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Ordered bases, order-preserving automorphisms and bi-orderable link groups†

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Abstract

We give a new criterion which guarantees that a free group admits a bi-ordering that is invariant under a given automorphism. As an application, we show that the fundamental group of the "magic manifold" is bi-orderable, answering a question of Kin and Rolfsen.

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1. *Introduction*

This work was motivated by a study of the Artin action [2] of the braid groups B_n upon the free group F_n of rank n, as in [11]. In the course of our investigation, we introduce a new method which may be applied more generally to establish that an automorphism of a free group preserves some bi-ordering. From this, we develop new methods of determining when a link L in S^3, thought of as the closure of a braid together with the braid axis, yields a bi-orderable link group $\pi_1(S^3 \setminus L)$. Some definitions are in order.

If the elements of a group G can be given a strict total ordering $<$ with the property that $f < g$ implies $hf < hg$ for all $f, g, h \in G$, then we say $<$ is a *left-ordering* of G and that G is *left-orderable*. However if we also have $f < g$ implies $fh < gh$ for all $f, g, h \in G$, then $<$ is called a *bi-ordering* of G, and G is said to be *bi-orderable*. For example, torsion-free abelian groups and also nonabelian free groups are bi-orderable [12, 14]. Let $\varphi : G \to G$ be an automorphism. If there is a bi-ordering $<$ of G such that $f < g$ if and only if $\varphi(f) < \varphi(g)$ for all $f, g \in G$, we say that φ is *order-preserving*, that $<$ is *invariant* under φ, and that φ *preserves* $<$.

Our main theorem for showing that an automorphism of a free group $\varphi : F \to F$ is order-preserving is the following. We say that an orbit $\mathcal{O}$ of a bijection $\sigma : I \to I$ is finite (respectively infinite) if the cardinality $|\mathcal{O}|$ of $\mathcal{O}$ is finite (respectively infinite).

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THEOREM 3·6. *Let F be the free group generated by $\{x_i \mid i \in I\}$ where I is a nonempty set. Let $\varphi : F \to F$ be an automorphism satisfying*

$$\varphi(x_i) = w_i x_{\sigma(i)} w_i^{-1}, \forall i \in I,$$

where $w_i \in F$ and σ is a bijection on I. Assume that $\sigma(i_0) = i_0$ for some $i_0 \in I$ and let $h : F \to \mathbb{Z}$ be the homomorphism defined by $h(x_{i_0}) = 1$ and $h(x_j) = 0$ for all $j \neq i_0$. Assume that for each finite orbit $\mathcal{O}$ of σ we have

$$\gcd\left(|\mathcal{O}|, \sum_{i \in \mathcal{O}} h(w_i)\right) = 1.$$

Then there exists a bi-ordering of F invariant under φ.

One source of automorphisms of F_n having a formula as in Theorem 3·6 is the Artin action of B_n on F_n [2]. The braid group B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to the relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{ for } 1 \leq i \leq n-2$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for } |i-j| > 1.$$

There is an embedding of B_n into the automorphism group of the free froup F_n on generators $\{x_1, \ldots, x_n\}$, which yields the Artin action of B_n on F_n. The embedding sends each generator σ_i of B_n to the automorphism which acts on each generator x_j of F_n according to

$$x_j^{\sigma_i} = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } j = i \\ x_i & \text{if } j = i+1 \\ x_j & \text{if } j \neq i, i+1. \end{cases}$$

In [11, Section 2·4] the following was observed. Consider the link $br(\beta) = \hat{\beta} \cup A$ in S^3, consisting of the braid closure $\hat{\beta}$ together with the braid axis A. Then the Artin action of a braid $\beta \in B_n$ on F_n yields an automorphism that preserves a bi-ordering of F_n if and only if the fundamental group $\pi_1(S^3 \setminus br(\beta))$ of the complement of the link $br(\beta)$ is bi-orderable.

An unanswered question in [11] was whether the braid $\beta = \sigma_1^2 \sigma_2^{-1} \in B_3$ yields an automorphism that preserves a bi-ordering of F_3. This is equivalent to the question of whether the fundamental group of the so-called "magic manifold" is bi-orderable; the magic manifold is a 3-cusped hyperbolic 3-manifold conjectured to be of minimal volume, see Figure 1. We are able to answer this question in the affirmative using Theorem 3·6 (see Proposition 4·1). We also generate many new examples of order-preserving braids, and therefore many bi-orderable link groups whose bi-orderability could not be determined with previously known techniques (e.g. see [4, 5, 7–9, 11, 18]).

