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BALANCE FOR TATE COHOMOLOGY WITH RESPECT TO SEMIDUALIZING MODULES

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Abstract

In this paper, we further study Tate cohomology of modules over a commutative ring with respect to semidualizing modules using the ideals of Sather-Wagstaff *et al*. ['Tate cohomology with respect to semidualizing modules', *J. Algebra* **324** (2010), 2336–2368]. In particular, we prove a balance result for the Tate cohomology $\widehat{\mathrm{Ext}}^n$ for any $n \in \mathbb{Z}$. This result complements the work of Sather-Wagstaff *et al*., who proved that the result holds for any $n \geq 1$. We also discuss some vanishing properties of Tate cohomology.

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1. Introduction

Tate cohomology was initially defined for representations of finite groups. Avramov and Martsinkovsky [1] extended the definition so that it can work well for finite modules of finite G-dimension over a Noetherian ring. They showed that if M is a finite R-module of finite G-dimension, then there is an exact sequence connecting the absolute cohomology functor $\mathrm{Ext}_R^*(M, -)$, the relative cohomology functor $\mathrm{Ext}_{\mathcal{G}}^*(M, -)$ (that are defined by a proper Gorenstein projective resolution of M), and the Tate cohomology functor $\widehat{\mathrm{Ext}}_R^*(M, -)$ (see [1, (7.1)]).

Balancedness of absolute cohomology Ext_R is well known. Holm [5, (3.6)] gave a balance result for the relative cohomlogy $\mathrm{Ext}_{\mathcal{G}}$ by showing that if M is an R-module of finite Gorenstein projective dimension and N is an R-module of finite Gorenstein injective dimension then $\mathrm{Ext}_{\mathcal{G}}^*(M, N)$ can also be computed using a proper Gorenstein injective resolution of N. Iacob [6, Theorem 2] proved a balance result for Tate

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cohomology $\widehat{\mathrm{Ext}}_R$ over Gorenstein rings. Recently, Christensen and Jorgensen [3] used the idea of a pinched complex to prove a general balance result for Tate cohomology, while Enochs *et al.* [4] gave a new way of constructing homology groups associated with a double complex, and with this result gave a new and elementary proof of balancedness of Tate cohomology.

Let $\mathcal{X}$ denote a subcategory of an abelian category $\mathcal{A}$ and $\mathcal{G}(\mathcal{X})$ denote the subcategory of $\mathcal{A}$ with objects of the form $M \cong \mathrm{Ker}(\delta^X_{-1})$ for some totally $\mathcal{X}$-acyclic complex X (see Section 2.3). Sather-Wagstaff *et al.* [9] constructed a theory of Tate cohomology in abelian categories. They proved the following balance result (see [9, (6.1)]).

THEOREM. *Let $\mathcal{W}$ and $\mathcal{V}$ be subcategories of $\mathcal{A}$. Assume that $\mathcal{W} \perp \mathcal{W}$ and $\mathcal{V} \perp \mathcal{V}$ and $\mathcal{G}(\mathcal{W}) \perp \mathcal{V}$ and $\mathcal{W} \perp \mathcal{G}(\mathcal{V})$. Assume that $\mathcal{W}$ is closed under kernels of epimorphisms and direct summands, and that $\mathcal{V}$ is closed under cokernels of monomorphisms and direct summands. Assume also that $\mathrm{Ext}^{\geq 1}_{\mathcal{W}\mathcal{A}}(\mathrm{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0 = \mathrm{Ext}^{\geq 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W}, \mathrm{cores}\widehat{\mathcal{V}})$. Then, for all $M \in \mathrm{res}\widehat{\mathcal{G}(\mathcal{W})}$, all $N \in \mathrm{cores}\widehat{\mathcal{G}(\mathcal{V})}$ and all $n \geq 1$,*

$$\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, N) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M, N).$$

We notice that when $\mathcal{W}$ is the subcategory of projective R-modules and $\mathcal{V}$ is the subcategory of injective R-modules, then the above theorem gives a balance result for Tate cohomology $\widehat{\mathrm{Ext}}^n_R$ for $n \geq 1$ over any associative ring R.

In this paper, we further study balancedness of Tate cohomology in abelian categories. We show that the result of Sather-Wagstaff*et al.* [9, (6.1)] is true for any $n \in \mathbb{Z}$ (see Corollary 3.10). More generally, we prove the following result (see Theorem 3.9).

THEOREM A. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{W}$ and $\mathcal{V}$ be subcategories of $\mathcal{A}$. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are exact, and $\mathcal{X}$ is closed under kernels of epimorphisms and $\mathcal{Y}$ is closed under cokernels of monomorphisms. Assume that $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$, and $\mathcal{V}$ is both an injective cogenerator and a projective generator for $\mathcal{Y}$. Assume also that $\mathcal{W}$ and $\mathcal{V}$ are closed under direct summands and satisfy $\mathcal{W} \perp \mathcal{Y}$, $\mathcal{X} \perp \mathcal{V}$ and $\mathrm{Ext}^{\geq 1}_{\mathcal{W}\mathcal{A}}(\mathrm{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0 = \mathrm{Ext}^{\geq 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W}, \mathrm{cores}\widehat{\mathcal{V}})$. Then, for all $M \in \mathrm{res}\widehat{\mathcal{X}}$ and $N \in \mathrm{cores}\widehat{\mathcal{Y}}$, and all $n \in \mathbb{Z}$,*

$$\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, N) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M, N).$$

As an application of Theorem A, we get the next balance result for Tate cohomology of modules with respect to semidualizing modules (see Corollary 3.12). This result was proved for each $n \geq 1$ in [9, Theorem D].

THEOREM B. *Let R be a commutative ring, and let B and C be semidualizing R-modules such that $B \in \mathcal{GP}_C(R)$. Set $B^\dagger = \mathrm{Hom}_R(B, C)$. Let M and N be R-modules such that $\mathcal{G}(\mathcal{P}_B)$-$\mathrm{pd}_R(M) < \infty$ and $\mathcal{G}(\mathcal{I}_{B^\dagger})$-$\mathrm{id}_R(N) < \infty$. Then, for each $n \in \mathbb{Z}$,*

$$\widehat{\mathrm{Ext}}^n_{\mathcal{P}_B\mathcal{M}}(M, N) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{M}\mathcal{I}_{B^\dagger}}(M, N).$$

Furthermore, under the hypothesis of Theorem B, we get that M has a proper $\mathcal{P}_{\mathcal{B}}$-resolution $W \xrightarrow{\simeq} M$ and a proper $\mathcal{G}(\mathcal{P}_B)$-resolution $X \xrightarrow{\simeq} M$ by Lemma 2.6. Set $\overline{\mathrm{id}_M}: W \longrightarrow X$ to be a lifting of the identity $\mathrm{id}_M: M \longrightarrow M$. Dually, one can construct $\overline{\mathrm{id}_N}$. Then the next result provides a new method to compute Tate cohomology of modules with respect to semidualizing modules (see Corollary 3.16).

Theorem C. *Let R be a commutative ring, and let B and C be semidualizing R-modules such that $B \in \mathcal{GP}_C(R)$. Set $B^{\dagger} = \mathrm{Hom}_R(B, C)$. Let M and N be R-modules such that $\mathcal{G}(\mathcal{P}_B)\text{-pd}_R(M) < \infty$ and $\mathcal{G}(\mathcal{I}_{B^{\dagger}})\text{-id}_R(N) < \infty$. Then, for each $n \geq 1$,*

$$\begin{aligned}\widehat{\mathrm{Ext}}^n_{\mathcal{P}_B\mathcal{M}}(M, N) &\cong \widehat{\mathrm{Ext}}^n_{\mathcal{M}\mathcal{I}_{B^{\dagger}}}(M, N)\\ &\cong \mathrm{H}_{-n-1}(\mathrm{Hom}_R(\mathrm{Cone}(\overline{\mathrm{id}_M}), N))\\ &\cong \mathrm{H}_{-n}(\mathrm{Hom}_R(M, \mathrm{Cone}(\overline{\mathrm{id}_N}))).\end{aligned}$$

As we will see, the vanishing properties of Tate cohomology play an important role in the proof of Theorem A. We prove the next vanishing result that encompasses the results of Sather-Wagstaff *et al.* [9, (5.2), (5.6) and (5.7)] (see Theorem 3.5 and Corollary 3.6).

