Bull. Aust. Math. Soc. **96** (2017), 245–255
doi:10.1017/S0004972717000272

STABLE, ALMOST STABLE AND ODD POINTS OF DYNAMICAL SYSTEMS

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(Received 19 January 2017; accepted 5 February 2017; first published online 26 April 2017)

Abstract

We consider stable and almost stable points of autonomous and nonautonomous discrete dynamical systems defined on the closed unit interval. Our considerations are associated with chaos theory by adding an additional assumption that an entropy of a function at a given point is infinite.

2010 *Mathematics subject classification*: primary 26A18; secondary 37B55, 37B25, 37C35.

Keywords and phrases: continuity, Darboux function, autonomous (nonautonomous) discrete dynamical system, (almost) stable point, odd point.

1. Introduction and preliminaries

Many papers associated with autonomous and nonautonomous discrete dynamical systems emphasise the close relationship between these systems and difference equations of the form $x_{n+1} = f(x_n)$ or $x_{n+1} = f_n(x_n)$ (see, for example, [1]). In this way, dynamical systems with discrete time observations have numerous practical applications in various fields, including economics, biology, information flow, or physics [3, 10, 17]. One of the main issues considered in this context is stability. Various concepts of this notion led to attempts at their standardisation, made in [10]. We will extend the concept presented there, combining it with an entropy of a function at a point. The notion of topological entropy is often connected with chaos theory and, in this context, there are relationships between the various definitions of chaos and the fact that a function has positive entropy (see, for example, [8]). In the classical approach, the topological aspects of discrete dynamical systems were considered in the context of continuous functions. If we restrict our considerations to continuous functions, of course, we limit the scope by eliminating, for example, derivatives. Therefore, since the beginning of the twenty-first century, there have appeared many papers showing that the classical topological considerations (for example, topological entropy, Sharkovskiĭ's theorem) may be extended to certain discontinuous functions [12, 15]. We therefore consider some issues related to discontinuous functions.

Throughout this paper, $\mathbb{I}$ will stand for the closed unit interval, $\mathbb{N}$ the set of all positive integers and $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$. We will only consider functions from $\mathbb{I}$ into $\mathbb{I}$, so from now on we will write f instead of $f : \mathbb{I} \to \mathbb{I}$. The symbol ρ_u will denote the metric of uniform convergence. The cardinality (respectively, the interior in the space $\mathbb{I}$ with the natural topology) of any set $A \subset \mathbb{I}$ will be denoted by $\#(A)$ (respectively, $\mathrm{int}(A)$). The restriction of f to $P \subset \mathbb{I}$ will be denoted by $f \upharpoonright P$. Moreover, if $f(P) \subset P$ then we will say that P is f-invariant.

In Theorem 2.6 we focus on the family of all *Darboux functions* f such that x_0 is a fixed point of f (that is, $f(x_0) = x_0$). We will denote this family by $\mathcal{D}\,\mathrm{Fix}_{x_0}$. Obviously, a function f is a Darboux function if the image of any connected set by f is a connected set. However, it is worth adding that a function f is a Darboux function if and only if each point $x \in \mathbb{I}$ is a Darboux point of f [2]. A point $x_0 \in \mathbb{I}$ is a *Darboux point of* f if $(\liminf_{x\to x_0^+} f(x), \limsup_{x\to x_0^+} f(x)) \subset R^+(f, x_0)$ and $(\liminf_{x\to x_0^-} f(x), \limsup_{x\to x_0^-} f(x)) \subset R^-(f, x_0)$, where $R^+(f, x_0)$ $(R^-(f, x_0))$ is the set of all points y such that for any $\varepsilon > 0$ there is a point $x \in [x_0, x_0 + \varepsilon)$ $(x \in (x_0 - \varepsilon, x_0])$ such that $f(x) = y$.

A *nonautonomous dynamical system* (or a dynamical system for short) is a pair $(\mathbb{I}, (f_{1,\infty}))$, where $(f_{1,\infty})$ is any sequence of functions $\{f_n\}_{n=1}^{\infty}$. We will identify a dynamical system $(\mathbb{I}, (f_{1,\infty}))$ with the sequence $\{f_n\}_{n=1}^{\infty}$ and denote it by $(f_{1,\infty})$. A dynamical system is called *autonomous* if $f_n = f$ for all $n \in \mathbb{N}$ and some function f (such a system will be denoted by (f)).

Let $\varepsilon > 0$, $i_0 \in \mathbb{N}$ and V be a set of functions from $\mathbb{I}$ into itself. We shall say that this set (i_0, ε)-*perturbs* the dynamical system (f) to a dynamical system having property P, if for any function $\xi \in V$ we have $\rho_u(f, \xi) < \varepsilon$ and the nonautonomous dynamical system $(f_{1,\infty})$ such that $f_i = f$ for $i \in \mathbb{N} \setminus \{i_0\}$ and $f_{i_0} = \xi$ has the property P.

The symbol $\mathrm{Fix}(f_{1,\infty})$ will stand for the set of all fixed points of $(f_{1,\infty})$, that is, $x_0 \in \mathrm{Fix}(f_{1,\infty})$ if $f_n(x_0) = x_0$ for $n \in \mathbb{N}$. By $\mathrm{Fix}(f)$, we mean the set of all fixed points of a function f. We denote the set of all continuity points of $(f_{1,\infty})$ by $C(f_{1,\infty})$. That is, $x_0 \in C(f_{1,\infty})$ if and only if for any $n \in \mathbb{N}$ the function f_n is continuous at x_0. For an autonomous dynamical system (f), we shorten the notation to $C(f)$. Similarly to [4], for a dynamical system $(f_{1,\infty})$ and $n, i \in \mathbb{N}$, the symbol f_n^i will stand for $f_{n+i-1} \circ f_{n+i-2} \circ \cdots \circ f_{n-1} \circ f_n$. In order to keep the symmetry of notation, in the case of an autonomous dynamical system (g) we will use the notation $(g)_1^i$ instead of g^i.

In Section 2 we will need the notion of an entropy of a function at a point, considered in [9, 13]. Let f be a function, $\mathcal{L}$ be a family of pairwise disjoint nonsingleton continuums in $\mathbb{I}$ and $J \subset \mathbb{I}$ be a connected set such that $J \subset f(A)$ for any $A \in \mathcal{L}$. A pair $B_f = (\mathcal{L}, J)$ is called an f-*bundle*. If $A \subset J$ for all $A \in \mathcal{L}$ then such an f-bundle will be called an f-*bundle with dominating fibre*.