1·1. *Organisation of the paper*

In Section 2 we introduce generalisations of the "positive eigenvalue" condition of [15], and use it to show certain automorphisms of infinitely generated abelian groups preserve a bi-ordering. In Section 3 we bootstrap this result to prove our main theorem. In Section 4 we apply our main theorem to analyse the Artin action on F_n and show that the fundamental group of the magic manifold is bi-orderable.

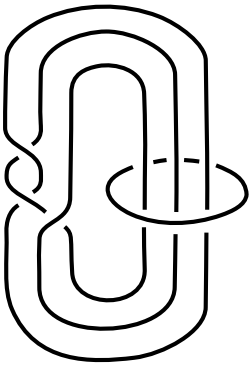

Fig. 1. The link $br(\beta)$ for $\beta = \sigma_1^2\sigma_2^{-1}$, whose complement is homeomorphic to the magic manifold.

2. *Ordered bases, modules and order-preserving homomorphisms*

Let $(S, <)$ be a set with a strict total ordering, and $f : S \to S$ a function. We say that f preserves the ordering $<$ if $a < b$ implies that $f(a) < f(b)$ for all $a, b \in S$, in this case we say that $<$ is invariant under f, and that the function f is order-preserving. Note that our definition of order-preserving is therefore context-sensitive: a bijection $f : S \to S$ from a set S to itself is order-preserving if there is a strict total ordering $<$ of S preserved by f, whereas an automorphism $\varphi : G \to G$ of a group is order-preserving if there is a bi-ordering $<$ of G preserved by φ.

In what follows, we let R be the ring $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, each equipped with its usual ordering. Given a free R-module M, an *ordered basis* for M is a pair $(\{v_i \mid i \in I\}, \prec)$ where $\{v_i \mid i \in I\}$ is a basis for the R-module M, and $\prec$ is a strict total ordering of the the index set I.

Definition 2·1. Given a free R-module M and an ordered basis $\mathcal{B} = (\{v_i \mid i \in I\}, \prec)$, the *lexicographic ordering of M associated with $\mathcal{B}$* is the bi-ordering $<_{\mathcal{B}}$ of the abelian group M defined as follows: Given $\alpha = \sum_i a_i v_i \in M \setminus \{0\}$, set $i(\alpha) = \min_{\prec}\{j \in I \mid a_j \neq 0\}$ and define $0 <_{\mathcal{B}} \alpha$ if only if $a_{i(\alpha)} > 0$.

Definition 2·2. Let M be a free R-module, $f : M \to M$ an R-module homomorphism, and $\mathcal{B} = (\{v_i \mid i \in I\}, \prec)$ an ordered basis. We say that f is *positively triangular with respect to $\mathcal{B}$* if there is an order-preserving map $\eta : I \to I$ with respect to $\prec$ such that for each $i \in I$,

$$f(v_i) = \lambda_i v_{\eta(i)} + \sum_{j \succ \eta(i)} c_{i,j} v_j,$$

where $\lambda_i, c_{i,j} \in R$ and $\lambda_i > 0$. We say that f is *positively triangular* if there exists an ordered basis $\mathcal{B}$ such that f is positively triangular with respect to $\mathcal{B}$.

Remark 2·3. Note that if M is a finite rank R-module, say with ordered basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ where the index set $\{1, \ldots, n\}$ is equipped with its natural ordering, then $f : M \to M$ is positively triangular with respect to $\mathcal{B}$ if and only if f is represented relative to $\mathcal{B}$ by an upper triangular matrix with positive entries on the diagonal.

The proof of the following result is routine and thus omitted.