Theorem D. *Let $\mathcal{X}$ and $\mathcal{W}$ be subcategories of $\mathcal{A}$. Assume that $\mathcal{X}$ is exact and closed under kernels of epimorphisms, and that $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$ and closed under direct summands. If $M \in \mathrm{res}\widehat{\mathcal{X}}$, then the following statements are equivalent.*

(1) $M \in \mathrm{res}\widehat{\mathcal{W}}$.
(2) $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(-, M) = 0$ *on* $\mathrm{res}\widehat{\mathcal{X}}$ *for each* $n \in \mathbb{Z}$.
(3) $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, -) = 0$ *for each* $n \in \mathbb{Z}$.
(4) $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(-, M) = 0$ *on* $\mathrm{res}\widehat{\mathcal{X}}$ *for some* $n \in \mathbb{Z}$.
(5) $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, -) = 0$ *for some* $n \in \mathbb{Z}$.
(6) $\widehat{\mathrm{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) = 0$.
(7) *The transformation* $\vartheta^i_{\mathcal{X}\mathcal{W}\mathcal{A}}(-, M): \mathrm{Ext}^i_{\mathcal{X}\mathcal{A}}(-, M) \longrightarrow \mathrm{Ext}^i_{\mathcal{W}\mathcal{A}}(-, M)$ *is an isomorphism on* $\mathrm{res}\widehat{\mathcal{X}}$ *for each* $i \in \mathbb{Z}$.
(8) *The transformation* $\vartheta^i_{\mathcal{X}\mathcal{W}\mathcal{A}}(M, -): \mathrm{Ext}^i_{\mathcal{X}\mathcal{A}}(M, -) \longrightarrow \mathrm{Ext}^i_{\mathcal{W}\mathcal{A}}(M, -)$ *is an isomorphism for each* $i \in \mathbb{Z}$.
(9) *The transformation* $\vartheta^i_{\mathcal{X}\mathcal{W}\mathcal{A}}(-, M): \mathrm{Ext}^i_{\mathcal{X}\mathcal{A}}(-, M) \longrightarrow \mathrm{Ext}^i_{\mathcal{W}\mathcal{A}}(-, M)$ *is an isomorphism on* $\mathrm{res}\widehat{\mathcal{X}}$ *for each* $1 \leq i \leq 2$.
(10) *The transformation* $\vartheta^i_{\mathcal{X}\mathcal{W}\mathcal{A}}(M, -): \mathrm{Ext}^i_{\mathcal{X}\mathcal{A}}(M, -) \longrightarrow \mathrm{Ext}^i_{\mathcal{W}\mathcal{A}}(M, -)$ *is an isomorphism either for two successive values of* i *with* $1 \leq i < d$ *or for a single value of* i *with* $i \geq d$, *where* $d = \mathcal{X}\text{-pd}(M) < \infty$.

The dual result is given in Theorem 3.7 and Corollary 3.8.

2. Preliminaries

We begin with some notation and terminology for use throughout this paper.

2.1. Throughout this work, $\mathcal{A}$ always denotes an abelian category, and given a ring R, $\mathcal{M}$ denotes the category of left R-modules. We use the term 'subcategory' for a 'full additive subcategory' that is closed under isomorphisms. A subcategory $\mathcal{X}$ of $\mathcal{A}$ is *exact* if it is closed under direct summands and extensions.

We fix subcategories $\mathcal{X}, \mathcal{Y}, \mathcal{W}$ and $\mathcal{V}$ of $\mathcal{A}$ such that $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{V} \subseteq \mathcal{Y}$. Write $\mathcal{X} \perp \mathcal{Y}$ if $\mathrm{Ext}^{\geq 1}_{\mathcal{A}}(X, Y) = 0$, and $\mathcal{X} \perp_1 \mathcal{Y}$ if $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0$ for any $X \in \mathcal{X}$ and any $Y \in \mathcal{Y}$. For an object M of $\mathcal{A}$, write $M \perp_1 \mathcal{Y}$ (respectively, $\mathcal{X} \perp_1 M$) if $\mathrm{Ext}^1_{\mathcal{A}}(M, Y) = 0$ for any $Y \in \mathcal{Y}$ (respectively, if $\mathrm{Ext}^1_{\mathcal{A}}(X, M) = 0$ for any $X \in \mathcal{X}$). We say that $\mathcal{W}$ is a *generator* for $\mathcal{X}$ if, for any $X \in \mathcal{X}$, there is an exact sequence $0 \longrightarrow X' \longrightarrow W \longrightarrow X \longrightarrow 0$ such that $W \in \mathcal{W}$ and $X' \in \mathcal{X}$. The subcategory $\mathcal{W}$ is a *projective generator* for $\mathcal{X}$ if $\mathcal{W}$ is a generator for $\mathcal{X}$ and $\mathcal{W} \perp \mathcal{X}$. Dually, one can give the concepts of *cogenerators* and *injective cogenerators*.

2.2. A *complex* $\cdots \longrightarrow X_1 \xrightarrow{\delta^X_1} X_0 \xrightarrow{\delta^X_0} X_{-1} \longrightarrow \cdots$ of objects of $\mathcal{A}$ will be denoted by (X, δ^X) or simply X. We frequently (and without warning) identify objects of $\mathcal{A}$ with complexes concentrated in degree zero. A complex X is *bounded above* if $X_n = 0$ for $n \gg 0$, and it is *bounded below* if $X_n = 0$ for $n \ll 0$. A complex X is *bounded* if it is both bounded above and bounded below. The nth *homology* of X is defined as $\mathrm{Ker}\delta^X_n/\mathrm{Im}\delta^X_{n+1}$ and it is denoted by $\mathrm{H}_n(X)$. For any $m \in \mathbb{Z}$, $\Sigma^m X$ denotes the complex with the degree-n term $(\Sigma^m X)_n = X_{n-m}$ and whose boundary operators are $(-1)^m \delta^X_{n-m}$. We set $\Sigma M = \Sigma^1 M$. The soft truncations of X at n are the complexes

$$X_{\subset n} \equiv \ 0 \longrightarrow \mathrm{Coker}(\delta^X_{n+1}) \xrightarrow{\overline{\delta^X_n}} X_{n-1} \xrightarrow{\delta^X_{n-1}} X_{n-2} \longrightarrow \cdots$$

and

$$X_{\supset n} \equiv \ \cdots \longrightarrow X_{n+2} \xrightarrow{\delta^X_{n+2}} X_{n+1} \xrightarrow{\delta^X_{n+1}} \mathrm{Ker}(\delta^X_n) \longrightarrow 0\ .$$

If X and Y are both complexes, then by a *morphism* $\alpha : X \longrightarrow Y$ we mean a sequence $\alpha_n : X_n \longrightarrow Y_n$ such that $\alpha_{n-1}\delta^X_n = \delta^Y_n \alpha_n$ for each $n \in \mathbb{Z}$. A *quasiisomorphism*, indicated by the symbol '$\simeq$', is a morphism of complexes that induces an isomorphism in homology. The *mapping cone* $\mathrm{Cone}(\alpha)$ of α is defined as $\mathrm{Cone}(\alpha)_n = Y_n \oplus X_{n-1}$ with nth boundary operator $\delta^{\mathrm{Cone}(\alpha)}_n = \left(\begin{smallmatrix} \delta^Y_n & \alpha_{n-1} \\ 0 & -\delta^X_{n-1} \end{smallmatrix}\right)$. It is well known that a morphism α is a quasiisomorphism if and only if its mapping cone $\mathrm{Cone}(\alpha)$ is exact. The *Hom-complex* $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ denotes the complex of abelian groups with the degree-n term $\mathrm{Hom}_{\mathcal{A}}(X, Y)_n = \prod_{t \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X_t, Y_{n+t})$ and whose nth boundary operator is given by $\{f_t\} \mapsto \{\delta^Y_{t+n} f_t - (-1)^n f_{t-1}\delta^X_t\}$. One can check that a morphism from X to Y is an element of $\mathrm{Ker}(\delta^{\mathrm{Hom}_{\mathcal{A}}(X,Y)}_0)$. A complex T is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-*exact* if $\mathrm{Hom}_{\mathcal{A}}(M, T)$ is exact for each object $M \in \mathcal{X}$. The term $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-*exact* is defined dually.

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2.3. An exact complex of objects in $\mathcal{X}$ is *totally $\mathcal{X}$-acyclic* if it is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact. Let $\mathcal{G}(\mathcal{X})$ denote the subcategory of $\mathcal{A}$ with objects of the form $M \cong \mathrm{Ker}(\delta_{-1}^X)$ for some totally $\mathcal{X}$-acyclic complex X.

Remark 2.4. If $\mathcal{W} \perp \mathcal{W}$, then, by [7, Theorem B and Corollary 4.7], $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{W})$, and $\mathcal{G}(\mathcal{W})$ is an exact subcategory of $\mathcal{A}$, and it is closed under kernels of epimorphisms (or cokernels of monomorphisms) if $\mathcal{W}$ is.

One can find the following definitions in [9].

2.5. Let M be an object of $\mathcal{A}$. An *$\mathcal{X}$-resolution* of M is a complex X of objects in $\mathcal{X}$ such that $X_{-n} = 0 = \mathrm{H}_n(X)$ for all $n > 0$ and $\mathrm{H}_0(X) \cong M$. The associated exact sequence

$$X^+ \equiv \ \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

is the *augmented $\mathcal{X}$-resolution* of M associated to X. Sometimes we call the quasiisomorphism $X \xrightarrow{\simeq} M$ an $\mathcal{X}$-resolution of M. A *bounded strict $\mathcal{W}\mathcal{X}$-resolution* X is a bounded $\mathcal{X}$-resolution such that $X_i \in \mathcal{W}$ for each $i \geq 1$. An $\mathcal{X}$-resolution X is *proper* if X^+ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact, and we let $\mathrm{res}\widetilde{\mathcal{X}}$ denote the subcategory of objects of $\mathcal{A}$ admitting a proper $\mathcal{X}$-resolution. The *$\mathcal{X}$-projective dimension* of M is the quantity

$$\mathcal{X}\text{-pd}(M) = \inf\{\sup\{n \geq 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

We let $\mathrm{res}\widehat{\mathcal{X}}$ denote the subcategory of objects of $\mathcal{A}$ of finite $\mathcal{X}$-projective dimension.

