a(n-2)+1 \equiv 0 \pmod{d},\ d \equiv -1 \pmod{a}, \\ 1 \le d \le \sqrt{a(n-2)+1},\ d \in \mathbb{N}\}. \end{aligned}$$

We also define

$$A_2(n) = \{(\tfrac{n-1+\frac{d+1}{2a}}{d}, \tfrac{d+1}{2a}, 2, \underbrace{1, 1, \ldots, 1}_{n-3 \text{ times}}) :$$
$$2a(n-1)+1 \equiv 0 \pmod{d},\ d \equiv -1 \pmod{2a},$$
$$4a-1 \le d \le \sqrt{2a(n-1)+1},\ d \in \mathbb{N}\},\ \text{when } n \ge 3.$$

We have $A_2(2) = \varnothing$. Moreover,

$$|A_1(n)| = |\{d :\ a(n-2)+1 \equiv 0 \pmod{d},\ d \equiv -1 \pmod{a},$$
$$1 \le d \le \sqrt{a(n-2)+1},\ d \in \mathbb{N}\}| = \left\lfloor \tfrac{d_{a-1}(a(n-2)+1)+1}{2} \right\rfloor.$$

In the case of the set $A_2(n)$, we have $d \ne 2a-1$; thus,

$$\begin{aligned} |A_2(n)| = |\{d :\ 2a(n-1)+1 \equiv 0 \pmod{d},\ d \equiv -1 \pmod{2a}, \\ 4a-1 \le d \le \sqrt{2a(n-1)+1},\ d \in \mathbb{N}\}| = \\ = |\{d :\ 2a(n-1)+1 \equiv 0 \pmod{d},\ d \equiv -1 \pmod{2a}, \\ 1 \le d \le \sqrt{2a(n-1)+1},\ d \in \mathbb{N}\}| \\ -|\{d : 2a(n-1)+1 \equiv 0 \pmod{d},\ d = 2a-1\}| = \\ \left\lfloor \tfrac{d_{2a-1}(2a(n-1)+1)+1}{2} \right\rfloor - \delta(2a-1|n). \end{aligned}$$

The sets $A_1(n)$, $A_2(n)$ are disjoint. Hence, $N_a(n) \ge |A_1(n)| + |A_2(n)|$. Thus, we get immediately (3.1). ■

Corollary 3.2 *If $n \in \mathbb{N}$, $n \ge 2$, then*

$$N_2(n) \ge \left\lfloor \tfrac{d(2n-3)+1}{2} \right\rfloor + \left\lfloor \tfrac{d_3(4n-3)+1}{2} \right\rfloor - \delta(3|n). \tag{3.2}$$

The following corollary is almost immediate.

Corollary 3.3 *If $n \in \mathbb{N}$, $n \ge 2$, then*

$$N_2(n) \ge \tfrac{1}{2} d(2n-3). \tag{3.3}$$

Proof Formula (3.3) follows at once from Corollary 3.2 and inequalities

$$\left\lfloor \tfrac{d_3(4n-3)+1}{2} \right\rfloor \ge \delta(3|n),\ \left\lfloor \tfrac{x+1}{2} \right\rfloor \ge \tfrac{1}{2}x,\ \text{where } x \in \mathbb{Z}.$$ ■

For the convenience of the reader, values of $N_2(n)$ for small values of n are presented in Table 2.

Table 2: The table lists the numbers $N_2(n)$ for $2 \le n \le 51$.

n	$N_2(n)$	n	$N_2(n)$	n	$N_2(n)$	n	$N_2(n)$	n	$N_2(n)$	n	$N_2(n)$	n	$N_2(n)$	n	$N_2(n)$	n	$N_2(n)$	n	$N_2(n)$
2	1	7	1	12	2	17	1	22	1	27	3	32	1	37	1	42	4	47	2
3	1	8	1	13	2	18	2	23	1	28	2	33	3	38	1	43	2	48	4
4	1	9	2	14	2	19	2	24	3	29	2	34	3	39	3	44	2	49	2
5	1	10	1	15	2	20	2	25	1	30	2	35	3	40	2	45	2	50	1
6	2	11	1	16	1	21	2	26	2	31	2	36	2	41	2	46	1	51	3

Corollary 3.4 *Let $n \in \mathbb{N}$, $n \geq 3$. If the equation*

$$x_1 + x_2 + \cdots + x_n = 2x_1 \cdot x_2 \cdot \ldots \cdot x_n \tag{3.4}$$

has exactly one solution $(n-1, \underbrace{1, 1, \ldots, 1}_{n-1 \text{ times}})$ in the natural numbers $x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$, then $2n-3$ is a prime number.