Let $\varepsilon > 0$, $n \in \mathbb{N}$, $B_f = (\mathcal{L}, J)$ be an f-bundle and $M \subset \bigcup \mathcal{L}$. We say that M is (B_f, n, ε)-*separated* if for each $x, y \in M$, $x \neq y$, there is $i \in \{0, 1, \ldots, n-1\}$ such that $f^i(x), f^i(y) \in J$ and $|f^i(x) - f^i(y)| > \varepsilon$. Define

$$\mathrm{maxsep}[B_f, n, \varepsilon] = \max\{\#(M) : M \subset \mathbb{I} \text{ is a } (B_f, n, \varepsilon)\text{-separated set}\}.$$

An *entropy of an f-bundle B_f* is the number given by the formula

$$h(B_f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \left[\frac{1}{n} \log(\operatorname{maxsep}[B_f, n, \varepsilon])\right].$$

LEMMA 1.1 [13]. *Let f be an arbitrary function and $B_f = (\mathcal{L}, J)$ be an f-bundle with dominating fibre. Then $h(B_f) \geq \log(\#(\mathcal{L}))$ whenever $\mathcal{L}$ is finite and $h(B_f) = +\infty$ whenever $\mathcal{L}$ is infinite.*

We shall say a *sequence of f-bundles* $B_f^k = (\mathcal{L}_k, J_k)$ *converges* to a point x_0, if for any $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that both $\bigcup \mathcal{L}_k \subset (x_0 - \varepsilon, x_0 + \varepsilon)$ and $(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \cap J_k \neq \emptyset$ for any $k \geq k_0$. In this way, we obtain a multifunction $E_f : \mathbb{I} \multimap \mathbb{R} \cup \{+\infty\}$ by putting $E_f(x) = \{\limsup_{n\to\infty} h(B_f^n) : B_f^n \longrightarrow x \text{ as } n \to \infty\}$.

An *entropy of a function f at* $x_0 \in \mathbb{I}$ [9] is the number $e_f(x_0) = \sup E_f(x_0)$.

2. Stable and almost stable points and perturbation

The notion of a stable point will be adopted in the version given in [10] and in a natural way it will be extended to the concept of an almost stable point.

We say that $x_0 \in \mathbb{I}$ is a *stable point* of a dynamical system $(f_{1,\infty})$ if $x_0 \in \operatorname{Fix}(f_{1,\infty})$ and for any $\varepsilon > 0$ there is $\delta > 0$ such that for each $i \in \mathbb{N}$ and $x \in \mathbb{I}$, if $|x - x_0| < \delta$ then $|f_1^i(x) - x_0| < \varepsilon$, and an *almost stable point* if for any $\varepsilon > 0$ there are $\delta > 0$ and $i_0 \in \mathbb{N}$ such that for each $i \geq i_0$ and $x \in \mathbb{I}$, if $|x - x_0| < \delta$ then $|f_1^i(x) - x_0| < \varepsilon$.

By a stable (almost stable) point of a function f, we mean a stable (almost stable) point of the autonomous dynamical system (f).

Before we get to more complex considerations, some basic properties and relationships between these concepts will be given. (The simple proofs are omitted.)

PROPERTY 2.1.

(i) *If $x_0 \in \mathbb{I}$ is a stable point of a dynamical system $(f_{1,\infty})$ then x_0 is an almost stable point of this system. The converse is not true in general.*

(ii) *If $x_0 \in \mathbb{I}$ and $x_0 \in C(f_{1,\infty})$ then x_0 is a stable point of the dynamical system $(f_{1,\infty})$ if and only if x_0 is its almost stable point.*

(iii) *If f is a function and $x_0 \in \mathbb{I}$ is a stable point of the autonomous dynamical system (f), then $x_0 \in C(f)$. The converse is not true.*

(iv) *There exist a dynamical system $(f_{1,\infty})$ and $x_0 \in \mathbb{I}$ such that x_0 is a stable point of $(f_{1,\infty})$ and $x_0 \notin C(f_{1,\infty})$.*

(v) *If $x_0 \in \mathbb{I}$ is a stable point of a dynamical system $(f_{1,\infty})$ then x_0 is a continuity point of f_1.*

(vi) *There exist a dynamical system $(f_{1,\infty})$ and $x_0 \in \mathbb{I}$ such that x_0 is a stable point of the function f_n for any $n \in \mathbb{N}$ and it is not a stable point of $(f_{1,\infty})$.*

PROPERTY 2.2. *If $(f_{1,\infty})$ is a dynamical system, $x_0 \in \operatorname{Fix}(f_{1,\infty})$ and there exists a nondegenerate interval $P \subset \mathbb{I}$ containing x_0 such that $f_1 \restriction P$ is a constant function or there exists $n_0 \in \mathbb{N} \setminus \{1\}$ such that P is $f_1^{n_0-1}$-invariant and $f_{n_0} \restriction P$ is a constant function, then*

https://doi.org/10.1017/S0004972717000272 Published online by Cambridge University Press

(a) *x_0 is an almost stable point of $(f_{1,\infty})$, and*

(b) *if additionally $x_0 \in C(f_{1,\infty})$ then x_0 is a stable point of $(f_{1,\infty})$.*

It is obvious that we should focus our attention on dynamical systems whose terms are not constant on any neighbourhood of x_0.

Let f be a function and $x_0 \in [0, 1)$ $(x_0 \in (0, 1])$. We shall say that f is *nowhere constant at x_0 from the right* (*from the left*) if for any $\varepsilon > 0$ there exists $x_1 \in (x_0, x_0 + \varepsilon)$ $(x_1 \in (x_0 - \varepsilon, x_0))$ such that $f(x_1) \neq f(x_0)$. Let $x_0 \in (0, 1)$. We shall say that f is *nowhere constant at x_0* if it is simultaneously nowhere constant at x_0 from the right and from the left.

For simplicity of notation, we adopt the convention that for $x_0 \in \{0, 1\}$ writing that f is nowhere constant at x_0 means that f is nowhere constant at 0 from the right and at 1 from the left.