We define *(proper) $\mathcal{Y}$-coresolutions* and *$\mathcal{Y}$-injective dimension*, $\mathcal{Y}$-id(M), of M dually. We let $\mathrm{cores}\widetilde{\mathcal{Y}}$ and $\mathrm{cores}\widehat{\mathcal{Y}}$ denote the subcategories of objects of $\mathcal{A}$ admitting a proper $\mathcal{Y}$-coresolution and objects of $\mathcal{A}$ of finite $\mathcal{Y}$-injective dimension, respectively. Similarly, a *bounded strict $\mathcal{Y}\mathcal{V}$-coresolution* Y of M is a bounded $\mathcal{Y}$-coresolution such that $Y_i \in \mathcal{V}$ for $i \leq -1$.

By [8, (3.3) and (3.4)], we have the following result.

Lemma 2.6. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions. Assume that $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$, and that $\mathcal{V}$ is both an injective cogenerator and a projective generator for $\mathcal{Y}$. Then $\mathrm{res}\widehat{\mathcal{X}} \subseteq \mathrm{res}\widetilde{\mathcal{W}} \cap \mathrm{res}\widetilde{\mathcal{X}}$ and $\mathrm{cores}\widehat{\mathcal{Y}} \subseteq \mathrm{cores}\widetilde{\mathcal{V}} \cap \mathrm{cores}\widetilde{\mathcal{Y}}$.*

2.7. Let M and N be objects of $\mathcal{A}$. If M admits a proper $\mathcal{X}$-resolution $X \xrightarrow{\simeq} M$, then the nth *relative cohomology group* $\mathrm{Ext}^n_{\mathcal{X}\mathcal{A}}(M, N)$ is

$$\mathrm{Ext}^n_{\mathcal{X}\mathcal{A}}(M, N) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(X, N)).$$

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If N admits a proper $\mathcal{Y}$-coresolution, the nth relative cohomology group $\mathrm{Ext}^n_{\mathcal{AY}}(M, N)$ is defined dually.

Assume that M admits a proper $\mathcal{W}$-resolution $W \xrightarrow{\gamma} M$ and a proper $\mathcal{X}$-resolution $X \xrightarrow{\gamma'} M$. Let $\overline{\mathrm{id}_M} : W \longrightarrow X$ be a lifting of the identity $\mathrm{id}_M : M \longrightarrow M$, then $\overline{\mathrm{id}_M}$ is a quasiisomorphism such that $\gamma = \gamma' \circ \overline{\mathrm{id}_M}$. We set

$$\vartheta^n_{\mathcal{XWA}}(M, -) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(\overline{\mathrm{id}_M}, -)) : \mathrm{Ext}^n_{\mathcal{XA}}(M, -) \longrightarrow \mathrm{Ext}^n_{\mathcal{WA}}(M, -).$$

When N admits a proper $\mathcal{V}$-coresolution and a proper $\mathcal{Y}$-coresolution, the map

$$\vartheta^n_{\mathcal{AYV}}(-, N) : \mathrm{Ext}^n_{\mathcal{AY}}(-, N) \longrightarrow \mathrm{Ext}^n_{\mathcal{AV}}(-, N)$$

is defined dually.

2.8. Let M and N be objects of $\mathcal{A}$. A *Tate $\mathcal{W}$-resolution* of M is a diagram $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$ of morphisms of complexes, where T is a totally $\mathcal{W}$-acyclic complex, γ is a proper $\mathcal{W}$-resolution of M, and α_n is an isomorphism for $n \gg 0$. We let $\mathrm{res}\overline{\mathcal{W}}$ denote the subcategory of objects of $\mathcal{A}$ admitting a Tate $\mathcal{W}$-resolution. A *Tate $\mathcal{V}$-coresolution* of N is defined dually, and we let $\mathrm{cores}\overline{\mathcal{V}}$ denote the subcategory of objects of $\mathcal{A}$ admitting a Tate $\mathcal{V}$-coresolution. Then $\mathrm{res}\overline{\mathcal{W}}$ and $\mathrm{cores}\overline{\mathcal{V}}$ are subcategories of $\mathcal{A}$, and $\mathcal{G}(\mathcal{W}) \subseteq \mathrm{res}\overline{\mathcal{W}} \subseteq \mathrm{res}\widetilde{\mathcal{W}}$ and $\mathcal{G}(\mathcal{V}) \subseteq \mathrm{cores}\overline{\mathcal{V}} \subseteq \mathrm{cores}\widetilde{\mathcal{V}}$. If $\mathcal{W} \perp \mathcal{W}$, then $\mathrm{res}\widehat{\mathcal{W}} \subseteq \mathrm{res}\overline{\mathcal{W}}$ and $\mathrm{cores}\widehat{\mathcal{W}} \subseteq \mathrm{cores}\overline{\mathcal{W}}$ (see [9, (3.2)]).

If M admits a Tate $\mathcal{W}$-resolution $T \longrightarrow W \longrightarrow M$, define the nth Tate cohomology group $\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}(M, N)$ as

$$\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}(M, N) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(T, N))$$

for each $n \in \mathbb{Z}$. It follows from [9, (3.8)] that this definition is independent (up to isomorphism) of the choice of Tate $\mathcal{W}$-resolution. Dually, if N admits a Tate $\mathcal{V}$-coresolution $N \longrightarrow V \longrightarrow S$, define the nth Tate cohomology group $\widehat{\mathrm{Ext}}^n_{\mathcal{AV}}(M, N)$ as

$$\widehat{\mathrm{Ext}}^n_{\mathcal{AV}}(M, N) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(M, S))$$

for each $n \in \mathbb{Z}$. This definition is also independent (up to isomorphism) of the choice of Tate $\mathcal{V}$-coresolution by [9, (3.8)].

3. Tate cohomology in Abelian categories

We begin with the following lemmas that are tools for the proof of Theorem 3.5.

LEMMA 3.1 [9, (4.5)]. *Assume that $\mathcal{W} \perp \mathcal{W}$ and $\mathcal{V} \perp \mathcal{V}$. Let M and N be objects of $\mathcal{A}$, then the following statements hold.*

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(1) *If* $M \in \operatorname{res}\widetilde{\mathcal{W}}$, *then* $\widehat{\operatorname{Ext}}^n_{\mathcal{W}\mathcal{A}}(-, M) = 0$ *on* $\operatorname{res}\overline{\mathcal{W}}$ *and* $\widehat{\operatorname{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, -) = 0$ *for all* $n \in \mathbb{Z}$.

(2) *If* $N \in \operatorname{cores}\widetilde{\mathcal{V}}$, *then* $\widehat{\operatorname{Ext}}^n_{\mathcal{A}\mathcal{V}}(N, -) = 0$ *on* $\operatorname{cores}\overline{\mathcal{V}}$ *and* $\widehat{\operatorname{Ext}}^n_{\mathcal{A}\mathcal{V}}(-, N) = 0$ *for all* $n \in \mathbb{Z}$.

Lemma 3.2. *Assume that* $\mathcal{W}$ *is closed under direct summands and* $\mathcal{W} \perp \mathcal{W}$, *and let* $M \in \operatorname{res}\overline{\mathcal{W}}$. *If* $\widehat{\operatorname{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) = 0$ *or* $\widehat{\operatorname{Ext}}^0_{\mathcal{A}\mathcal{W}}(M, M) = 0$, *then* $M \in \operatorname{res}\widetilde{\mathcal{W}}$.

