INTEGRAL BASES FOR QUADRATIC FORMS

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1. Let

$$Q_n(\mathbf{x}) = Q_n(x_1, \ldots, x_n) = \sum_{r,s=1}^{n} a_{rs}x_rx_s$$

be an indefinite quadratic form in the integer variables $x_1, \ldots, x_n$ with real coefficients of determinant $D = ||a_{rs}||_{(n)} \neq 0$. The homogeneous minimum $M_H(Q_n)$ and the inhomogeneous minimum $M_I(Q_n)$ of $Q_n(\mathbf{x})$ are defined as follows:

$$M_H(Q_n) = \inf_{\mathbf{x}\neq\mathbf{o}} |Q_n(\mathbf{x})|, \tag{1}$$

$$M_I(Q_n) = \sup_{\mathbf{x}_0} \inf_{\mathbf{x}} |Q_n(\mathbf{x} + \mathbf{x}_0)|, \tag{2}$$

where the upper bound in (2) is over all real $x_0 = (x_1^{(0)}, \ldots, x_n^{(0)})$. By a theorem of Blaney **(2**, Theorem 2**)**, it has been known for some time that there is a constant C_n, depending only on n, such that $M_I(Q_n) \leqslant C_n|D|^{1/n}$. The least such value of C_n is known for $n = 2, 3$ and, recently, Birch **(1)** has proved that, when $n = 2m$ and Q_{2m} is any quadratic form of signature $s(Q_{2m}) = 0$, then

$$M_I(Q_{2m}) \leqslant |\tfrac{1}{4}D|^{1/2m}, \tag{3}$$

thus generalizing the special case $m = 1$, due to Minkowski. Although a similar bound $M_H(Q_n) \leqslant C_n' \, |D|^{1/n}$ holds for the homogeneous minimum, the situation is not strictly analogous. A classical theorem of Meyer asserts that every $Q_n(\mathbf{x})$ with rational coefficients in at least 5 variables represents 0 with $\mathbf{x} \neq \mathbf{o}$ and this, in part, has given rise to the conjecture that $M_H(Q_n) = 0$ for every *real* Q_n in at least 5 variables. The most important advance in this direction was made by Davenport **(3)**, with subsequent improvements by others, and we now know that $M_H(Q_n) = 0$ when $n \geqslant 21$. However, a connection between the two minima was exhibited by Birch (*loc. cit.*) in the course of his paper, in a relatively easy way, when Q_n has at least 3 variables and represents arbitrarily small non-zero values (as, for example, when $M_H(Q_n) = 0$ and is not attained). Under these conditions he showed that $M_I(Q_n) = 0$. In an attempt to find a closer relation between $M_I(Q_n)$ and the homogeneous problem, I propose to introduce another "homogeneous" minimum $M_B(Q_n) \geqslant M_H(Q_n)$ of Q_n, associated with a set of n integral basis

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vectors: let $\mathbf{x} = \mathbf{x}_r = (x_1^{(r)}, \ldots, x_n^{(r)})$, $r = 1, 2, \ldots, n$, be a set of such vectors with

$$\det(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \pm 1 \tag{4}$$

and define

$$M_B(Q_n) = \inf\left\{ \max_{r=1,\ldots,n} |Q_n(\mathbf{x}_r)| \right\}, \tag{5}$$

where the bound is over all sets of n integral vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ satisfying (4). To summarize information for $n = 2, 3$, we have

$$M_H(Q_2) \leqslant \frac{1}{2\sqrt{5}} |D|^{1/2} \text{ (Markoff)*},$$

$$M_B(Q_2) \leqslant |D|^{1/2} \text{ (Minkowski)*}, \tag{6}$$

$$M_I(Q_2) \leqslant \frac{1}{2} |D|^{1/2} \text{ (Minkowski)*}, \tag{7}$$

and

$$M_H(Q_3) \leqslant \left|\frac{2}{3} D\right|^{1/3} \text{ (Markoff)*},$$

$$M_B(Q_3) \leqslant \left|\frac{27}{25} D\right|^{1/3} \text{ (Foster 5)}, \tag{8}$$

$$M_I(Q_3) \leqslant \left|\frac{27}{100} D\right|^{1/3} \text{ (Davenport 4)}, \tag{9}$$