Lemma 2.3. *If f is nowhere constant at $x_0 \in \mathbb{I}$ from the left (from the right) and there is a nondegenerate interval $P \subset \mathbb{I}$ such that $x_0 \in P$ and $f \restriction P$ is a Darboux function, then for any $x_1 \in P$ such that $x_1 < x_0$ $(x_1 > x_0)$ the image $f([x_1, x_0])$ $(f([x_0, x_1]))$ is a nondegenerate interval.*

Lemma 2.4. *Let $n_0 \in \mathbb{N}$, $f_1, f_2, \ldots, f_{n_0}$ be functions and $x_0 \in \mathbb{I}$ be a fixed point of each function f_n for $n \in \{1, \ldots, n_0\}$. If there is a nondegenerate interval $P \subset \mathbb{I}$ such that $x_0 \in \operatorname{int}(P)$ and for any $n \leq n_0$ the function f_n is nowhere constant at x_0, P is f_n-invariant and $f_n \restriction P$ is a Darboux function, then $f_1^{n_0}$ is nowhere constant at x_0.*

Proof. Assume that $x_0 \in (0, 1)$. If $x_0 \in \{0, 1\}$ the proof proceeds in the same way. Let $\varepsilon > 0$. Without loss of generality we can assume that $\varepsilon < \min\{x_0 - \inf P; \sup P - x_0\}$. If $n_0 = 1$ the lemma is obvious, so assume that $n_0 > 1$. It is sufficient to show that

$$\text{there are } x_1 \in (x_0 - \varepsilon, x_0), x_2 \in (x_0, x_0 + \varepsilon) \text{ such that } f_1^{n_0}(x_1) \neq x_0, f_1^{n_0}(x_2) \neq x_0. \quad (2.1)$$

Consider first the function f_1. There are $y_1^1 \in (x_0 - \varepsilon, x_0)$, $y_2^1 \in (x_0, x_0 + \varepsilon)$ such that $f_1(y_1^1) \neq x_0$ and $f_1(y_2^1) \neq x_0$. By Lemma 2.3, $f_1([y_1^1, x_0])$ and $f_1([x_0, y_2^1])$ are nondegenerate intervals containing x_0.

Next, there are $y_1^2 \in f_1([y_1^1, x_0]) \cap (x_0 - \varepsilon, x_0 + \varepsilon)$, $y_2^2 \in f_1([x_0, y_2^1]) \cap (x_0 - \varepsilon, x_0 + \varepsilon)$ such that $f_2(y_1^2) \neq x_0$ and $f_2(y_2^2) \neq x_0$. So we can find $y_1^{1,2} \in [y_1^1, x_0]$, $y_2^{1,2} \in [x_0, y_2^1]$ such that $f_1(y_1^{1,2}) = y_1^2$, $f_1(y_2^{1,2}) = y_2^2$, $f_2(f_1(y_1^{1,2})) \neq x_0$ and $f_2(f_1(y_2^{1,2})) \neq x_0$. Obviously, $y_1^{1,2} \in (x_0 - \varepsilon, x_0)$ and $y_2^{1,2} \in (x_0, x_0 + \varepsilon)$.

The sets $f_2(f_1([y_1^{1,2}, x_0]))$ and $f_2(f_1([x_0, y_2^{1,2}]))$ are nondegenerate intervals containing x_0 (see Lemma 2.3). So there are $y_1^{1,2,3} \in (x_0 - \varepsilon, x_0)$ and $y_2^{1,2,3} \in (x_0, x_0 + \varepsilon)$ such that $f_3(f_2(f_1(y_1^{1,2,3}))) \neq x_0$ and $f_3(f_2(f_1(y_2^{1,2,3}))) \neq x_0$. We continue in this fashion obtaining points $y_1^{1,2,\ldots,n_0} \in (x_0 - \varepsilon, x_0)$ and $y_2^{1,2,\ldots,n_0} \in (x_0, x_0 + \varepsilon)$ such that $f_1^{n_0}(y_1^{1,2,\ldots,n_0}) \neq x_0$ and $f_1^{n_0-1}(y_2^{1,2,\ldots,n_0}) \neq x_0$. Putting $x_1 = y_1^{1,2,\ldots,n_0}$ and $x_2 = y_2^{1,2,\ldots,n_0}$ yields (2.1), which means that $f_1^{n_0}$ is nowhere constant at x_0. □

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LEMMA 2.5. *Let $(f_{1,\infty})$ be a dynamical system and $x_0 \in \mathrm{Fix}(f_{1,\infty})$. If there are $n_0 \in \mathbb{N}$ and a nondegenerate interval $P \subset \mathbb{I}$ such that $x_0 \in \mathrm{int}(P)$, f_{n_0} is not continuous at x_0 from both sides (with the obvious qualification if $x_0 \in \{0, 1\}$) and for any $n < n_0$ the function f_n is nowhere constant at x_0, P is f_n-invariant and $f_n \restriction P$ is a Darboux function, then x_0 is not a stable point of $(f_{1,\infty})$.*

PROOF. If $n_0 = 1$ then Proposition 2.1(iv) implies that x_0 is not a stable point of $(f_{1,\infty})$.

Assume that $n_0 > 1$. Then one can find sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \subset \mathbb{I}$ such that $x_k \le x_0 \le y_k$ for any $k \in \mathbb{N}$, $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = x_0$ and x_0 is not a limit of both sequences $\{f_{n_0}(x_k)\}_{k\in\mathbb{N}}$ and $\{f_{n_0}(y_k)\}_{k\in\mathbb{N}}$. There is no loss of generality in assuming that $\lim_{k\to\infty} f_{n_0}(x_k) = \alpha$ and $\lim_{k\to\infty} f_{n_0}(y_k) = \beta$. Put $\varepsilon = \min\{\frac{1}{2}|x_0 - \alpha|, \frac{1}{2}|y_0 - \beta|\}$ and suppose that x_0 is a stable point of $(f_{1,\infty})$. Thus there is $\delta > 0$ such that

$$\text{if } |x - x_0| < \delta \text{ then } |x_0 - f_1^{n_0}(x)| < \varepsilon. \tag{2.2}$$

Let $k_0 \in \mathbb{N}$ be such that $|x_{k_0} - x_0| < \delta$, $x_{k_0} \in P$, $f_{n_0}(x_k) \notin (x_0 - \varepsilon, x_0 + \varepsilon)$ and $f_{n_0}(y_k) \notin (x_0 - \varepsilon, x_0 + \varepsilon)$ for $k > k_0$. By Lemmas 2.4 and 2.3, $f_1^{n_0-1}([x_{k_0}, x_0])$ is a nondegenerate interval containing x_0. Let $k_1 > k_0$ be such that $x_{k_1} \in f_1^{n_0-1}([x_{k_0}, x_0])$ or $y_{k_1} \in f_1^{n_0-1}([x_{k_0}, x_0])$. Without any restriction of generality, we can assume that $y_{k_1} \in f_1^{n_0-1}([x_{k_0}, x_0])$, so there exists $t_0 \in [x_{k_0}, x_0]$ such that $f_1^{n_0-1}(t_0) = y_{k_1}$. Therefore $|t_0 - x_0| < \delta$ and $f_1^{n_0}(t_0) \notin (x_0 - \varepsilon, x_0 + \varepsilon)$, which contradicts (2.2). □

In the next theorem, we show that under some natural assumptions we can perturb an autonomous dynamical system (f) so that a given point will be an almost stable point of a new system and will not be its stable point.