LEMMA 2·4. *Let M be a free R-module, and $f : M \to M$ and R-module homomorphism. Assume that M is the direct sum of free sub-modules $\{M_j\}_{j\in J}$ and that $f(M_j) \subseteq M_j$ for all $j \in J$. Assume that for each $j \in J$, there is an ordered basis $\mathcal{B}_j = (\{v_{j,i} \mid i \in I_j\}, \prec_j)$ of M_j with respect to which $f|_{M_j}$ is positively triangular. Fix a total order $<$ on J and let $\mathcal{B} = (\{v_{j,i} \mid j \in J, i \in I_j\}, \prec)$, where $\prec$ is defined by $(j, i) \prec (j', i')$ if and only if $j < j'$ or $j = j'$ and $i \prec_j i'$. Then f is positively triangular with respect to $\mathcal{B}$.*

LEMMA 2·5. *Let M be a free R-module and $\mathcal{B} = (\{v_i \mid i \in I\}, \prec)$ an ordered basis for M. If $f : M \to M$ is an R-module homomorphism that is positively triangular with respect to $\mathcal{B}$, then f preserves the bi-ordering $<_{\mathcal{B}}$.*

Proof. Suppose f is positively triangular with respect to $\mathcal{B}$ and let $\prec$, η, λ_i and $c_{i,j}$ be as given in Definition 2·2. Recall that if $\alpha = \sum_i a_i v_i \in M \setminus \{0\}$ then $i(\alpha) := \min_{\prec}\{j \in I \mid a_j \neq 0\}$. To check that $<_{\mathcal{B}}$ is invariant under f, assume that $\alpha = \sum_i a_i v_i >_{\mathcal{B}} 0$ and we want to prove that $f(\alpha) >_{\mathcal{B}} 0$. We have:

$$\begin{aligned} f(\alpha) &= a_{i(\alpha)} f(v_{i(\alpha)}) + \sum_{j \succ i(\alpha)} a_j f(v_j) \\ &= a_{i(\alpha)} \left(\lambda_{i(\alpha)} v_{\eta(i(\alpha))} + \sum_{j \succ \eta(i(\alpha))} c_{i(\alpha),j} v_j \right) \\ &\qquad + \sum_{j \succ i(\alpha)} a_j \left(\lambda_j v_{\eta(j)} + \sum_{k \succ \eta(j)} c_{j,k} v_k \right) \\ &= a_{i(\alpha)} \lambda_{i(\alpha)} v_{\eta(i(\alpha))} + \sum_{k \succ \eta(i(\alpha))} c'_{\alpha,k} v_k. \end{aligned}$$

To reach the last equality, we have regrouped terms to arrive at new coefficients $c'_{\alpha,k}$ for $k \succ \eta(i(\alpha))$, where we used the fact that $k \succeq \eta(j)$ and $j \succ i(\alpha)$ implies $k \succ \eta(i(\alpha))$, as $\succ$ is invariant under η.

Since both $a_{i(\alpha)}$ and $\lambda_{i(\alpha)}$ are positive, we conclude that $f(\alpha) >_{\mathcal{B}} 0$, as desired.

Definition 2·6. Let $\mathcal{B} = (\{v_i \mid i \in I\}, \prec)$ and $\mathcal{C} = (\{w_j \mid j \in J\}, \lhd)$ be ordered bases of R-modules of M and N respectively. We define the *tensor* $\mathcal{B} \otimes \mathcal{C}$ to be the ordered basis $(\{v_i \otimes w_j \mid (i, j) \in I \times J\}, <)$ of R-module $M \otimes N$, where the total ordering $<$ on $I \times J$ is defined lexicographically: $(i, j) < (s, t)$ if $i \prec s$ or $i = s$ and $j \lhd t$.

LEMMA 2·7. *Let $f : M \to M$ and $g : N \to N$ be R-module homomorphisms. Assume that f and g are positively triangular with respect to ordered bases $\mathcal{B}$ and $\mathcal{C}$ respectively. Then the R-module homomorphism $f \otimes g : M \otimes N \to M \otimes N$ is positively triangular with respect to $\mathcal{B} \otimes \mathcal{C}$.*

Proof. Let $\mathcal{B} = (\{v_i \mid i \in I\}, \prec)$ and $\mathcal{C} = (\{w_j \mid j \in J\}, \lhd)$. For f, let $\eta : I \to I$ be as in Definition 2·2, and for g, we will use the function $\xi : J \to J$. Define the map $\sigma : I \times J \to I \times J$: $\sigma(i, j) = (\eta(i), \xi(j))$, which is clearly preserves the lexicographic ordering $<$ of $I \times J$.