where the numerical constant in each of these inequalities is best possible. It is also known (see, for example, Lemma 1) that there is some constant C_n'', depending only on n, such that

$$M_B(Q_n) \leqslant C_n'' |D|^{1/2} \tag{10}$$

for all indefinite forms Q_n with determinant $D \neq 0$. For forms in 4 or more variables, the signature assumes importance and I conjecture that

$$\sup M_B(Q_{n,s}) = 4^{1/n} \sup M_I(Q_{n,s}), \qquad n = 2, 3, \ldots, \tag{11}$$

where both bounds are over all forms $Q_{n,s}$ with fixed signature s and fixed determinant $D \neq 0$. In support of this conjecture we have (6), (7) and (8), (9) which settle it for $n = 2, 3$. In this paper, I prove that for all forms of signature 0,

$$M_B(Q_{2m,0}) \leqslant |D|^{1/2m}, \qquad n = 2m. \tag{12}$$

*For these classical results, see, for example, Koksma, *Diophantische Approximationen* (Chelsea), Kap. III, §§ 2, 4; VI, § 2. A short proof of Markoff's inequality for Q_3 is given by H. Davenport, J. London Math. Soc., *22* (1947), 96–99.

Since the equality signs in both (3) and (12) are essential for the special cases

$$(13)\qquad \sum_{i=1}^{m-1} x_{2i-1}x_{2i} + 2x_{2m-1}x_{2m},$$

$$x_1 \equiv \ldots \equiv x_{2m-2} \equiv 0, x_{2m-1} \equiv x_{2m} \equiv \tfrac{1}{2} \pmod 1$$

and

$$(14)\qquad \sum_{i=1}^{m-1} (x_{2i-1}^2 - x_{2i}^2) + 2x_{2m-1}x_{2m},$$

respectively, the conjecture is thus established for the case $s = 0$. In the course of the proof of (12), I also prove that

$$(15)\qquad M_B(Q_n) = 0$$

for any form Q_n in at least 3 variables which represents arbitrarily small non-zero values (Theorem 1). Having established this, the proof of (12) for $2m \geqslant 4$ may be conveniently divided into two cases:

Case I: $M_H(Q_{2m,0}) > 0$.

Case II: $Q_{2m,0}$ represents 0 with $\mathbf{x} \neq \mathbf{o}$, but does not represent arbitrarily small non-zero values (from the work of Oppenheim **(7)** we know that such forms have commensurable coefficients). For convenience, we shall state our main result in a different way. Clearly, (12) is an immediate consequence of the following theorem.

THEOREM. *If Q_{2m} is any quadratic form of signature* 0 *and determinant $D \neq 0$, it is equivalent, by an integral unimodular substitution, to a form with coefficients a_{ij}, say, which satisfy*

$$|a_{ii}| \leqslant |D|^{1/2m}, \qquad i = 1, 2, \ldots, 2m.$$

The proof in Case I (see Theorem 2) depends on a reduction* of Q_{2m} used by Birch, the relevant details of which are assembled in Lemma 3. A similar sort of reduction is available in Case II (see Lemma 4 and Theorem 3).

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2. The critical case (14). Suppose, if possible, that the form Q in (14) satisfies $M_B(Q) < |D|^{1/2m} = 1$; we shall deduce a contradiction. Since Q has integral coefficients, we see that there are integral vectors $\mathbf{x}_r = (x_1^{(r)}, \ldots, x_n^{(r)})$. $r = 1, 2, \ldots, 2m$, with determinant ± 1 for which $Q(\mathbf{x}_r) = 0, r = 1, 2, \ldots, 2m$, But since $x^2 \equiv x \pmod 2$,

*The reduction theory developed by Birch in his work on the inhomogeneous problem is the foundation for my proof of (12) and I have borrowed freely from his paper **(1)** to avoid tedious repetition.

†Further properties of forms satisfying the hypotheses of Theorem 1 are contained in **(9)**.

$$Q(\mathbf{x}_r) \equiv x_1^{(r)} + \ldots + x_{2m-3}^{(r)} \pmod 2,$$

whence

$$x_1^{(r)} + \ldots + x_{2m-3}^{(r)} \equiv 0 \pmod 2, \qquad r = 1, 2, \ldots, 2m.$$

Clearly, then, $\det(\mathbf{x}_1, \ldots, \mathbf{x}_{2m}) \equiv 0 \pmod 2$, a contradiction. Hence $M_B(Q) \geqslant |D|^{1/2m}$, and the equality sign in (12) is necessary.