Proof. We prove the case when $\widehat{\operatorname{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) = 0$; the proof of the other case is dual. Since $M \in \operatorname{res}\overline{\mathcal{W}}$, without loss of generality, we may assume that there is a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \xrightarrow{\gamma} M$ of M such that α_n is an isomorphism for each $n \geq t$, where $t \geq 1$. Let $M_i = \operatorname{Im}(\delta_i^W)$ for $i \geq 1$, then $M_i \in \operatorname{res}\overline{\mathcal{W}}$. Note that the exact sequence

$$\cdots \longrightarrow W_t \longrightarrow \cdots \longrightarrow W_0 \xrightarrow{\gamma} M \longrightarrow 0$$

is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact and $W_i \in \mathcal{W}$ for $i \geq 0$, so

$$\widehat{\operatorname{Ext}}^j_{\mathcal{W}\mathcal{A}}(A, W_i) = 0 = \widehat{\operatorname{Ext}}^j_{\mathcal{W}\mathcal{A}}(W_i, B)$$

for any $j \in \mathbb{Z}$, any $i \geq 0$, any object B of $\mathcal{A}$ and any $A \in \operatorname{res}\widetilde{\mathcal{W}}$ by Lemma 3.1. Thus, by [9, (4.6) and (4.7)],

$$\widehat{\operatorname{Ext}}^0_{\mathcal{W}\mathcal{A}}(M_t, M_t) \cong \widehat{\operatorname{Ext}}^t_{\mathcal{W}\mathcal{A}}(M, M_t) \cong \widehat{\operatorname{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) = 0.$$

Note that $M_t \in \mathcal{G}(\mathcal{W})$, so that $M_t \in \mathcal{W}$ by [9, (5.1)], and hence $M \in \operatorname{res}\widetilde{\mathcal{W}}$. $\square$

Lemma 3.3. *Assume that* $\mathcal{X}$ *and* $\mathcal{Y}$ *are exact,* $\mathcal{W}$ *is a generator for* $\mathcal{X}$ *and* $\mathcal{V}$ *is a cogenerator for* $\mathcal{Y}$. *Consider the exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of objects of $\mathcal{A}$, *then the following statements hold.*

(1) *If* $M'', M \in \mathcal{X}$ *and* $\mathcal{W} \perp_1 M'$, *then* $M' \in \mathcal{X}$; *if* $\mathcal{W}$ *is closed under direct summands,* $M', M \in \mathcal{X}$ *and* $M'' \perp_1 \mathcal{X}$, *then* $M'' \in \mathcal{W}$.

(2) *If* $M', M \in \mathcal{Y}$ *and* $M'' \perp_1 \mathcal{V}$, *then* $M'' \in \mathcal{Y}$; *if* $\mathcal{V}$ *is closed under direct summands,* $M'', M \in \mathcal{Y}$ *and* $\mathcal{Y} \perp_1 M'$, *then* $M' \in \mathcal{V}$.

Proof. We prove part (1); the proof of part (2) is dual. Since $M'' \in \mathcal{X}$, there is an exact sequence

$$0 \longrightarrow X \longrightarrow W \longrightarrow M'' \longrightarrow 0$$

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with $W \in \mathcal{W}$ and $X \in \mathcal{X}$. Consider the following pullback diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & M' \arrow[r, equal] \arrow[d] & M' \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & W \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $M, X \in \mathcal{X}$, the exactness of the middle row, with the fact that $\mathcal{X}$ is closed under extensions, implies that $D \in \mathcal{X}$. Note that $\mathrm{Ext}^1_{\mathcal{A}}(W, M') = 0$ since $W \in \mathcal{W}$, so the middle column is split, and hence $M' \in \mathcal{X}$.

For the other part, since $M \in \mathcal{X}$, there is an exact sequence

$$0 \longrightarrow X \longrightarrow W \longrightarrow M \longrightarrow 0$$

with $W \in \mathcal{W}$ and $X \in \mathcal{X}$. Consider the following pullback diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & M' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & W \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & M'' \arrow[r, equal] \arrow[d] & M'' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $M', X \in \mathcal{X}$, the exactness of the top row and the fact that $\mathcal{X}$ is closed under extensions imply that $D \in \mathcal{X}$, and hence $\mathrm{Ext}^1_{\mathcal{A}}(M'', D) = 0$. Thus the middle column is split, and so $M'' \in \mathcal{W}$ since $W \in \mathcal{W}$. □

Lemma 3.4. *Assume that $\mathcal{X}$ is exact and $\mathcal{W}$ is a projective generator for $\mathcal{X}$. Let $M \in \mathrm{res}\widehat{\mathcal{X}}$ with $\mathcal{X}\text{-pd}(M) = t < \infty$. If $M \in \mathrm{res}\widetilde{\mathcal{W}}$ with $W \xrightarrow{\simeq} M$ a proper $\mathcal{W}$-resolution of M, then $K_t = \mathrm{Im}(\ W_t \longrightarrow W_{t-1}\ ) \in \mathcal{X}$ with $W_{-1} = M$.*

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Proof. If $t = 0$, then $K_0 = M \in \mathcal{X}$. Let $t > 0$, and let

$$0 \longrightarrow X_t \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

be an augmented $\mathcal{X}$-resolution of M, then it is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact since $\mathcal{W} \perp \mathcal{X}$. Thus we get the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & K_t \arrow[r] \arrow[d] & W_{t-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & W_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
0 \arrow[r] & X_t \arrow[r] & X_{t-1} \arrow[r] & \cdots \arrow[r] & X_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Since each row is exact and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact, we get that the mapping cone

$$0 \longrightarrow K_t \longrightarrow X_t \oplus W_{t-1} \longrightarrow \cdots \longrightarrow X_1 \oplus W_0 \longrightarrow X_0 \oplus M \longrightarrow M \longrightarrow 0$$

is exact and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. Thus the sequence

$$0 \longrightarrow K_t \longrightarrow X_t \oplus W_{t-1} \longrightarrow \cdots \longrightarrow X_1 \oplus W_0 \longrightarrow X_0 \longrightarrow 0$$

is exact and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. Now, repeated application of Lemma 3.3 yields $K_t \in \mathcal{X}$. □

The next result encompasses [9, (5.2)]. Notice that, even when $\mathcal{X}$ is exact and closed under kernels of epimorphisms, and $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$ and closed under direct summands, one may have $\mathcal{X} \subsetneq \mathcal{G}(\mathcal{W})$ (see [9, (3.12)]).

Theorem 3.5. *Assume that $\mathcal{X}$ is exact and closed under kernels of epimorphisms, and that $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$ and closed under direct summands. Let $M \in \mathrm{res}\widehat{\mathcal{X}}$. Then the following statements are equivalent:*

(1) $M \in \mathrm{res}\widehat{\mathcal{W}}$;
(2) $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(-, M) = 0$ *on* $\mathrm{res}\widehat{\mathcal{X}}$ *for each* $n \in \mathbb{Z}$;
(2′) $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, -) = 0$ *for each* $n \in \mathbb{Z}$;
(3) $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(-, M) = 0$ *on* $\mathrm{res}\widehat{\mathcal{X}}$ *for some* $n \in \mathbb{Z}$;
(3′) $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, -) = 0$ *for some* $n \in \mathbb{Z}$;
(4) $\widehat{\mathrm{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) = 0$.

Proof. (1) ⇒ (2) follows from Lemma 3.1 and [9, (3.4)].

(2) ⇒ (3) is trivial.

(3) ⇒ (4). Assume that $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(-, M) = 0$ on $\mathrm{res}\widehat{\mathcal{X}}$ for some $n \in \mathbb{Z}$. If $n = 0$, then the condition (4) holds immediately.

Let $n < 0$, and let $n = -d$ with $d > 0$. Since $M \in \mathrm{res}\widehat{\mathcal{X}}$, we get $M \in \mathrm{res}\widetilde{\mathcal{W}}$ by [8, (3.4)]. Let $W \xrightarrow{\simeq} M$ be a proper $\mathcal{W}$-resolution of M, and let $M_i \in \mathrm{Im}(\,W_i \longrightarrow W_{i-1}\,)$

for $i \geq 1$, then $M_i \in \operatorname{res}\widehat{\mathcal{X}}$ by Lemmas 3.3(1) and 3.4. Note that, for any $t \in \mathbb{Z}$ and any $i \geq 0$, $\widehat{\operatorname{Ext}}^t_{\mathcal{W}\mathcal{A}}(W_i, M) = 0$ by Lemma 3.1, so

$$\widehat{\operatorname{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) \cong \widehat{\operatorname{Ext}}^{-d}_{\mathcal{W}\mathcal{A}}(M_d, M) = 0$$

by [9, (4.6)] since $M_d \in \operatorname{res}\widehat{\mathcal{X}}$.

Let $n > 0$. By [8, (3.3)], there is an exact sequence

$$0 \longrightarrow M \longrightarrow W_{-1} \longrightarrow M_{-1} \longrightarrow 0 \tag{$*$}$$

with $W_{-1} \in \operatorname{res}\widehat{\mathcal{W}}$ and $M_{-1} \in \mathcal{X}$. Since $\mathcal{W} \perp \operatorname{res}\widehat{\mathcal{X}}$, the sequence ($*$) is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. Note that $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, so there is an exact sequence

$$0 \longrightarrow M_{-1} \longrightarrow W_{-2} \longrightarrow W_{-3} \longrightarrow \cdots \tag{$**$}$$

with $W_i \in \mathcal{W}$ for $i \leq -2$, such that $M_{-i} = \operatorname{Im}(\, W_{-i} \longrightarrow W_{-i-1}\,) \in \mathcal{X}$ for $i \geq 2$. Obviously, the sequence ($**$) is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact since $\mathcal{W} \perp \mathcal{X}$. Then

$$\widehat{\operatorname{Ext}}^0_{\mathcal{W}\mathcal{A}}(M, M) \cong \widehat{\operatorname{Ext}}^{n}_{\mathcal{W}\mathcal{A}}(M_{-n}, M) = 0$$

by [9, (4.6)], since $M_{-n} \in \operatorname{res}\widehat{\mathcal{X}}$ and $\widehat{\operatorname{Ext}}^t_{\mathcal{W}\mathcal{A}}(W_i, M) = 0$ for any $t \in \mathbb{Z}$ and any $i \leq -1$ by Lemma 3.1.