Writing $u_{(i,j)}$ for $v_i \otimes w_j$, we have

$$\begin{aligned}(f \otimes g)(u_{(i,j)}) = (f \otimes g)(v_i \otimes w_j) &= f(v_i) \otimes g(w_j)\\ &= \left(\lambda_i v_{\eta(i)} + \sum_{s \succ \eta(i)} a_{i,s} v_s\right) \otimes \left(\mu_j w_{\xi(j)} + \sum_{t \rhd \xi(j)} b_{j,t} w_t\right)\\ &= \lambda_i \mu_j u_{(\eta(i),\xi(j))} + \sum_{t \rhd \xi(j)} \lambda_i b_{j,t} u_{(\eta(i),t)}\\ &\quad + \sum_{s \succ \eta(i)} a_{i,s} \mu_j u_{(s,\xi(j))}\\ &\quad + \sum_{s \succ \eta(i), t \rhd \xi(j)} a_{i,s} b_{j,t} u_{(s,t)}\\ &= \lambda_i \mu_j u_{\sigma(i,j)} + \sum_{(s,t) > \sigma(i,j)} c_{i,j,s,t} u_{(s,t)},\end{aligned}$$

for some $c_{i,j,s,t} \in R$. As $\lambda_i \mu_j > 0$, $f \otimes g$ is thus positively triangular with respect to $\mathcal{B} \otimes \mathcal{C}$.

3. *Order-preserving automorphisms of free groups*

We now aim to build upon the results of the last section, bootstrapping to yield order-preserving automorphisms of free groups of arbitrary rank.

Our first lemma is a generalisation of [**15**, Lemma 4·5], wherein we confirm that the free group F in that proof need not be finitely generated. The proof remains largely unchanged, with two exceptions: (1) the summations appearing below over the variable j are sums of terms drawn from an infinite set (though each summation is still finite), and (2) the Equation (3·3) is demonstrated directly without referring to the Jacobian matrix as in [**15**]. Despite the changes being minor, we present the proof here for the sake of completeness, and so that the reader can confirm for themselves that the conclusion of the lemma is independent of the cardinality of the generating set of F.

Here is the setup for the lemma. Let F be a free group generated by $\{z_i \mid i \in J\}$, where J is not necessarily finite (unlike [**15**, Lemma 4·5]). Let F_k be the kth term of the lower central series of F, defined by $F_1 = F$ and $F_{k+1} = [F_k, F]$. Let H be the abelianisation $F/[F,F]$ of F, written additively. Let $\mathbb{Z}F$ be the group ring of F over $\mathbb{Z}$, and $\epsilon : \mathbb{Z}F \to \mathbb{Z}$ the homomorphism of rings defined by $\epsilon(\sum_{i=1}^n k_i g_i) = \sum_{i=1}^n k_i$ for $k_i \in \mathbb{Z}$ and $g_i \in F$. Set $I = \ker(\epsilon)$, which is an ideal of the ring $\mathbb{Z}F$. For $g \in F$ and $k \in \{1, 2, \dots\}$, we have $g \in F_k$ if and only if $g - 1 \in I^k$ [**6**, first paragraph of p. 557]. Therefore, there is an injective homomorphism $\psi_k : F_k/F_{k+1} \to I^k/I^{k+1}$, sending $[g]$ to $[g-1]$, where the square brackets are to be interpreted as equivalence classes in the appropriate quotient. Note that a basis of I^k/I^{k+1} is[1]

$$\{[(z_{\ell_1} - 1) \cdots (z_{\ell_k} - 1)] \mid \ell_1, \dots, \ell_k \in J\}.$$

[1] This generates the group because of the Taylor expansion formula for the Fox derivation. To prove that it is $\mathbb{Z}$-independent, we use the the fact that $D_{i_1} \cdots D_{i_k}((z_{\ell_1} - 1) \cdots (z_{\ell_{k-1}} - 1)) \neq 0$ if and only if $(i_1, \dots, i_k) = (\ell_1, \dots, \ell_k)$, which can easily be proved by induction.