3. For the proof of (15), or (26) in Theorem 1, we use the following reduction of Q_n.

LEMMA 1. *For $n \geqslant 2$, let Q_n be an indefinite quadratic form of determinant $D \neq 0$. Then Q_n is equivalent to a form whose coefficients a_{ij}, say, satisfy*

(16) $$|a_{ij}| \ll |D|^{1/n}, \qquad i, j = 1, 2, \ldots, n,$$

and

(17) $$a_{11} > 0, \ldots, a_{nn} > 0,$$

(18) $$a_{ii} \gg |D|^{1/n} \qquad (i = 1, 2, \ldots, n).$$

The constant implied by the Vinogradov symbol $\ll$ depends only on n. A proof of the reduction of Q_n to one satisfying (16) has been given recently by Watson (**8**, Theorem 1), in the course of which he shows that Q_n is equivalent to

(19) $$\sum_{i=1}^{n} a_i(x_i + l_i)^2, \qquad a_i \neq 0,$$

where l_i is a linear form in $x_j (j > i)$, l_n is identically zero, and where

(20) $$|a_i| \ll |D|^{1/n} \ll |a_i|, \qquad a_{n-1} < 0,$$

(21) $$a_n > 0.$$

Since the coefficients of l_i may be taken to lie between $\pm\frac{1}{2}$, this is sufficient for (16). Starting from this point, our proof is confined to further reductions of Q_n which can be made to obtain (17), (18) without disturbing (16).

Proof. The first step is a preliminary transformation to change the form into one for which

(22) $$a_{11} > 0, \ |D|^{1/n} \ll a_{11} \ll |D|^{1/n}, \ a_{ij} \ll |D|^{1/n} \ (i, j = 1, 2, \ldots, n).$$

Let

(23) $$t_n = [a_n^{-1/2} (|a| + \ldots + |a_{n-1}|)^{1/2} + 1]$$

and choose integers t_r $(r = n-1, \ldots, 1)$ successively with

(24) $$|t_r + l_r(t_{r+1}, \ldots, t_n)| \leqslant \tfrac{1}{2}.$$

By (20), we see that

(25) $t_r \ll 1 \quad (r = 1, \ldots, n), \; t_n^2 \geqslant \max\{1, a_n^{-1}(|a_1| + \ldots + |a_{n-1}|)\}.$

Then

$$a_n t_n^2 + \tfrac{1}{4}(|a_1| + \ldots + |a_{n-1}|) \geqslant Q_n(t_1, \ldots, t_n) \geqslant a_n t_n^2 - \tfrac{1}{4}(|a_1| + \ldots + |a_{n-1}|) > 0;$$

hence, by (25) and (20),

$$|D|^{1/n} \gg Q_n(t_1, \ldots, t_n) \gg |D|^{1/n}.$$

Let $\delta = \text{g.c.d.}(t_1, \ldots, t_n)$ and put $\delta x_r^* = t_r$, then

$$\text{g.c.d.}(x_1^*, \ldots, x_n^*) = 1$$

and $1 \leqslant t_n = \delta x_n^* \ll 1$, whence $\delta \ll 1$. Since

$$Q_n(t_1, \ldots, t_n) = \delta^2 Q_n(x_1^*, \ldots, x_n^*),$$

we also have

$$|D|^{1/n} \gg Q_n(x_1^*, \ldots, x_n^*) \gg |D|^{1/n}, \qquad Q_n(x_1^*, \ldots, x_n^*) > 0.$$