THEOREM 2.6. *Let $x_0 \in \mathbb{I}$ and $f \in \mathcal{D}\,\mathrm{Fix}_{x_0}$ be such that x_0 is its stable point and $f'(x_0) \in (0, 1)$. (If $x_0 \in \{0, 1\}$, this is a one-sided derivative.) Then for any $\varepsilon > 0$ there is an open (in the space $(\mathcal{D}\,\mathrm{Fix}_{x_0}, \rho_u)$) set $V_\varepsilon \subset \mathcal{D}\,\mathrm{Fix}_{x_0}$ such that for any $i \in \mathbb{N}$ the dynamical system (f) is (i, ε)-perturbed by V_ε to a dynamical system for which x_0 is an almost stable point and is not its stable point.*

PROOF. Let $\varepsilon > 0$. Assume that $x_0 \in (0, 1)$. If $x_0 \in \{0, 1\}$ the proof is analogous.

According to Proposition 2.1(iii), there is $\delta_0 \in (0, \min\{\varepsilon/3, x_0, 1 - x_0\})$ such that $f([x_0 - \delta_0, x_0 + \delta_0]) \subset (x_0 - \varepsilon/3, x_0 + \varepsilon/3)$. Since $f'(x_0) \in (0, 1)$, one can find $\sigma \in (0, 1)$ and $\alpha \in (0, \delta_0)$ such that, for any $x \in (x_0 - \alpha, x_0 + \alpha) \setminus \{x_0\}$,

$$0 < \frac{f(x) - f(x_0)}{x - x_0} < \sigma. \tag{2.3}$$

Thus, if $x \in (x_0, x_0 + \alpha)$ then $f(x) \in (x_0, x)$. From this and the fact that $f(x_0) = x_0$,

$$(f)_1^n(x) \in (x_0, (f)_1^{n-1}(x)) \subset (x_0, x) \quad \text{for } x \in (x_0, x_0 + \alpha) \text{ and } n \in \mathbb{N}. \tag{2.4}$$

In the same manner,

$$(f)_1^n(x) \in ((f)_1^{n-1}(x), x_0) \subset (x, x_0) \quad \text{for } x \in (x_0 - \alpha, x_0) \text{ and } n \in \mathbb{N}. \tag{2.5}$$

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Conditions (2.3), (2.4) and (2.5) yield

$$|(f)_1^n(x) - x_0| < \frac{3}{4}\alpha\sigma^n \quad \text{for } x \in \Big(x_0 - \frac{\alpha}{4}, x_0 + \frac{3}{4}\alpha\Big] \text{ and } n \in \mathbb{N}. \tag{2.6}$$

For $n = 1$ the above inequality is obvious. If $x \in (x_0 - \frac{1}{4}\alpha, x_0 + \frac{3}{4}\alpha]$ and $n \in \mathbb{N} \setminus \{1\}$ then $|(f)_1^n(x) - x_0| = |f((f)_1^{n-1}(x)) - x_0| < \sigma|(f)_1^{n-1}(x) - x_0| < \frac{3}{4}\alpha\sigma^n$.

Let $\{a_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive numbers such that $a_1 \le \frac{1}{2}\alpha$ and $\lim_{n\to\infty} a_n = 0$. Put $b_n = x_0 - a_n$, $c_n = x_0 + a_n$, $d_n = \frac{1}{2}(c_{2n+1} + c_{2n})$ and $r_n = \frac{1}{2}(b_{2n+1} + b_{2n})$ for $n \in \mathbb{N}$. Moreover, let $g_0(x) = f(x)$ if $x \in \mathbb{I} \setminus (x_0 - \alpha, x_0 + \alpha) \cup \{x_0\}$, $g_0(x) = x_0$ if $x \in \bigcup_{n=1}^{\infty}((b_{2n-1}, b_{2n}) \cup (c_{2n}, c_{2n-1}))$, $g_0(x) = x_0 + \frac{1}{2}\alpha$ if $x = d_n$ or $x = r_n$ for $n \in \mathbb{N}$ and let g_0 be linear otherwise.

It is easy to see that each point of $\mathbb{I}$ is a Darboux point of g_0 and $x_0 \in \mathrm{Fix}(g_0)$, so $g_0 \in \mathcal{D}\mathrm{Fix}_{x_0}$. Moreover, $\rho_u(g_0, f) \le \frac{2}{3}\varepsilon$. So, if $V_\varepsilon = \{\phi \in \mathcal{D}\mathrm{Fix}_{x_0} : \rho_u(g_0, \phi) < \frac{1}{4}\alpha\}$ and $\xi \in V_\varepsilon$, then $\rho_u(f, \xi) \le \rho_u(f, g_0) + \rho_u(g_0, \xi) < \varepsilon$.

Let $\xi \in V_\varepsilon$. Fix $i_0 \in \mathbb{N}$ and put $\beta = \frac{1}{8}\alpha$. Suppose that x_0 is a stable point of $(f_{1,\infty})$, where $f_n = f$ for $n \in \mathbb{N} \setminus \{i_0\}$ and $f_{i_0} = \xi$. Then there exists $\gamma > 0$ such that for any $k \in \mathbb{N}$ and $x \in (x_0 - \gamma, x_0 + \gamma)$,

$$f_1^k(x) \in (x_0 - \beta, x_0 + \beta). \tag{2.7}$$

If $i_0 = 1$, consider $n_0 \in \mathbb{N}$ such that $d_{n_0} \in (x_0 - \gamma, x_0 + \gamma)$. Thus $f_1(d_{n_0}) = \xi(d_{n_0}) \in (x_0 + \frac{1}{4}\alpha, x_0 + \frac{3}{4}\alpha)$, contrary to (2.7).

If $i_0 > 1$, let $x_1 \in (x_0, x_0 + \min\{\gamma, \alpha\})$. Obviously $x_0 \in \mathrm{Fix}(f_1^{i_0-1})$ and, by (2.4), $f_1^{i_0-1}(x_1) \in (x_0, x_1)$. Let $n_1 \in \mathbb{N}$ be such that $d_{n_1} \in (x_0, f_1^{i_0-1}(x_1))$. There is $x_* \in (x_0, x_1)$ such that $f_1^{i_0-1}(x_*) = d_{n_1}$. Then $f_1^{i_0}(x_*) \in (x_0 + \frac{1}{4}\alpha, x_0 + \frac{3}{4}\alpha)$, contrary to (2.7).