(4) $\Rightarrow$ (1) holds by Lemma 3.2 and [9, (3.4)].

Similarly, we can prove (1) $\Rightarrow$ (2′) $\Rightarrow$ (3′) $\Rightarrow$ (4). □

The next corollary encompasses [9, (5.6) and (5.7)] by noting that if $\mathcal{W}$ is closed under kernels of epimorphisms and $\mathcal{W} \perp \mathcal{W}$ then $\operatorname{res}\widehat{\mathcal{G}(\mathcal{W})} = \operatorname{res}\widehat{\mathcal{W}}$ (see [9, (3.6)]). The equivalence of (1), (2′) and (3′) of the following result was proved in [9, (5.6)] by using [9, (5.2)]. However, we see that [9, (5.2)] is in the special case when $\mathcal{X} = \mathcal{G}(\mathcal{W})$. Now we can prove it using Theorem 3.5.

Corollary 3.6. *Assume that $\mathcal{X}$ is exact and closed under kernels of epimorphisms, and that $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$ and closed under direct summands. Let $M \in \operatorname{res}\widehat{\mathcal{X}}$ with $\mathcal{X}$-pd$(M) = d < \infty$. Then the following statements are equivalent:*

(1) $M \in \operatorname{res}\widehat{\mathcal{W}}$;

(2) *The transformation* $\vartheta^i_{\mathcal{X}\mathcal{W}\mathcal{A}}(-, M) : \operatorname{Ext}^i_{\mathcal{X}\mathcal{A}}(-, M) \longrightarrow \operatorname{Ext}^i_{\mathcal{W}\mathcal{A}}(-, M)$ *is an isomorphism on* $\operatorname{res}\widehat{\mathcal{X}}$ *for each* $i \in \mathbb{Z}$;

(2′) *The transformation* $\vartheta^i_{\mathcal{X}\mathcal{W}\mathcal{A}}(M, -) : \operatorname{Ext}^i_{\mathcal{X}\mathcal{A}}(M, -) \longrightarrow \operatorname{Ext}^i_{\mathcal{W}\mathcal{A}}(M, -)$ *is an isomorphism for each* $i \in \mathbb{Z}$;

(3) *The transformation* $\vartheta^i_{\mathcal{X}\mathcal{W}\mathcal{A}}(-, M) : \operatorname{Ext}^i_{\mathcal{X}\mathcal{A}}(-, M) \longrightarrow \operatorname{Ext}^i_{\mathcal{W}\mathcal{A}}(-, M)$ *is an isomorphism on* $\operatorname{res}\widehat{\mathcal{X}}$ *for each* $1 \leq i \leq 2$;

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(3′) *The transformation* $\vartheta^i_{\mathcal{XWA}}(M,-)\colon \operatorname{Ext}^i_{\mathcal{XA}}(M,-) \longrightarrow \operatorname{Ext}^i_{\mathcal{WA}}(M,-)$ *is an isomorphism either for two successive values of* i *with* $1 \le i < d$ *or for a single value of* i *with* $i \ge d$;

Proof. (1) $\Leftrightarrow$ (2′) $\Leftrightarrow$ (3′) can be proved as in the proof of [9, (5.6)] using Theorem 3.5.

(1) $\Rightarrow$ (2) follows from [8, (4.10)] since $\operatorname{res}\widehat{\mathcal{X}} \subseteq \operatorname{res}\widetilde{\mathcal{W}} \cap \operatorname{res}\widetilde{\mathcal{X}}$ by Lemma 2.6.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let $N \in \operatorname{res}\widehat{\mathcal{X}}$, and let $\mathcal{X}\text{-pd}(N) = t < \infty$. If $t = 0$, then $N \in \mathcal{X}$, and so $\operatorname{Ext}^1_{\mathcal{XA}}(N, M) = 0$. Thus

$$\operatorname{Ext}^1_{\mathcal{WA}}(N, M) \cong \operatorname{Ext}^1_{\mathcal{XA}}(N, M) = 0.$$

This implies that $\widehat{\operatorname{Ext}}^1_{\mathcal{WA}}(N, M) = 0$ by [9, (4.10)]. Let $t = 1$. Since $\vartheta^1_{\mathcal{XWA}}(N, M)$ is an isomorphism, we get $\widehat{\operatorname{Ext}}^1_{\mathcal{WA}}(N, M) = 0$ by [9, (4.10)]. Let $t \ge 2$. Since $\vartheta^1_{\mathcal{XWA}}(N, M)$ and $\vartheta^2_{\mathcal{XWA}}(N, M)$ are isomorphisms, we get that $\widehat{\operatorname{Ext}}^1_{\mathcal{WA}}(N, M) = 0$ by [9, (4.10)]. Therefore, $\widehat{\operatorname{Ext}}^1_{\mathcal{WA}}(-, M) = 0$ on $\operatorname{res}\widehat{\mathcal{X}}$, and so $M \in \operatorname{res}\widehat{\mathcal{W}}$ by Theorem 3.5. □

The proofs of the next two results are dual to the previous two.

Theorem 3.7. *Assume that* $\mathcal{Y}$ *is exact and closed under cokernels of monomorphisms, and that* $\mathcal{V}$ *is both an injective cogenerator and a projective generator for* $\mathcal{Y}$ *and closed under direct summands. Let* $N \in \operatorname{cores}\widehat{\mathcal{Y}}$. *Then the following statements are equivalent:*

(1) $N \in \operatorname{cores}\widehat{\mathcal{V}}$;

(2) $\widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(N, -) = 0$ *on* $\operatorname{cores}\widehat{\mathcal{Y}}$ *for each* $n \in \mathbb{Z}$;

(2′) $\widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(-, N) = 0$ *for each* $n \in \mathbb{Z}$;

(3) $\widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(N, -) = 0$ *on* $\operatorname{cores}\widehat{\mathcal{Y}}$ *for some* $n \in \mathbb{Z}$;

(3′) $\widehat{\operatorname{Ext}}^n_{\mathcal{AV}}(-, N) = 0$ *for some* $n \in \mathbb{Z}$;

(4) $\widehat{\operatorname{Ext}}^0_{\mathcal{AV}}(N, N) = 0$.

Corollary 3.8. *Assume that* $\mathcal{Y}$ *is exact and closed under cokernels of monomorphisms, and that* $\mathcal{V}$ *is both an injective cogenerator and a projective generator for* $\mathcal{Y}$ *and closed under direct summands. Let* $N \in \operatorname{cores}\widehat{\mathcal{Y}}$ *with* $\mathcal{Y}\text{-id}(N) = d < \infty$. *Then the following statements are equivalent.*

(1) $N \in \operatorname{cores}\widehat{\mathcal{V}}$.

(2) *The transformation* $\vartheta^i_{\mathcal{AVY}}(N,-)\colon \operatorname{Ext}^i_{\mathcal{AY}}(N,-) \longrightarrow \operatorname{Ext}^i_{\mathcal{AV}}(N,-)$ *is an isomorphism on* $\operatorname{cores}\widehat{\mathcal{Y}}$ *for each* $i \in \mathbb{Z}$.

(2′) *The transformation* $\vartheta^i_{\mathcal{AVY}}(-,N)\colon \operatorname{Ext}^i_{\mathcal{AY}}(-,N) \longrightarrow \operatorname{Ext}^i_{\mathcal{AV}}(-,N)$ *is an isomorphism for each* $i \in \mathbb{Z}$.

(3) *The transformation* $\vartheta^i_{\mathcal{AVY}}(N,-)\colon \operatorname{Ext}^i_{\mathcal{AY}}(N,-) \longrightarrow \operatorname{Ext}^i_{\mathcal{AV}}(N,-)$ *is an isomorphism on* $\operatorname{cores}\widehat{\mathcal{Y}}$ *for each* $1 \le i \le 2$.

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(3′) *The transformation* $\vartheta^i_{\mathcal{AVY}}(-, N)$: $\mathrm{Ext}^i_{\mathcal{AY}}(-, N) \longrightarrow \mathrm{Ext}^i_{\mathcal{AV}}(-, N)$ *is an isomorphism either for two successive values of i with* $1 \le i < d$ *or for a single value of i with* $i \ge d$.

The next theorem is the main result of this paper, which was proved by Sather-Wagastaff *et al.* in the special case when $\mathcal{X} = \mathcal{G}(\mathcal{W})$, $\mathcal{Y} = \mathcal{G}(\mathcal{V})$ and $n \ge 1$ (see [9, (6.1)]).