We now form an integral unimodular matrix X^* with $(x_1^*, \ldots, x_n^*)$ as the first column, this being possible since $\text{g.c.d.}(x_1^*, \ldots, x_n^*) = 1$. Moreover, $t_r \ll 1$ implies $x_r^* \ll 1$ and so we can complete X^* with elements $\ll 1$. Applying the substitution $\mathbf{x} = X^* \mathbf{x}'$ to $Q_n(\mathbf{x})$ the coefficient of $x_1'^2$ in the new form is equal to $Q_n(x_1^*, \ldots, x_n^*)$. Hence we can assume that Q_n satisfies (22). Now, with a_{11} fixed, it is possible to modify a_{jj} $(j \neq 1)$, if necessary, by a substitution of the type $x_1 = x_1' + n_j x_j$ without violating (22) or affecting the coefficient of x_k^2 when $k \neq j$. The new coefficient a'_{jj} of x_j^2 is given by

$$\begin{aligned} a'_{jj} &= a_{11} n_j^2 + 2a_{1j} n_j + a_{jj} \\ &= a_{11}^{-1}\{(a_{11}n_j + a_{1j})^2 + (a_{11}a_{jj} - a_{1j}^2)\}. \end{aligned}$$

We select n_j as the integer determined by

$$n_j = a_{11}^{-1}\,|a_{1j}^2 - a_{11}a_{jj}|^{1/2} - a_{1j}a_{11}^{-1} + \theta_j, \qquad 1 < \theta_j \leqslant 2.$$

Then $n_j \ll 1$ and so the conditions in (22) are maintained. Moreover, we have

$$a'_{jj} = \begin{cases} 2\theta_j|a_{1j}^2 - a_{11}a_{jj}|^{1/2} + a_{11}\theta_j^2, & \text{if } a_{11}a_{jj} - a_{1j}^2 \leqslant 0, \\ 2\theta_j|a_{1j}^2 - a_{11}a_{jj}|^{1/2} + a_{11}\theta_j^2 + 2(a_{11}a_{jj} - a_{1j}^2)a_{11}^{-1} & \text{if } a_{11}a_{jj} - a_{1j}^2 > 0 \end{cases}$$

and so, in either case,

$$|D|^{1/n} \ll a'_{jj} \ll |D|^{1/n}, \qquad a'_{jj} > 0.$$

Applying this in turn, we secure the remaining conditions in (17) and (18).

THEOREM 1. *For $n \geqslant 3$, let Q_n be an indefinite quadratic form of determinant $D \neq 0$, which represents arbitrarily small non-zero values. Then, for any $\epsilon > 0$, Q_n is equivalent to a form with coefficients a_{ij}, say, where*

(26) $$|a_{ii}| < \epsilon \qquad (i = 1, 2, \ldots, n).$$

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Proof. By considering $-Q_n$ in place of Q_n, if necessary, we may suppose that the signature $s(Q_n)$ is non-negative. By Oppenheim's work **(6)**, we know that an indefinite form in at least 3 variables, which assumes arbitrarily small values, does so with *both* signs. Let δ be any positive number $<\epsilon$. Then, we can suppose that, after a suitable integral unimodular substitution,

$$Q_n = a(x_1 + l_1)^2 - Q_{n-1}(x_2, \ldots, x_n),$$

where

$$0 < a < \delta \tag{27}$$

and $s(Q_n) \geqslant 0$. Observe that l_1 is a linear form in $x_2, \ldots, x_n$ and that Q_{n-1} is a quadratic form with determinant $-D/a \neq 0$. Since Q_{n-1} is non-singular, the conditions $s(Q_n) \geqslant 0$ and $a > 0$ together imply that Q_{n-1} is indefinite. Thus, Lemma 1 may be applied to

$$Q_{n-1} = \sum_{i,j=2}^{n} b_{ij} x_i x_j,$$

say, and we can suppose (after a suitable transformation) that, in particular,

$$0 < b_{ii} \ll |D/a|^{1/n-1} \qquad (i = 2, \ldots, n).$$

Putting $x_i = 1$, $x_j = 0$ if $j \geqslant 2$, $j \neq i$, Q_n reduces to $Q_n^{(i)}$, say, where

$$Q_n^{(i)} = a(x_1 + \alpha_i)^2 - b_{ii} \tag{28}$$

and

$$0 < ab_{ii} \ll a|D/a|^{1/n-1} \ll a^{1/2}\, |D|^{1/n-1}, \tag{29}$$

since $n \geqslant 3$. Selecting $x_1 = x_{1i}$ $(i \geqslant 2)$ to be an integer for which