Hence we can define a $\mathbb{Z}$-module isomorphism $i_k : I^k/I^{k+1} \to H^{\otimes k}$ by

$$i_k([(z_{\ell_1} - 1) \cdots (z_{\ell_k} - 1)]) = a_{\ell_1} \otimes \cdots \otimes a_{\ell_k},$$

where $a_\ell := [z_\ell]$ is the image of z_ℓ in H. Moreover, there is an injective $\mathbb{Z}$-module homomorphism j_k from $H^{\otimes k}$ to $(H \otimes \mathbb{R})^k$, given by

$$j_k(a_1 \otimes \cdots \otimes a_k) = (a_1 \otimes 1) \otimes \cdots \otimes (a_k \otimes 1), \forall a_1, \ldots, a_k \in H.$$

LEMMA 3·1. *For every automorphism $\varphi : F \to F$ and every positive integer k, the following diagram commutes*:

$$\begin{array}{ccccccc} F_k/F_{k+1} & \xrightarrow{\psi_k} & I^k/I^{k+1} & \xrightarrow{i_k} & H^{\otimes k} & \xrightarrow{j_k} & (H \otimes \mathbb{R})^{\otimes k} \\ \downarrow{\scriptstyle \varphi_k} & & \downarrow{\scriptstyle \varphi'_k} & & \downarrow{\scriptstyle \varphi_{ab}^{\otimes k}} & & \downarrow{\scriptstyle (\varphi_{ab} \otimes Id)^{\otimes k}} \\ F_k/F_{k+1} & \xrightarrow{\psi_k} & I^k/I^{k+1} & \xrightarrow{i_k} & H^{\otimes k} & \xrightarrow{j_k} & (H \otimes \mathbb{R})^{\otimes k} \end{array}$$

where the vertical maps are induced from φ.

Proof. It is easy to check that the first rectangle and the last rectangle are commutative, because all the maps involved are canonical. Therefore we focus on commutativity of the middle rectangle. Let $\alpha := [(z_{\ell_1} - 1) \cdots (z_{\ell_k} - 1)] \in I^k/I^{k+1}$, it is enough to prove that $i_k(\varphi'_k(\alpha)) = \varphi_{ab}^{\otimes k}(i_k(\alpha))$ for all such α.

First, $\varphi'_k(\alpha) = [(\varphi(z_{\ell_1}) - 1) \cdots (\varphi(z_{\ell_k}) - 1)]$. Note that we have

$$\varphi(z_{\ell_s}) - 1 = \sum_\ell D_\ell^0(\varphi(z_{\ell_s}))(z_\ell - 1) + O(2),$$

where $D_\ell = \partial/\partial z_\ell$ is the derivation defined in [6], $D_\ell^0(w) = \epsilon(D_\ell(w))$, and $O(2)$ is a term in I^2. We compute

$$\begin{aligned} i_k(\varphi'_k(\alpha)) &= i_k\left(\left[\Big(\sum_\ell D_\ell^0(\varphi(z_{\ell_1}))(z_\ell - 1)\Big) \cdots \Big(\sum_\ell D_i^0(\varphi(z_{\ell_k}))(z_\ell - 1)\Big)\right]\right) \qquad (3{\cdot}1) \\ &= \Big(\sum_\ell D_\ell^0(\varphi(z_{\ell_1}))a_\ell\Big) \otimes \cdots \otimes \Big(\sum_\ell D_\ell^0(\varphi(z_{\ell_k}))a_\ell\Big) \\ &= \Big[\sum_\ell D_\ell^0(\varphi(z_{\ell_1}))z_\ell\Big] \otimes \cdots \otimes \Big[\sum_\ell D_\ell^0(\varphi(z_{\ell_k}))z_\ell\Big]. \end{aligned}$$

Second, we have

$$\begin{aligned} \varphi_{ab}^{\otimes k}(i_k(\alpha)) &= \varphi_{ab}^{\otimes k}\left(a_{\ell_1} \otimes \cdots \otimes a_{\ell_k}\right) \qquad (3{\cdot}2) \\ &= \varphi_{ab}(a_{\ell_1}) \otimes \cdots \otimes \varphi_{ab}(a_{\ell_k}) \\ &= [\varphi(z_{\ell_1})] \otimes \cdots \otimes [\varphi(z_{\ell_k})]. \end{aligned}$$

Comparing (3·1) and (3·2), we only need to prove that for $w \in F$, we have

$$[w] = \left[\sum_\ell D_\ell^0(w)z_\ell\right] \qquad (3{\cdot}3)$$

in H. But this is true, because $D^0_\ell(w)$ is the sum of the exponents of z_ℓ that appear in w, and recall that we write the operation in H additively.