Theorem 3.9. *Assume that* $\mathcal{X}$ *and* $\mathcal{Y}$ *are exact, and* $\mathcal{X}$ *is closed under kernels of epimorphisms and* $\mathcal{Y}$ *is closed under cokernels of monomorphisms. Assume that* $\mathcal{W}$ *is both an injective cogenerator and a projective generator for* $\mathcal{X}$ *and* $\mathcal{V}$ *is both an injective cogenerator and a projective generator for* $\mathcal{Y}$. *Assume also that* $\mathcal{W}$ *and* $\mathcal{V}$ *are closed under direct summands and satisfy* $\mathcal{W} \perp \mathcal{Y}$, $\mathcal{X} \perp \mathcal{V}$ *and* $\mathrm{Ext}^{\ge 1}_{\mathcal{WA}}(\mathrm{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0 = \mathrm{Ext}^{\ge 1}_{\mathcal{AV}}(\mathcal{W}, \mathrm{cores}\widehat{\mathcal{V}})$. *Then, for all* $M \in \mathrm{res}\widehat{\mathcal{X}}$ *and* $N \in \mathrm{cores}\widehat{\mathcal{Y}}$, *and all* $n \in \mathbb{Z}$,

$$\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}(M, N) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{AV}}(M, N).$$

Proof. We first prove the case when $n \ge 1$ using a method similar to that of [9, (6.1)]. We give the proof here for the sake of completeness.

Note that $M \in \mathrm{res}\widehat{\mathcal{X}}$, so there is a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \longrightarrow M$ of M such that each $\mathrm{Coker}(\delta^T_i)$ is in $\mathcal{X}$ and each α_i is a split surjection for $i \in \mathbb{Z}$ by [9, (3.4)]. Thus there exists a degree-wise split exact sequence

$$0 \longrightarrow \Sigma^{-1}X \longrightarrow \widetilde{T} \longrightarrow W \longrightarrow 0$$

of complexes by [9, (3.10)], where $\widetilde{T} = T_{\supset -1}$ is exact with $\widetilde{T}_{-1} \in \mathcal{X}$, and X is a bounded strict $\mathcal{WX}$-resolution of M. Then, for $n \ge 1$,

$$\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}(M, N) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(T, N)) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(\widetilde{T}, N)).$$

Similarly, let $N \longrightarrow V \xrightarrow{\beta} S$ be a Tate $\mathcal{V}$-resolution of N such that each $\mathrm{Ker}(\delta^S_i)$ is in $\mathcal{Y}$ and each β_i is a split injection for $i \in \mathbb{Z}$. Then there exists a degree-wise split exact sequence

$$0 \longrightarrow V \longrightarrow \widetilde{S} \longrightarrow \Sigma Y \longrightarrow 0$$

of complexes by [9, (3.11)], where $\widetilde{S} = S_{\subset 1}$ is exact with $\widetilde{S}_1 \in \mathcal{Y}$, and Y is a bounded strict $\mathcal{YV}$-coresolution of N. Thus, for $n \ge 1$,

$$\widehat{\mathrm{Ext}}^n_{\mathcal{AV}}(M, N) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(M, S)) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(M, \widetilde{S})).$$

In the following, we show that $\mathrm{H}_i(\mathrm{Hom}_{\mathcal{A}}(\widetilde{T}, N)) \cong \mathrm{H}_i(\mathrm{Hom}_{\mathcal{A}}(M, \widetilde{S}))$ for any $i \in \mathbb{Z}$.

Note that $\widetilde{S}$ is an exact bounded above complex of objects in $\mathcal{Y}$, so $\mathrm{Hom}_{\mathcal{A}}(W_i, \widetilde{S})$ is exact for each i since $\mathcal{W} \perp \mathcal{Y}$, and hence $\mathrm{Hom}_{\mathcal{A}}(W, \widetilde{S})$ is exact by [2, (2.4)]. Now consider the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(W, \widetilde{S}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\widetilde{T}, \widetilde{S}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\Sigma^{-1}X, \widetilde{S}) \longrightarrow 0,$$

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then we get that $\mathrm{Hom}_{\mathcal{A}}(\widetilde{T}, \widetilde{S}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\Sigma^{-1}X, \widetilde{S})$ is a quasiisomorphism. On the other hand, notice that X is a bounded strict $\mathcal{W}\mathcal{X}$-resolution of M, so X is a proper $\mathcal{X}$-resolution of M. Thus the morphism

$$\mathrm{Hom}_{\mathcal{A}}(M, \widetilde{S}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, \widetilde{S})$$

is a quasiisomorphism by [8, (6.6)]. Therefore, for any $i \in \mathbb{Z}$,

$$\begin{aligned} \mathrm{H}_i(\mathrm{Hom}_{\mathcal{A}}(M, \widetilde{S})) &\cong \mathrm{H}_i(\mathrm{Hom}_{\mathcal{A}}(X, \widetilde{S})) \\ &\cong \mathrm{H}_{i+1}(\mathrm{Hom}_{\mathcal{A}}(\Sigma^{-1}X, \widetilde{S})) \\ &\cong \mathrm{H}_{i+1}(\mathrm{Hom}_{\mathcal{A}}(\widetilde{T}, \widetilde{S})). \end{aligned}$$

Similarly, we get that $\mathrm{H}_i(\mathrm{Hom}_{\mathcal{A}}(\widetilde{T}, N)) \cong \mathrm{H}_{i+1}(\mathrm{Hom}_{\mathcal{A}}(\widetilde{T}, \widetilde{S}))$ for any $i \in \mathbb{Z}$. This implies that $\mathrm{H}_i(\mathrm{Hom}_{\mathcal{A}}(\widetilde{T}, N)) \cong \mathrm{H}_i(\mathrm{Hom}_{\mathcal{A}}(M, \widetilde{S}))$ for any $i \in \mathbb{Z}$. Thus, for $n \geq 1$,

$$\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, N) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M, N). \tag{$\natural$}$$

Now let $n = -d$ with $d \geq 0$, and we will prove that $\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{A}}(M, N) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{A}\mathcal{V}}(M, N)$. Since $M \in \mathrm{res}\widehat{\mathcal{X}}$, there is an exact sequence

$$0 \longrightarrow M \longrightarrow W_{-1} \longrightarrow M_{-1} \longrightarrow 0 \tag{¶}$$

with $W_{-1} \in \mathrm{res}\widehat{\mathcal{W}}$ and $M_{-1} \in \mathcal{X}$ by [8, (3.3)]. Note that $\mathcal{W} \perp \mathcal{X}$, then $\mathcal{W} \perp \mathrm{res}\widehat{\mathcal{X}}$, and so the sequence (¶) is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. Since $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, we get an exact sequence

$$0 \longrightarrow M_{-1} \longrightarrow W_{-2} \longrightarrow W_{-3} \longrightarrow \cdots \tag{¶¶}$$

with each $W_i \in \mathcal{W}$ for $i \leq -2$, such that $M_{-i} = \mathrm{Im}(\, W_{-i} \longrightarrow W_{-i-1}\,) \in \mathcal{X}$ for $i \geq 2$. Thus the sequence (¶¶) is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. Notice that $\widehat{\mathrm{Ext}}^j_{\mathcal{W}\mathcal{A}}(W_{-s}, A) = 0$ for any object A of $\mathcal{A}$, any $s \geq 1$ and any $j \in \mathbb{Z}$ by Lemma 3.1, and hence

$$\widehat{\mathrm{Ext}}^i_{\mathcal{W}\mathcal{A}}(M, A) \cong \widehat{\mathrm{Ext}}^{i+k}_{\mathcal{W}\mathcal{A}}(M_{-k}, A) \tag{§}$$

for any $k \geq 1$ and $i \in \mathbb{Z}$ by [9, (4.7)].

On the other hand, by [8, (3.3)], there is an exact sequence

$$0 \longrightarrow N_1 \longrightarrow V_1 \longrightarrow N \longrightarrow 0 \tag{†}$$

with $V_1 \in \mathrm{cores}\widehat{\mathcal{V}}$ and $N_1 \in \mathcal{Y}$. Note that $\mathcal{Y} \perp \mathcal{V}$, then $\mathrm{cores}\widehat{\mathcal{Y}} \perp \mathcal{V}$, and so the sequence (†) is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{V})$-exact. Since $\mathcal{V}$ is a projective generator for $\mathcal{Y}$, we get an exact sequence

$$\cdots \longrightarrow V_3 \longrightarrow V_2 \longrightarrow N_1 \longrightarrow 0 \tag{‡}$$

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with each $V_i \in \mathcal{V}$ for $i \geq 2$ such that $N_i = \mathrm{Im}(\ V_{i+1} \longrightarrow V_i\ ) \in \mathcal{Y}$ for $i \geq 2$. Thus the sequence (‡) is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{V})$-exact. Notice that $\widehat{\mathrm{Ext}}^j_{\mathcal{A}\mathcal{V}}(B, V_s) = 0$ for any object B of $\mathcal{A}$, any $s \geq 1$ and any $j \in \mathbb{Z}$ by Lemma 3.1, and therefore

$$\widehat{\mathrm{Ext}}^i_{\mathcal{A}\mathcal{V}}(B, N) \cong \widehat{\mathrm{Ext}}^{i+k}_{\mathcal{A}\mathcal{V}}(B, N_k) \tag{§§}$$

for any $k \geq 1$ and $i \in \mathbb{Z}$ by [9, (4.7)].