These contradictions show that x_0 is not a stable point of $(f_{1,\infty})$.

Now we will show that x_0 is an almost stable point of $(f_{1,\infty})$. Let $\beta_1 > 0$. There is $\delta_0 > 0$ such that $|(f)_1^n(x) - x_0| < \beta_1$ for any $n \in \mathbb{N}$ and $|x - x_0| < \delta_0$. Put $\delta_* = \min\{\delta_0, \frac{1}{2}\alpha\}$ and consider the following two possibilities.

Suppose $i_0 = 1$. Let $n_0 > 2$ be such that $\frac{3}{4}\alpha\sigma^{n_0-2} < \beta_1$. Let $n > n_0$ and $|x - x_0| < \delta_*$. If $x \in (x_0 - \delta_*, x_0 + \delta_*)$ then $f_1(x) = \xi(x) \in (x_0 - \frac{1}{4}\alpha, x_0 + \frac{3}{4}\alpha)$. From this and (2.6) we obtain $|f_1^n(x) - x_0| < \beta_1$.

Suppose $i_0 > 1$. Let $n_0 > i_0 + 1$ be such that $\frac{3}{4}\alpha\sigma^{n_0-i_0-1} < \beta_1$. Let $n > n_0$ and $|x - x_0| < \delta_*$. If $x \in (x_0 - \delta_*, x_0]$ then, by (2.5), $f_1^{i_0-1}(x) = (f)_1^{i_0-1}(x) \in (x_0 - \delta_*, x_0]$. If $x \in (x_0, x_0 + \delta_*)$ then, by (2.4), $f_1^{i_0-1}(x) = (f)_1^{i_0-1}(x) \in (x_0, x_0 + \delta_*)$. Therefore, if $x \in (x_0 - \delta_*, x_0 + \delta_*]$ then $f_1^{i_0-1}(x) \in (x_0 - \delta_*, x_0 + \delta_*)$. Thus $f_1^{i_0}(x) \in (x_0 - \frac{1}{4}\alpha, x_0 + \frac{3}{4}\alpha)$. From this and (2.6) we conclude that $|f_1^n(x) - x_0| = |f_{i_0+1}^{n-i_0}(f_1^{i_0}(x)) - x_0| < \beta_1$.

In both cases, we see that x_0 is an almost stable point of $(f_{1,\infty})$. □

In the above theorem, the set $\mathcal{D}\mathrm{Fix}_{x_0}$ can be replaced by the family of all Darboux Baire-one functions such that x_0 is their fixed point if we start with a Darboux Baire-one function f. Also, if we assume that the function f is almost continuous in the sense of Stallings (this kind of function was introduced in [14]) then, using Lemma 2.3 and

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Theorems 2.2 and 2.4 from [11], we can prove that the function g_0 constructed in the above proof is almost continuous in the sense of Stallings. Therefore, we can replace $\mathcal{D}\,\mathrm{Fix}_{x_0}$ by the set of all Stallings almost continuous functions such that x_0 is their fixed point. Moreover, if we assume that f is an approximately continuous function (as defined in [16]) and, in addition, require that the sequence $\{a_n\}_{n\in\mathbb{N}}$ considered in the above proof is such that x_0 is a density point of the set $\bigcup_{n=1}^{\infty}((b_{2n-1}, b_{2n}) \cup (c_{2n}, c_{2n-1}))$ then we obtain immediately that g_0 is an approximately continuous function. Thus in the above theorem the set $\mathcal{D}\,\mathrm{Fix}_{x_0}$ can be replaced by the set of all approximately continuous functions such that x_0 is their fixed point.

3. Odd points and approximation

The analysis of different examples of functions leads us to the interesting observation that entropy of a function may be focused at one point. The problematic question here is the meaning of the expression 'entropy is focused around a point' [6, 13, 18]. Although positive entropy at a given point can be understood as 'unpredictable' behaviour of the function around this point, it turns out that there are situations where the function at a given point is stable, but an entropy of the function at this point is equal to infinity. This leads to distinguishing so-called odd points.

We shall say that $x_0 \in \mathbb{I}$ is an *odd point of a dynamical system* $(f_{1,\infty})$ if x_0 is an almost stable point of the dynamical system $(f_{1,\infty})$ and for any $n \in \mathbb{N}$ an entropy of the function f_n at the point x_0 is infinite. By an odd point of a function f, we mean an odd point of the autonomous dynamical system (f).

Let $\mathrm{Odd}_c(x_0)$ denote a family of all continuous at x_0 functions f such that x_0 is an odd point of f. Clearly, $\mathrm{Odd}_c(x_0) \subset \mathrm{St}(x_0)$, where $\mathrm{St}(x_0)$ is the family of all functions f such that x_0 is a stable point of f. We can prove even more.

Theorem 3.1. *Let* $x_0 \in [0, 1]$. *The set* $\mathrm{Odd}_c(x_0)$ *is a dense set with empty interior in the space* $(\mathrm{St}(x_0), \rho_u)$.

Proof. Assume that $x_0 \in (0, 1)$. Similar arguments apply to the case $x_0 \in \{0, 1\}$.

We first prove that $\mathrm{Odd}_c(x_0)$ is dense in the space $(\mathrm{St}(x_0), \rho_u)$. Let $f \in \mathrm{St}(x_0)$ and $\varepsilon > 0$. By Proposition 2.1(iii), there is $\delta \in (0, \min\{\frac{1}{3}\varepsilon, x_0, 1 - x_0\})$ such that $f([x_0 - \delta, x_0 + \delta]) \subset (x_0 - \frac{1}{3}\varepsilon, x_0 + \frac{1}{3}\varepsilon)$. Put $b_i^n = x_0 + \delta(2^n + 1 + i)/2^{2n}(2^n + 1)$ for $n \in \mathbb{N}$ and $i \in \{0, \dots, 2^n + 1\}$.