$$|x_{1i} + \alpha_i - a^{-1/2} b_{ii}^{1/2}| \leqslant \tfrac{1}{2},$$

we have

$$\begin{aligned} Q_n^{(i)} &\ll a + (ab_{ii})^{1/2} \\ &\ll \delta + |D|^{1/2n-2}\, \delta^{1/4}, \end{aligned}$$

by (29) and (27). Thus with δ chosen sufficiently small, initially, we can ensure that $|Q_n^{(i)}| < \epsilon$ $(i = 2, \ldots, n)$. Since the set

$$(1, 0, \ldots, 0),\ (x_{12}, 1, 0, \ldots, 0), \ldots,\ (x_{1n}, 0, \ldots, 0, 1)$$

has determinant 1, Q_n can be transformed into a form whose diagonal elements are $a, Q_n^{(2)}, \ldots, Q_n^{(n)}$ and the conclusion follows.

4. For the proof of Cases I and II, we recall the results of Birch on the reduction of Q_{2m} (Lemmas 3, 4), together with an estimate for the minimum of a binary quadratic polynomial (Lemma 2).

LEMMA 2. *Let ϕ be an indefinite binary form of determinant $-d$. Then, for any x^*, y^* and any μ, there are $(x, y) \equiv (x^*, y^*)$ (mod 1) such that*

$$|\phi(x, y) + \mu| \leqslant \max\{2^{-1/2}\, d^{1/2},\ d^{1/4}\, |\mu|^{1/2}\}. \tag{30}$$

Proof. See Birch **(1**, Lemma 4**)**.

LEMMA 3. *Let Q_{2m} be a quadratic form in at least 4 variables of determinant $D \neq 0$ with $s(Q_{2m}) = 0$ and with $|Q_{2m}|$ bounded below.* Then Q_{2m} is equivalent to*

$$(31) \qquad \psi(x_1 + a_{12}x_2 + \ldots, x_2 + \ldots) + Q_{2m-2}(x_3, \ldots, x_{2m}),$$

where ψ is an indefinite binary quadratic form of determinant $-d$, say, where

$$(32) \qquad 0 < d \leqslant (\tfrac{5}{6})^{m-1} |D|^{1/m}$$

and $|Q_{2m-2}|$ is bounded below.

Proof. See Birch **(1)**; this follows from his Lemmas 9, 10, and 11.

LEMMA 4. *For $m \geqslant 1$, let Q_{2m} be a rational quadratic form with determinant $D \neq 0$ and signature 0, that represents 0 non-trivially. Then it can be expressed, equivalently, as*

$$(33) \qquad Q_{2m} = \psi(x_1 + a_{12}x_2 + \ldots, x_2 + \ldots) + Q_{2m-2}(x_3, \ldots, x_{2m}),$$

where either

$$(34) \qquad \psi = 2a(x_1 + \ldots)x_2 \quad \text{and} \quad 0 < a \leqslant |D|^{1/2m}$$

or

$$(35) \qquad d(\psi) < |D|^{1/m} \quad \text{and} \quad m \geqslant 2.$$

Proof. See Birch **(1**, Lemma 12**)**. This result is not stated explicitly, although it is an easy deduction from Lemma 12 and the argument of the Corollary.

5. Case I.

THEOREM 2. *For $m \geqslant 1$, let Q_{2m} be a quadratic form in $2m$ variables of determinant $D \neq 0$ with signature 0 and with $|Q_{2m}|$ bounded below. Then Q_{2m} is equivalent to a form which satisfies*

$$(36) \qquad |a_{ii}| \leqslant (\tfrac{5}{6})^{\nu_m} |D|^{1/2m} \quad (i = 1, 2, \ldots, 2m),$$

where

$$(37) \qquad \nu_m = \begin{cases} \tfrac{1}{2}(m-3) + 2^{-(m-1)} & \text{for } m \geqslant 3, \\ 0 & \text{for } m = 1, 2. \end{cases}$$