Now we get the following isomorphisms:

$$\begin{aligned}
\widehat{\mathrm{Ext}}^{-d}_{\mathcal{W}\mathcal{A}}(M, N) &\cong \widehat{\mathrm{Ext}}^{1}_{\mathcal{W}\mathcal{A}}(M_{-d-1}, N)\\
&\cong \widehat{\mathrm{Ext}}^{1}_{\mathcal{A}\mathcal{V}}(M_{-d-1}, N)\\
&\cong \widehat{\mathrm{Ext}}^{d+2}_{\mathcal{A}\mathcal{V}}(M_{-d-1}, N_{d+1})\\
&\cong \widehat{\mathrm{Ext}}^{d+2}_{\mathcal{W}\mathcal{A}}(M_{-d-1}, N_{d+1})\\
&\cong \widehat{\mathrm{Ext}}^{1}_{\mathcal{W}\mathcal{A}}(M, N_{d+1})\\
&\cong \widehat{\mathrm{Ext}}^{1}_{\mathcal{A}\mathcal{V}}(M, N_{d+1})\\
&\cong \widehat{\mathrm{Ext}}^{-d}_{\mathcal{A}\mathcal{V}}(M, N),
\end{aligned}$$

where the first and the fifth isomorphisms follow from (§), the third and the seventh hold by (§§), and the remaining ones follow from (♮) since $d \geq 0$. Thus we get that $\widehat{\mathrm{Ext}}^{n}_{\mathcal{W}\mathcal{A}}(M, N) \cong \widehat{\mathrm{Ext}}^{n}_{\mathcal{A}\mathcal{V}}(M, N)$ for $n \leq 0$.

Therefore, we have $\widehat{\mathrm{Ext}}^{n}_{\mathcal{W}\mathcal{A}}(M, N) \cong \widehat{\mathrm{Ext}}^{n}_{\mathcal{A}\mathcal{V}}(M, N)$ for all $M \in \mathrm{res}\widehat{\mathcal{X}}$ and $N \in \mathrm{cores}\widehat{\mathcal{Y}}$, and all $n \in \mathbb{Z}$. □

Corollary 3.10. *Assume that $\mathcal{W} \perp \mathcal{W}$, $\mathcal{V} \perp \mathcal{V}$, $\mathcal{G}(\mathcal{W}) \perp \mathcal{V}$ and $\mathcal{W} \perp \mathcal{G}(\mathcal{V})$. Assume that $\mathcal{W}$ is closed under kernels of epimorphisms and direct summands and that $\mathcal{V}$ is closed under cokernels of monomorphisms and direct summands. Assume also that $\mathrm{Ext}^{\geq 1}_{\mathcal{W}\mathcal{A}}(\mathrm{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0 = \mathrm{Ext}^{\geq 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W}, \mathrm{cores}\widehat{\mathcal{V}})$. Then, for all $M \in \mathrm{res}\widehat{\mathcal{G}(\mathcal{W})}$, all $N \in \mathrm{cores}\widehat{\mathcal{G}(\mathcal{V})}$ and all $n \in \mathbb{Z}$,*

$$\widehat{\mathrm{Ext}}^{n}_{\mathcal{W}\mathcal{A}}(M, N) \cong \widehat{\mathrm{Ext}}^{n}_{\mathcal{A}\mathcal{V}}(M, N).$$

Proof. Immediately by Theorem 3.9 and Remark 2.4. □

We write $\mathcal{P}$ and $\mathcal{I}$ for the subcategories of projective left R-modules and injective left R-modules, respectively. One can check easily that $\mathcal{W} = \mathcal{P}$ and $\mathcal{V} = \mathcal{I}$ satisfy the hypotheses of Corollary 3.10, thus we have the next corollary that can be found in [3, Theorem 5.4] and [4, Corollary 3.4].

Corollary 3.11. *Let M and N be left R-modules such that*

$$\mathcal{G}(\mathcal{P})\text{-pd}_R(M) < \infty \quad \textit{and} \quad \mathcal{G}(\mathcal{I})\text{-id}_R(N) < \infty.$$

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Then, for each $n \in \mathbb{Z}$,

$$\widehat{\mathrm{Ext}}^n_R(M,N) = \widehat{\mathrm{Ext}}^n_{\mathcal{PM}}(M,N) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{MI}}(M,N).$$

Let R be a commutative ring. An R-module C is called *semidualizing* if C admits a degree-wise finite projective resolution, $\mathrm{Ext}^{\geq 1}_R(C,C) = 0$ and the natural homothety map $R \longrightarrow \mathrm{Hom}_R(C,C)$ is an isomorphism. Examples include the rank-one free R-modules and a dualizing (canonical) R-module (when one exists). We let $\mathcal{P}_C$ (respectively, $\mathcal{I}_C$) denote the subcategory of R-modules $C \otimes_R P$ (respectively, $\mathrm{Hom}_R(C,I)$) with P (respectively, I) projective (respectively, injective). Modules in $\mathcal{P}_C$ and $\mathcal{I}_C$ are called *C-projective* and *C-injective*, respectively. A *complete $\mathcal{PP}_C$-resolution* is an exact and $\mathrm{Hom}_R(-,\mathcal{P}_C(R))$-exact complex X of R-modules with X_i projective for $i \geq 0$ and X_j C-projective for $j < 0$. An R-module M is G_C*-projective* if there exists a complete $\mathcal{PP}_C$-resolution X such that $M \cong \mathrm{Ker}(\delta^X_{-1})$. We let $\mathcal{GP}_C(R)$ denote the subcategory of G_C-projective R-modules.

Let B and C be semidualizing R-modules such that $B \in \mathcal{GP}_C(R)$. Set $B^\dagger = \mathrm{Hom}_R(B,C)$, then $B^\dagger$ is a semidualizing R-module. Now $\mathcal{W} = \mathcal{P}_B(R)$ and $\mathcal{V} = \mathcal{I}_{B^\dagger}(R)$ satisfy the hypotheses of Corollary 3.10 by the proof of [9, (6.2)]. Thus we have the next result that was proved by Sather-Wagstaff *et al.* for $n \geq 1$ (see [9, Theorem D]).

Corollary 3.12. *Let R be a commutative ring, and let B and C be semidualizing R-modules such that $B \in \mathcal{GP}_C(R)$. Set $B^\dagger = \mathrm{Hom}_R(B,C)$. Let M and N be R-modules such that*

$$\mathcal{G}(\mathcal{P}_B)\text{-}\mathrm{pd}_R(M) < \infty \quad \textit{and} \quad \mathcal{G}(\mathcal{I}_{B^\dagger})\text{-}\mathrm{id}_R(N) < \infty.$$

Then, for each $n \in \mathbb{Z}$,

$$\widehat{\mathrm{Ext}}^n_{\mathcal{P}_B\mathcal{M}}(M,N) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{MI}_{B^\dagger}}(M,N).$$

In the following, we let $\mathcal{X}, \mathcal{Y}, \mathcal{W}$ and $\mathcal{V}$ denote subcategories of $\mathcal{M}$ (the category of left R-modules).

Assume that $\mathcal{X}$ is closed under extensions, and $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$. Let $M \in \mathrm{res}\widehat{\mathcal{X}}$. Then M has a proper $\mathcal{X}$-resolution $X \xrightarrow{\simeq} M$ and a proper $\mathcal{W}$-resolution $W \xrightarrow{\simeq} M$ by Lemma 2.6. Set $\overline{\mathrm{id}_M} : W \longrightarrow X$ a lifting of the identity $\mathrm{id}_M : M \longrightarrow M$. Then we have the following result that provides a new method to compute Tate cohomology.

Proposition 3.13. *Assume that $\mathcal{X}$ is exact and closed under kernels of epimorphisms, and $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$ and closed under direct summands. Let $M \in \mathrm{res}\widehat{\mathcal{X}}$. Then*

$$\widehat{\mathrm{Ext}}^n_{\mathcal{WM}}(M,N) \cong \mathrm{H}_{-n-1}(\mathrm{Hom}_R(\mathrm{Cone}(\overline{\mathrm{id}_M}),N))$$

for any left R-module N and any $n \geq 1$.