Define the function $g : \mathbb{I} \to \mathbb{I}$ by $g(x) = f(x)$ for $x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta)$, $g(x) = x$ for $x \in (\mathbb{I} \cap [x_0 - \frac{2}{3}\delta, x_0 + \frac{1}{2}\delta]) \setminus \bigcup_{n=1}^{\infty}[x_0 + \delta/2^{2n}, x_0 + \delta/2^{2n-1}]$, $g(x) = x_0 + \delta/2^{2n}$ for $x \in \{b_i^n : n \in \mathbb{N} \text{ and } i = 0, 2, \dots, 2^n\}$, $g(x) = x_0 + \delta/2^{2n-1}$ for $x \in \{b_i^n : n \in \mathbb{N}$ and $i = 1, 3, \dots, 2^n + 1\}$ and g linear otherwise. It is easy to see that

$$g([b_0^n, b_{2^n+1}^n]) = [b_0^n, b_{2^n+1}^n] \quad \text{for } n \in \mathbb{N}, \tag{3.1}$$

and

$$g([b_i^n, b_{i+1}^n]) = [b_0^n, b_{2^n+1}^n] \quad \text{for } n \in \mathbb{N} \text{ and } i \in \{0, 2, \dots, 2^n\}. \tag{3.2}$$

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We will show that $\rho_u(f,g) < \varepsilon$ and $g \in \mathrm{Odd_c}(x_0)$. Obviously, $x_0 \in \mathrm{Fix}(g)$ and g is continuous at x_0. Let $\alpha > 0$. There is $n_0 \in \mathbb{N}$ such that $\delta/2^{2n-1} < \alpha$ for $n \geq n_0$. Put $\beta_0 = \min\{\delta/2^{2n_0-1}, \alpha\}$. We claim that

$$|x_0 - (g)_1^i(x)| < \alpha \quad \text{for } i \in \mathbb{N} \text{ whenever } |x_0 - x| < \beta_0.$$

Let $x \in (x_0 - \beta_0, x_0 + \beta_0)$ and $i \in \mathbb{N}$. If there is $n > n_0$ such that $x \in [b_0^n, b_{2^n+1}^n]$ then condition (3.1) gives $(g)_1^i(x) \in [b_0^n, b_{2^n+1}^n] \subset (x_0 - \alpha, x_0 + \alpha)$. If for any $n > n_0$ we have $x \notin [b_0^n, b_{2^n+1}^n]$, then $g(x) = x$ and, in consequence, $(g)_1^i(x) = x \in (x_0 - \alpha, x_0 + \alpha)$.

Observe that $e_g(x_0) = \infty$. Indeed, put $\mathcal{F}_n = \{[b_{2i}^n, b_{2i+1}^n] : i = 0, 1, \ldots, 2^{n-1}\}$ for $n \in \mathbb{N}$. By condition (3.2), it is easy to show that the $(\mathcal{F}_n, [b_0^n, b_{2^n+1}^n])$ are g-bundles with dominating fibre. Moreover, we check at once that the sequence of bundles $\{(\mathcal{F}_n, [b_0^n, b_{2^n+1}^n])\}_{n\in\mathbb{N}}$ converges to x_0. Since $\#(\mathcal{F}_n) = 2^{n-1} + 1$ for $n \in \mathbb{N}$, Lemma 1.1 implies that $\infty \in E_g(x_0)$. Thus an entropy of g at x_0 is infinite.

Finally, note that $\rho_u(f,g) < \varepsilon$. Indeed, if $x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta)$ then $g(x) = f(x)$, so $|g(x) - f(x)| = 0$. If $x \in [x_0 - \frac{2}{3}\delta, x_0 + \frac{1}{2}\delta]$ then $g(x) \in [x_0 - \frac{2}{3}\delta, x_0 + \frac{1}{2}\delta]$. Moreover, for $x \in (x_0 - \delta, x_0 - \frac{2}{3}\delta) \cup (x_0 + \frac{1}{2}\delta, x_0 + \delta)$ we have $g(x) \in (x_0 - \frac{1}{3}\varepsilon, x_0 + \frac{1}{3}\varepsilon)$. Now, we see at once that $|g(x) - f(x)| < \frac{2}{3}\varepsilon$ for $x \in (x_0 - \delta, x_0 + \delta)$. Finally, we obtain $\rho_u(f,g) = \sup_{x\in\mathbb{I}} |g(x) - f(x)| < \varepsilon$.

Since ε is arbitrary, these considerations show that $\mathrm{Odd_c}(x_0)$ is dense in the space $(\mathrm{St}(x_0), \rho_u)$.

We will now show that $\mathrm{Odd_c}(x_0)$ has empty interior in the space $(\mathrm{St}(x_0), \rho_u)$. For this purpose it is sufficient to show that for any $f \in \mathrm{St}(x_0)$ and $\varepsilon > 0$ there exists $g_* \in \mathrm{St}(x_0)$ such that $g_* \notin \mathrm{Odd_c}(x_0)$ and $\rho_u(f, g_*) < \varepsilon$.

Let us fix $f \in \mathrm{St}(x_0)$ and $\varepsilon > 0$. One can find $\delta \in (0, \min\{\frac{1}{3}\varepsilon, x_0, 1 - x_0\})$ such that $f([x_0 - \delta, x_0 + \delta]) \subset (x_0 - \frac{1}{3}\varepsilon, x_0 + \frac{1}{3}\varepsilon)$. Put $g_*(x) = f(x)$ if $x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta)$ and $g_*(x) = x_0$ if $[x_0 - \frac{1}{2}\delta, x_0 + \frac{1}{2}\delta]$ and let g_* be linear otherwise. Obviously, $\rho(g_*, f) < \varepsilon$. What is more, if $|x_0 - x| < \frac{1}{2}\delta$ then $(g_*)_1^i(x) = x_0$ for $i \in \mathbb{N}$, which gives that $g_* \in \mathrm{St}(x_0)$. Since g_* is a constant function on the set $[x_0 - \frac{1}{2}\delta, x_0 + \frac{1}{2}\delta]$, we have $e_{g_*}(x_0) = 0$, which means that $g_* \notin \mathrm{Odd_c}(x_0)$. □

We now focus our attention on an approximation of a function by (nonautonomous) dynamical systems. The first theorem is related to an approximation by a dynamical system consisting of discontinuous functions and the second is connected with a dynamical system consisting of functions continuous at some point. For other kinds of approximation by functions with an entropy at a special point, see [5, 7, 13].

Theorem 3.2. *Let $x_0 \in \mathbb{I}$ and $f \in \mathrm{St}(x_0)$. For any $\varepsilon > 0$ there exists a dynamical system $(f_{1,\infty}^{\varepsilon})$ such that:*

- (W1) *f_n^{ε} is not continuous at x_0 from both sides (so also is nowhere constant at x_0) for any $n \in \mathbb{N}$ (with the obvious one-sided interpretation if $x_0 \in \{0, 1\}$),*
- (W2) *for any $n \in \mathbb{N}$ the point x_0 is not an almost stable point of each function f_n^{ε},*
- (W3) *x_0 is not a stable point of $(f_{1,\infty}^{\varepsilon})$,*
- (W4) *x_0 is an odd point of $(f_{1,\infty}^{\varepsilon})$, so it is also an almost stable point of this system,*
- (W5) *$\rho_u(f, f_n^{\varepsilon}) < \varepsilon$ for any $n \in \mathbb{N}$.*

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PROOF. Let $f \in \mathrm{St}(x_0)$ and $\varepsilon > 0$. We will construct the dynamical system having the above properties for $x_0 \in (0, 1)$. The proofs in other cases proceed in a similar way.