Remarks. In the proof, we put $\lambda_m = (\tfrac{5}{6})^{\nu_m}$ and use the relations

$$(38) \qquad \lambda_1 = \lambda_2 = 1 \quad \text{and} \quad \lambda_m^2 = (\tfrac{5}{6})^{\frac{1}{2}(m-2)} \lambda_{m-1} \ (m \geqslant 2),$$

which are easily verified for $m = 1, 2$, while for $m \geqslant 3$, we have

$$2\nu_m - \tfrac{1}{2}(m-2) = m - 3 + 2^{-(m-2)} - \tfrac{1}{2}(m-2) = \tfrac{1}{2}(m-4) + 2^{-(m-2)} = \nu_{m-1}.$$

Note also that

$$(39) \qquad \lambda_m = (\tfrac{5}{6})^{\frac{1}{2}(m-3)+2^{(-m+2)}} \geqslant (\tfrac{5}{6})^{\frac{1}{2}(m-1)} \quad \text{for } m \geqslant 2.$$

*I.e., $M_H(Q_{2m}) > 0$.

Proof. The case $m = 1$ is well known, having been established by Minkowski. For $m \geqslant 2$, we use Lemma 3 to reduce Q_{2m} to the form

$$\psi(x_1 + a_{12}x_2 + \ldots, x_2 + \ldots) + Q_{2m-2}(x_3, \ldots, x_{2m}),$$

where ψ is an indefinite binary quadratic form of determinant $-d$, say, satisfying

$$0 < d \leqslant (\tfrac{5}{6})^{m-1}|D|^{1/m} \tag{40}$$

and where $|Q_{2m-2}|$ is bounded below. Since Q_{2m-2} has signature 0 and determinant $-D/d \neq 0$, we may proceed by induction on m. Suppose then that the theorem holds for all such forms in $2m - 2$ variables; we shall deduct that it then holds for $2m$ variables. Thus, by a suitable reduction of Q_{2m-2} we may suppose that

$$|Q_{2m-2}^{(r)}| = |Q_{2m-2}(x_3{}^{(r)}, \ldots, x_{2m}{}^{(r)})| \leqslant \lambda_{m-1}(|D|/d)^{1/2m-2} \tag{41}$$

for $r = 3, \ldots, 2m$, where

$$x_s{}^{(r)} = \begin{cases} 1 & \text{if } r = s, \\ 0 & \text{if } r \neq s. \end{cases}$$

Now, for each $r \geqslant 3$, we select integers $x_1{}^{(r)}$, $x_2{}^{(r)}$ such that

$$|Q_{2m}(x_1{}^{(r)}, \ldots, x_{2m}{}^{(r)})| = |\psi(x_1{}^{(r)} + a_{12}x_2{}^{(r)} + \alpha_r, x_2{}^{(r)} + \beta_r) + Q_{2m-2}^{(r)}|,$$

say, is small. By Lemma 2, we can arrange that this does not exceed

$$\max\{(\tfrac{1}{2}d)^{1/2}, |Q_{2m-2}^{(r)}|^{1/2}d^{1/4}\} \leqslant \max\left\{\frac{1}{\sqrt{2}}(\tfrac{5}{6})^{(m-1)/2}|D|^{1/2m}, \lambda_{m-1}^{1/2}d^{(m-2)/4}|D|^{1/2m}\right\}$$
$$= \lambda_m|D|^{1/2m}$$

by (40), (41), and (38). Having chosen $(x_1{}^{(r)}, \ldots, x_{2m}{}^{(r)})$ for $r \geqslant 3$ with $x_r{}^{(r)} = 1$, $x_s{}^{(r)} = 0$ if $s \neq r$, $r \geqslant 3$, $s \geqslant 3$, it suffices to take

$$x_3{}^{(r)} = \ldots = x_{2m}{}^{(r)} = 0 \qquad \text{and} \qquad \begin{vmatrix} x_1{}^{(1)} & x_2{}^{(1)} \\ x_1{}^{(2)} & x_2{}^{(2)} \end{vmatrix} = \pm 1$$

for $r = 1, 2$. Then

$$Q_{2m}{}^{(r)} = \psi(x_1{}^{(r)} + a_{12}x_2{}^{(r)}, x_2{}^{(r)}), \qquad r = 1, 2,$$

and we appeal to the case $m = 1$ of the theorem to obtain

$$|Q_{2m}^{(r)}| \leqslant d^{1/2} \leqslant (\tfrac{5}{6})^{(m-1)/2}|D|^{1/2m} \leqslant \lambda_m|D|^{1/2m},$$

by (40) and (38). This completes the proof.