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Proof. By [9, (3.4)], there is a Tate $\mathcal{W}$-resolution $T \xrightarrow{\alpha} W \xrightarrow{\eta} M$ of M such that $\mathrm{Coker}(\delta_1^T) \in \mathcal{X}$ and α_n are split surjections for all $n \in \mathbb{Z}$. Using [9, (3.10)] we get a degree-wise split exact sequence

$$0 \longrightarrow \Sigma^{-1}X \xrightarrow{\lambda} \widetilde{T} \xrightarrow{\alpha} W \longrightarrow 0 \tag{||}$$

of complexes with $\widetilde{T}$ exact, where $X \xrightarrow{\simeq} M$ is a bounded strict $\mathcal{W}\mathcal{X}$-resolution of M and $W \xrightarrow{\simeq} M$ is a proper $\mathcal{W}$-resolution of M. Since $\mathcal{X} \perp \mathcal{W}$, we get that $X \xrightarrow{\simeq} M$ is a proper $\mathcal{X}$-resolution of M. By the proof of [9, (3.10)], we can rewrite the sequence (||) as follows.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\Sigma^{-1}X \equiv \arrow[d, "\lambda"] & \cdots \arrow[r] & \mathrm{Ker}(\alpha_1) \arrow[r] \arrow[d] & \mathrm{Ker}(\alpha_0) \arrow[r] \arrow[d] & \mathrm{Coker}(\delta_1^T) \arrow[r] \arrow[d, "="] & 0 \arrow[r] \arrow[d] & \cdots \\
\widetilde{T} \equiv \arrow[d, "\alpha"] & \cdots \arrow[r] & T_1 \arrow[r, "\delta_1^T"] \arrow[d, "\alpha_1"] & T_0 \arrow[r, "\pi"] \arrow[d, "\alpha_0"] & \mathrm{Coker}(\delta_1^T) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
W \equiv & \cdots \arrow[r] & W_1 \arrow[r, "\delta_1^W"] \arrow[d] & W_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
 & & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Since the sequence (||) is degree-wise split, there is $\alpha_i' : W_i \longrightarrow T_i$ for $i \geq 0$ such that $\alpha_i \alpha_i' = \mathrm{id}_{W_i}$. Thus we get the following commutative diagram

$$\begin{tikzcd}
W^+ \equiv \cdots \arrow[r] \arrow[d, "\tau^+"] & W_2 \arrow[r] \arrow[d, "\tau_2"] & W_1 \arrow[r] \arrow[d, "\tau_1"] & W_0 \arrow[r, "\eta"] \arrow[d, "\tau_0"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
X^+ \equiv \cdots \arrow[r] & \mathrm{Ker}(\alpha_1) \arrow[r] & \mathrm{Ker}(\alpha_0) \arrow[r] & \mathrm{Coker}(\delta_1^T) \arrow[r, "f"] & M \arrow[r] & 0
\end{tikzcd}$$

with the first row an augmented proper $\mathcal{W}$-resolution of M and the second row an augmented proper $\mathcal{X}$-resolution of M, where $\tau_0 = \pi\alpha_0'$ and $\tau_i = (-1)^i(\delta_i^T \alpha_i' - \alpha_{i-1}'\delta_i^W)$ for $i \geq 1$, and $f(x + \mathrm{Im}\delta_1^T) = \eta\alpha_0(x)$ for any $x \in T_0$. Now one can check that $\widetilde{T} \cong \Sigma^{-1}\mathrm{Cone}(\tau)$. Thus, for $n \geq 1$,

$$\begin{aligned}
\widehat{\mathrm{Ext}}^n_{\mathcal{W}\mathcal{M}}(M, N) &= \mathrm{H}_{-n}(\mathrm{Hom}_R(T, N)) \\
&\cong \mathrm{H}_{-n}(\mathrm{Hom}_R(\widetilde{T}, N)) \\
&\cong \mathrm{H}_{-n}(\mathrm{Hom}_R(\Sigma^{-1}\mathrm{Cone}(\tau), N)) \\
&\cong \mathrm{H}_{-n-1}(\mathrm{Hom}_R(\mathrm{Cone}(\tau), N)) \\
&\cong \mathrm{H}_{-n-1}(\mathrm{Hom}_R(\mathrm{Cone}(\overline{\mathrm{id}_M}), N)),
\end{aligned}$$

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where the second isomorphism holds since $\widetilde{T} = T_{\supset -1}$, and the last isomorphism follows from the fact that $\mathrm{Cone}(\overline{\mathrm{id}_M})$ and $\mathrm{Cone}(\tau)$ are homotopy equivalent [6, page 392]. □

The next result is proved dually by noting that if $\mathcal{Y}$ is closed under extensions and $\mathcal{V}$ is both an injective cogenerator and a projective generator for $\mathcal{Y}$, then $\mathrm{cores}\widehat{\mathcal{Y}} \subseteq \mathrm{cores}\widetilde{\mathcal{V}} \cap \mathrm{cores}\widetilde{\mathcal{Y}}$ by Lemma 2.6.

Proposition 3.14. *Assume that $\mathcal{Y}$ is exact and closed under cokernels of monomorphisms, and $\mathcal{V}$ is both an injective cogenerator and a projective generator for $\mathcal{Y}$ and closed under direct summands. Let $N \in \mathrm{cores}\widehat{\mathcal{Y}}$, and let $N \xrightarrow{\simeq} Y$ be a proper $\mathcal{Y}$-coresolution and $N \xrightarrow{\simeq} V$ a proper $\mathcal{V}$-coresolution of N, and let $\overline{\mathrm{id}_N} : Y \longrightarrow V$ be a lifting of the identity $\mathrm{id}_N : N \longrightarrow N$. Then*

$$\widehat{\mathrm{Ext}}^n_{\mathcal{MV}}(M, N) \cong \mathrm{H}_{-n}(\mathrm{Hom}_R(M, \mathrm{Cone}(\overline{\mathrm{id}_N})))$$

for any left R-module M and any $n \geq 1$.

The next corollary is immediate by Theorem 3.9 and Propositions 3.13 and 3.14.

Corollary 3.15. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are exact, $\mathcal{X}$ is closed under kernels of epimorphisms and $\mathcal{Y}$ is closed under cokernels of monomorphisms. Assume that $\mathcal{W}$ is both an injective cogenerator and a projective generator for $\mathcal{X}$, and $\mathcal{V}$ is both an injective cogenerator and a projective generator for $\mathcal{Y}$. Assume also that $\mathcal{W}$ and $\mathcal{V}$ are closed under direct summands and satisfy $\mathcal{W} \perp \mathcal{Y}$, $\mathcal{X} \perp \mathcal{V}$ and $\mathrm{Ext}^{\geq 1}_{\mathcal{WA}}(\mathrm{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0 = \mathrm{Ext}^{\geq 1}_{\mathcal{AV}}(\mathcal{W}, \mathrm{cores}\widehat{\mathcal{V}})$. Then, for all $M \in \mathrm{res}\widehat{\mathcal{X}}$ and $N \in \mathrm{cores}\widehat{\mathcal{Y}}$, and all $n \geq 1$,*

$$\begin{aligned}\widehat{\mathrm{Ext}}^n_{\mathcal{WA}}(M, N) &\cong \widehat{\mathrm{Ext}}^n_{\mathcal{AV}}(M, N)\\ &\cong \mathrm{H}_{-n-1}(\mathrm{Hom}_R(\mathrm{Cone}(\overline{\mathrm{id}_M}), N))\\ &\cong \mathrm{H}_{-n}(\mathrm{Hom}_R(M, \mathrm{Cone}(\overline{\mathrm{id}_N}))).\end{aligned}$$

Let R be a commutative ring, and let B and C be semidualizing R-modules such that $B \in \mathcal{GP}_C(R)$. Set $B^\dagger = \mathrm{Hom}_R(B, C)$. Let M and N be R-modules such that $\mathcal{G}(\mathcal{P}_B)\text{-pd}_R(M) < \infty$ and $\mathcal{G}(\mathcal{I}_{B^\dagger})\text{-id}_R(N) < \infty$. Then M has a proper $\mathcal{G}(\mathcal{P}_B)$-resolution $X \xrightarrow{\simeq} M$ and a proper $\mathcal{P}_B$-resolution $W \xrightarrow{\simeq} M$ by Lemma 2.6. Set $\overline{\mathrm{id}_M} : W \longrightarrow X$ a lifting of the identity $\mathrm{id}_M : M \longrightarrow M$. Dually, one can construct $\overline{\mathrm{id}_N}$. Then we have the next result by Corollary 3.15.

Corollary 3.16. *Let R be a commutative ring, and let B and C be semidualizing R-modules such that $B \in \mathcal{GP}_C(R)$. Set $B^\dagger = \mathrm{Hom}_R(B, C)$. Let M and N be R-modules such that*

$$\mathcal{G}(\mathcal{P}_B)\text{-pd}_R(M) < \infty \quad \textit{and} \quad \mathcal{G}(\mathcal{I}_{B^\dagger})\text{-id}_R(N) < \infty.$$

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Then, for each $n \geq 1$,

$$\begin{aligned}\widehat{\mathrm{Ext}}^{n}_{\mathcal{P}_B\mathcal{M}}(M,N) &\cong \widehat{\mathrm{Ext}}^{n}_{\mathcal{M}\mathcal{I}_{B^\dagger}}(M,N)\\ &\cong \mathrm{H}_{-n-1}(\mathrm{Hom}_R(\mathrm{Cone}(\overline{\mathrm{id}_M}),N))\\ &\cong \mathrm{H}_{-n}(\mathrm{Hom}_R(M,\mathrm{Cone}(\overline{\mathrm{id}_N}))).\end{aligned}$$

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