We may assume that $\varepsilon < \min\{x_0, 1 - x_0\}$. Proposition 2.1(iii) implies that there exists $\delta < \min\{\frac{1}{3}\varepsilon, x_0, 1 - x_0\}$ such that $f([x_0 - \delta, x_0 + \delta]) \subset (x_0 - \frac{1}{3}\varepsilon, x_0 + \frac{1}{3}\varepsilon)$. Put $a_k = \delta/2^k$, $b_k = x_0 + a_k$ and $c_k = x_0 - a_k$ for $k \in \mathbb{N}$.

Now fix $n \in \mathbb{N}$ and put $f_n(x) = f(x)$ for $x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta)$, $f_n(x) = x_0$ for $x \in \{b_{2k-1}, c_{2k-1} : k \in \mathbb{N}\} \cup \{x_0\}$, $f_n(x) = b_n$ for $x \in \{b_{2k} : k \in \mathbb{N}\}$ and $f_n(x) = c_n$ for $x \in \{c_{2k} : k \in \mathbb{N}\}$ and let f_n be linear otherwise.

We will show that the dynamical system $(f_{1,\infty})$, where f_n is defined by the above formula, has properties (W1)–(W5). Indeed, conditions (W1) and (W5) are obvious. Condition (W1) and Lemma 2.5 yield (W3).

Let $n \in \mathbb{N}$. For any $\sigma > 0$, $(f_n)_1^k([x_0 - \frac{1}{2}\delta, x_0 + \frac{1}{2}\delta] \cap (x_0 - \sigma, x_0 + \sigma)) = [c_n, b_n]$. Fix $\varepsilon_0 = a_{n+1}$. For any $\beta > 0$ and any $k \in \mathbb{N}$, we can find $x_1 \in (x_0 - \beta, x_0 + \beta)$ such that $(f_n)_1^k(x_1) = b_n$, so $|x_0 - (f_n)_1^k(x_1)| = a_n > \varepsilon_0$. This means that x_0 is not an almost stable point of the function f_n, which gives (W2).

To prove (W4) we show first that x_0 is an almost stable point of $(f_{1,\infty})$. Let $\varepsilon_1 > 0$. There is $n_0 \in \mathbb{N}$ such that $a_n < \frac{1}{2}\varepsilon_1$ for any $n \geq n_0$. Moreover, $f_1^n([x_0 - \frac{1}{2}\delta, x_0 + \frac{1}{2}\delta]) = [c_n, b_n]$. Thus for any $n \geq n_0$, if $|x - x_0| < \frac{1}{2}\delta$ then $|f_1^n(x) - x_0| < a_n < \varepsilon_1$. Since $x_0 \in \mathrm{Fix}(f_{1,\infty})$, x_0 is an almost stable point of $(f_{1,\infty})$.

Now we will show that $e_{f_n}(x_0) = \infty$ for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. One can find $k_0 \in \mathbb{N}$ such that $b_{2k} < b_n$ for any $k \geq k_0$. Put $\mathcal{F}_k = \{[b_{2(k+k_0+i)+1}, b_{2(k+k_0+i)}] : i \in \mathbb{N}\}$ for each $k \in \mathbb{N}$. For any $k \in \mathbb{N}$ the pair $(\mathcal{F}_k, [x_0, b_n])$ is an f_n-bundle with dominating fibre. Moreover, the sequence of bundles $(\mathcal{F}_k, [x_0, b_n])$ converges to the point x_0. Since for any $k \in \mathbb{N}$ the family $\mathcal{F}_k$ is infinite, Lemma 1.1 implies that $\infty \in E_{f_n}(x_0)$, so an entropy of f_n at x_0 is equal to ∞.

Finally, x_0 is an odd point of $(f_{1,\infty})$. □

THEOREM 3.3. *Let $x_0 \in \mathbb{I}$ and $f \in \mathrm{St}(x_0)$. For any $\varepsilon > 0$ there exists a dynamical system $(f^{\varepsilon}_{1,\infty})$ such that:*

(C1) *f_n^{ε} is continuous at x_0 and nowhere constant at x_0 for any $n \in \mathbb{N}$,*
(C2) *for any $n \in \mathbb{N}$ the point x_0 is not an almost stable point of the function f_n^{ε},*
(C3) *x_0 is a stable point of $(f^{\varepsilon}_{1,\infty})$,*
(C4) *x_0 is an odd point of $(f^{\varepsilon}_{1,\infty})$,*
(C5) *$\rho_u(f, f_n^{\varepsilon}) < \varepsilon$ for any $n \in \mathbb{N}$.*

PROOF. Assume that $x_0 \in (0, 1)$. Similar arguments apply to the case $x_0 = 0$ or $x_0 = 1$.

Let $f \in \mathrm{St}(x_0)$ and $\varepsilon \in (0, \min\{x_0, 1 - x_0\})$. Proposition 2.1(iii) shows that there is $\delta \in (0, \min\{\frac{1}{3}\varepsilon, x_0, 1 - x_0\})$ such that $f([x_0 - \delta, x_0 + \delta]) \subset (x_0 - \frac{1}{3}\varepsilon, x_0 + \frac{1}{3}\varepsilon)$.

Consider the dynamical system $(f_{1,\infty})$, where for any $n \in \mathbb{N}$, the function f_n is defined in the following way: $f_n(x) = f(x)$ if $x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta) \cup \{x_0\}$; $f_n(x) = x_0 + \delta/2^{k-1}$ if $x = x_0 + \delta/2^k$ or $x = x_0 - \delta/2^k$ and $k > n$ and $k \in \mathbb{N}$; $f_n(x) = x_0 + \delta/2^k$ if $x = x_0 + \delta/2^{k+1} + s \cdot \delta/(2^{k+1} - 1)2^{k+2}$ or $x = x_0 - \delta/2^{k+1} - s \cdot \delta/(2^{k+1} - 1)2^{k+2}$ and $k > n$ and $k \in \mathbb{N}$ and $s = 2, 4, \ldots, 2^{k+1} - 2$; $f_n(x) = x_0$ if $x = x_0 + 3\delta/2^{k+2}$ or $x = x_0 - 3\delta/2^{k+2}$ and $k > n$ and $k \in \mathbb{N}$; $f_n(x) = x_0 + \delta/2^k$ if $x = x_0 + \delta/2^{k+1} + s \cdot \delta/(2^{k+1} - 1)2^{k+2}$ or

$x = x_0 - \delta/2^{k+1} - s \cdot \delta/(2^{k+1} - 1)2^{k+2}$ and $k > n$ and $k \in \mathbb{N}$ and $s = 1, 3, \ldots, 2^{k+1} - 3$; $f_n(x) = x_0 + \delta/2^n$ if $x = x_0 + \frac{2}{3}\delta$ or $x = x_0 - \frac{2}{3}\delta$ and f_n is linear otherwise.