6. Case II.

THEOREM 3. *For $m \geqslant 1$, let Q_{2m} be a rational quadratic form of determinant*

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$D \neq 0$ with signature 0, *which represents* 0, *non-trivially. Then Q_{2m} is equivalent to a form which satisfies*

$$(42) \qquad |a_{ii}| \leqslant |D|^{1/2m} \qquad (i = 1, 2, \ldots, 2m).$$

Proof. By Lemma 4, we can reduce Q_{2m} to the form

$$Q_{2m} = \psi(x_1 + a_{12}x_2 + \ldots, x_2 + \ldots) + Q_{2m-2}(x_3, \ldots, x_{2m}),$$

where either

$$\text{(a)} \qquad \psi = 2a(x_1 + a_{12}x_2 + \ldots)x_2 \quad \text{and} \quad 0 < a \leqslant |D|^{1/2m}$$

or

$$\text{(b)} \qquad d(\psi) < |D|^{1/m} \quad \text{and } m \geqslant 2.$$

In case (a) we select

$$(x_1^{(r)}, \ldots, x_{2m}^{(r)}) = \begin{cases} (1, 0, \ldots, 0) & \text{if } r = 1, \\ (x_1^{(2)}, 1, 0, \ldots, 0) & \text{if } r = 2, \\ x_2^{(r)} = x_r^{(r)} = 1, x_s^{(r)} = 0 & \text{if } s \neq r,\ r \geqslant 3,\ s \geqslant 3. \end{cases}$$

For $r = 1$, $Q_{2m} = 0$ and for $r \geqslant 2$, Q_{2m} takes the value

$$2a(x_1^{(r)} + a_{12}) + \alpha^{(r)}, \text{ say.}$$

Then, by a suitable choice of $x_1^{(r)}$, we have

$$(43) \qquad |Q_{2m}| \leqslant a \leqslant |D|^{1/2m}.$$

To complete the proof, we proceed by induction on m. Suppose then that the theorem is true for $2m - 2$ variables, we shall deduce that it then holds for $2m$ variables. We know that it is true for $m = 1$ (Minkowski), so we may assume that $m \geqslant 2$. Since we have dealt with case (a), it suffices to consider case (b). Applying our inductive hypothesis to Q_{2m-2}, we can assume, after a suitable reduction, that $x_r^{(r)} = 1$, $x_s^{(r)} = 0$, $s \neq r$, $r \geqslant 3$, $s \geqslant 3$ gives

$$(44) \qquad |Q_{2m-2}^{(r)}| = |Q_{2m-2}(x_3^{(r)}, \ldots, x_{2m}^{(r)})| \leqslant |D/d|^{1/2m-2},$$

whenever Q_{2m-2} represents 0 non-trivially. However, by Theorems 1 and 2, we know that this holds, even if Q_{2m-2} does not represent 0. Hence, arguing as in Theorem 2 and using Lemma 2, we can choose $x_1^{(r)}$, $x_2^{(r)}$ $(r \geqslant 3)$ so that

$$\begin{aligned} |Q_{2m}(x_1^{(r)}, \ldots, x_{2m}^{(r)})| &= |\psi(x_1^{(r)} + a_{12}x_2^{(r)} + \alpha_r, x_2^{(r)} + \beta_r) + Q_{2m-2}^{(r)}|, \text{ say,} \\ &\leqslant \max\{(\tfrac{1}{2}d)^{1/2}, |Q_{2m-2}^{(r)}|^{1/2}d^{1/4}\}. \\ &\leqslant |D|^{1/2m} \qquad (r = 3, \ldots, 2m), \end{aligned}$$

by (b) and (44). Similarly, with $x_3^{(r)} = \ldots = x_{2m}^{(r)} = 0$ $(r = 1, 2)$ we have

$$|Q_{2m}^{(r)}| = |\psi(x_1^{(r)} + a_{12}x_2^{(r)}, x_2^{(r)})| \leqslant d^{1/2} \leqslant |D|^{1/2m},$$

on appealing to the known result for two variables. This completes the proof.

Having established Theorems 1, 2, and 3 the main theorem is also completed.

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