Proof If $N_2(n) = 1$, then by Corollary 3.3 we get $2 \geq d(2n-3)$. Since $2n-3 \geq 3$, it follows that $2n-3$ is a prime number. ∎

Remark 3.5 If $N_1(n) = 1$, then $n-1$ must be a Sophie Germain prime number (see [8]).

4 The set of exceptional values

Let $E^2_{\leq k} = \{n : N_2(n) \leq k,\ n \geq 2\}$, where $k \in \mathbb{N}$. In particular, $E^2_{\leq 1} = \{n : N_2(n) = 1, n \geq 2\}$.

Theorem 4.1 *The set $E^2_{\leq k}$ has natural density 0, i.e., the ratio*

$$\tfrac{1}{x}|E^2_{\leq k} \cap [1, x]|$$

tends to 0 as $x \to \infty$.

Proof Let $\Omega(m)$ count the total number of prime factors of m. We have $\Omega(m) \leq d(m) - 1$ for every natural m. Let $\pi_i(x) = |\{m : \Omega(m) = i,\ 1 \leq m \leq x\}|$, i.e., the number of $1 \leq m \leq x$ with i prime factors (not necessarily distinct). By Corollary 3.3, we have $N_2(n) \geq \frac{1}{2}d(2n-3)$. Thus, if $n \in E^2_{\leq k}$, then $d(2n-3) \leq 2k$ and consequently $\Omega(2n-3) \leq 2k-1$. Therefore,

$$|E^2_{\leq k} \cap [1, x]| \leq \sum_{i=0}^{2k-1} \pi_i(2x-3),$$

where $x \geq 2$. Using the sieve of Eratosthenes, one can show that (see [2, p. 75])

$$\pi_i(x) \leq \tfrac{1}{i!} x \frac{(A \log\log x + B)^i}{\log x}$$

for some constants $A, B > 0$. There follows that

$$0 \leq \tfrac{1}{x}|E^2_{\leq k} \cap [1, x]| \leq \tfrac{2x-3}{x} \sum_{i=0}^{2k-1} \tfrac{1}{i!} \frac{(A \log\log (2x-3) + B)^i}{\log (2x-3)}.$$

For a fixed k, the right-hand side tends to 0, as $x \to \infty$. Thus,

$$\lim_{x\to\infty} \tfrac{1}{x}|E^2_{\leq k} \cap [1, x]| = 0.$$

This completes the proof. ∎

The above theorem implies that the set $E^2_k = \{n :\ N_2(n) = k,\ n \geq 2\}$ has zero natural density for any fixed $k \geq 1$. This observation might suggest that the set $E^2_k = \{n :\ N_2(n) = k,\ n \geq 2\}$ is finite for any fixed $k \geq 1$ and that the number $N_2(n) \to \infty$ as $n \to \infty$. In the next theorem, we study the average behavior of $N_2(n)$.

Theorem 4.2 *If $\varepsilon > 0$, then for sufficiently large x, we have*

$$\sum_{1<n\leq x} N_2(n) \geq \tfrac{1-\varepsilon}{8} x \log x.$$

Proof By [9, 14], there exists constant $c > 0$ such that

$$\left| \sum_{\substack{1\leq n\leq x,\\ n\equiv 1 \pmod 2}} d(n) - \tfrac{x}{4}\log x \right| \leq cx,$$

for sufficiently large $x > x_0$. It follows that

$$\sum_{\substack{1\leq n\leq x,\\ n\equiv 1 \pmod 2}} d(n) \geq \tfrac{x}{4}\log(x) - cx$$

for $x > x_0$. By Corollary 3.3, for $n \geq 2$, we have $N_2(n) \geq \frac{1}{2} d(2n-3)$. Therefore,

$$\begin{aligned}\frac{1}{x}\sum_{1<n\leq x} N_2(n) \geq \frac{1}{x}\sum_{1<n\leq x} \tfrac{1}{2} d(2n-3) &= \frac{1}{2x} \sum_{\substack{1\leq m\leq 2x-3\\ m\equiv 1 \pmod 2}} d(m)\\ &\geq \frac{1}{8}\log(2x-3) - c\,\tfrac{2x-3}{2x}\end{aligned}$$

for $2x - 3 > x_0$. Let $\varepsilon > 0$, then for sufficiently large x, we have

$$\frac{1}{x}\sum_{1<n\leq x} N_2(n) \geq (1-\varepsilon)\frac{1}{8}\log x.$$ ∎

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