We will show that for this dynamical system conditions (C1)–(C5) are fulfilled. Conditions (C1) and (C5) are obvious.

Fix $n \in \mathbb{N}$. We see at once that $f_n \restriction [x_0 + \delta/2^{n+1}, x_0 + 2\delta/3]$ is a constant function equal to $x_0 + \delta/2^n$.

Moreover, for $\beta < \delta/2^n$ and any $\sigma > 0$ there exist $x_1 \in (x_0, x_0 + \sigma)$ and $i_0 \in \mathbb{N}$ such that $(f_n)_1^i(x_1) \notin (x_0 - \beta, x_0 + \beta)$ for each $i > i_0$. Indeed, fix $\beta < \delta/2^n$ and $\sigma > 0$. There is $k_0 \in \mathbb{N}$ such that $x_0 + \delta/2^{k_0} \in (x_0, x_0 + \sigma)$ and $k_0 > n + 1$. Putting $x_1 = x_0 + \delta/2^{k_0}$ and $i_0 = k_0 - (n + 1)$, we obtain $f_n(x_1) = x_0 + \delta/2^{k_0 - 1}$. Thus $(f_n)_1^2(x_1) = x_0 + \delta/2^{k_0-2}$, $(f_n)_1^3(x_1) = x_0 + \delta/2^{k_0-3}$ and in general $(f_n)_1^s(x_1) = x_0 + \delta/2^{k_0-s}$ for $s \in \{1, \ldots, k_0 - n + 1\}$. Hence $(f_n)_1^{i_0+1}(x_1) = f_n(x_0 + \delta/2^{k_0-i_0}) = x_0 + \delta/2^{k_0-i_0-1} = x_0 + \delta/2^n$. Obviously, $x_0 + \delta/2^n \in (x_0 + \delta/2^{n+1}, x_0 + 2\delta/3)$. Thus for any $i > i_0$ we have $(f_n)_1^i(x_1) = (f_n)_{i-i_0-1}^{i_0+2}((f_n)_1^{i_0+1}(x_1)) = x_0 + \delta/2^n$, so $|(f_n)_1^i(x_1) - x_0| > \beta$, which shows that x_0 is not an almost stable point of the function f_n, and (C2) is proved.

To show (C3), note first that $x_0 \in \mathrm{Fix}(f_{1,\infty})$. Moreover, for each $n \in \mathbb{N}$ we have $f_n([x_0 - \delta/2, x_0 + \delta/2]) = [x_0, x_0 + \delta/2^n]$. Thus

$$f_1^k([x_0 - \delta/2, x_0 + \delta/2]) \subset [x_0, x_0 + \delta/2^k] \quad \text{for } k \in \mathbb{N}. \tag{3.3}$$

Let $\beta > 0$. One can find $k_0 \in \mathbb{N}$ such that $\delta/2^{k_0} < \delta$ and $k_0 > 1$. Then if $|x_0 - x| < \delta/2$, by (3.3), we get $f_1^k(x) \in [x_0, x_0 + \delta/2^{k_0}]$ for $k \geq k_0$. Thus $|x_0 - f_1^k(x)| < \beta$.

For each $k \in \{1, \ldots, k_0 - 1\}$, we can find $\sigma_k > 0$ such that if $|x_0 - x| < \sigma_k$ then $|x_0 - f_1^k(x)| < \beta$. Putting $\sigma_0 = \min\{\frac{1}{2}\delta, \sigma_1, \ldots, \sigma_{k_0-1}\}$, we see that for any $k \in \mathbb{N}$ if $|x_0 - x| < \sigma_0$ then $|x_0 - f_1^k(x)| < \beta$. Thus x_0 is a stable point of $(f_{1,\infty})$.

By Proposition 2.1(i) and (C3), x_0 is an almost stable point of $(f_{1,\infty})$. We only need to show that for any $n \in \mathbb{N}$, an entropy of f_n at the point x_0 is equal to ∞.

For this, let $n \in \mathbb{N}$. For any $k \in \mathbb{N}$ and $s \in \{0, 2, \ldots, 2^{n+k+1} - 2\}$, we consider the set $J_k^s = [x_0 + \delta/2^{n+k+1} + s\delta/(2^{n+k+1} - 1)2^{n+k+2}, x_0 + \delta/2^{n+k+1} + (s + 1)\delta/(2^{n+k+1} - 1)2^{n+k+2}]$. For $k \in \mathbb{N}$, the pair $B_{f_n}^k = (\mathcal{F}_k, [x_0, \delta/2^{n+k}])$, where $\mathcal{F}_k = \{J_k^s : s = 0, 2, \ldots, 2^{n+k+1} - 2\}$, is an f_n-bundle.

Moreover, $J_k^{2^{n+k+1}-2} = [3\delta/2^{n+k+2} - \delta/(2^{n+k+1} - 1)2^{n+k+2}, 3\delta/2^{n+k+2}] \subset [x_0, \delta/2^{n+k}]$, so $J_k^s \subset [x_0, \delta/2^{n+k}]$ for any $k \in \mathbb{N}$ and $s = 0, 2, \ldots, 2^{n+k+1} - 2$. What is more, $f_n(J_k^s) = [x_0, \delta/2^{n+k}]$ for $k \in \mathbb{N}$ and $s = 0, 2, \ldots, 2^{n+k+1} - 2$. So, for each $k \in \mathbb{N}$, the pair $B_{f_n}^k$ is an f_n-bundle with dominating fibre.

Since the sequence $(B_{f_n}^k)_{k \in \mathbb{N}}$ is convergent to x_0 and for each $k \in \mathbb{N}$ the cardinality of the family $\mathcal{F}_k$ is equal to 2^{n+k}, Lemma 1.1 gives $h(B_{f_n}^k) \geq n + k$ for any $k \in \mathbb{N}$. Thus $\limsup_{k\to\infty} h(B_{f_n}^k) = \infty$ and, in consequence, $e_{f_n}(x_0) = \infty$. This proves (C4). □

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https://doi.org/10.1017/S0004972717000272 Published online by Cambridge University Press