We are now ready to prove the main result of this section, which by Remark 2·3, is a generalisation of [**15**, Theorem 2·6] from finitely generated free groups to free groups of arbitrary rank.

PROPOSITION 3·2. *Let F be a free group, $\varphi : F \to F$ an automorphism, set $H := F/[F,F]$, and let $\varphi_{ab} : H \to H$ be the automorphism induced by φ. If $\varphi_{ab} \otimes Id : H \otimes \mathbb{R} \to H \otimes \mathbb{R}$ is positively triangular, then there is a bi-ordering $<$ on F which is invariant under φ.*

Proof. Let k be an arbitrary positive integer. Let F_k be the kth term in the lower central series of F and $\varphi_k : F_k/F_{k+1} \to F_k/F_{k+1}$ the automorphism induced by φ. By Lemma 2·7, $(\varphi_{ab} \otimes Id)^{\otimes k} : (H \otimes \mathbb{R})^{\otimes k} \to (H \otimes \mathbb{R})^{\otimes k}$ is positively triangular. By Lemma 2·5, there is a bi-ordering $<'_k$ of $(H \otimes \mathbb{R})^{\otimes k}$ which is invariant under $(\varphi_{ab} \otimes Id)^{\otimes k}$.

By Lemma 3·1, the ordering $<'_k$ can be used to define a bi-ordering $<_k$ of F_k/F_{k+1} that is invariant under φ_k. The ordering is defined by declaring that for $a, b \in F_k/F_{k+1}$, we have $a <_k b$ if and only if $j_k(i_k(\psi_k(a))) <'_k j_k(i_k(\psi_k(b)))$.

Finally, we use these orderings $<_k$ $(k = 1, 2, \dots)$ to define a bi-ordering $<$ on F. It is known that $\cap_{k=1}^{\infty} F_k = \{1\}$, thus for $a \neq 1$ in F, there is a largest k, denoted as $k(a)$, such that $a \in F_k$. Then we define $a > 1$ if and only if $[a] >_{k(a)} 1$ in $F_{k(a)}/F_{k(a)+1}$, where $[a]$ indicates the equivalence class in the quotient $F_{k(a)}/F_{k(a)+1}$ which contains a. This defines a bi-ordering of F and it is invariant under φ by construction.

Proposition 3·2 tells us that for automorphisms φ of a free group F, a sufficient condition for φ to be order-preserving is that $\varphi_{ab} \otimes Id$ is positively triangular. However, this condition is not a necessary condition for φ to be order-preserving. We will see in Example 3·8 that there exist homomorphisms φ such that $\varphi_{ab} \otimes Id$ is not positively triangular, but φ is still order-preserving.

Remark 3·3. Let φ be an automorphism of a free group F and set $H = F/[F,F]$. If $\varphi_{ab} : H \to H$ is positively triangular, then $\varphi_{ab} \otimes Id : H \otimes \mathbb{R} \to H \otimes \mathbb{R}$ is positively triangular. However, the reverse is not true, and it is for this reason that we insist $\varphi_{ab} \otimes Id : H \otimes \mathbb{R} \to H \otimes \mathbb{R}$ be positively triangular in Proposition 3·2.

As an example of this behaviour, let F be the free group generated by x_1, x_2. Let φ be the automorphism of F given by

$$\varphi(x_1) = x_1^2 x_2, \text{ and } \varphi(x_2) = x_1^2 x_2 x_1 x_2.$$

Writing the abelian group $H = F/[F,F]$ additively and the cosets of x_1 and x_2 as $\overline{x_1}$ and $\overline{x_2}$ respectively, φ_{ab} sends $\overline{x_1}$ to $2\overline{x_1} + \overline{x_2}$ and $\overline{x_2}$ to $3\overline{x_1} + 2\overline{x_2}$ respectively. Then matrix of $\varphi_{ab} \otimes Id : H \otimes \mathbb{R} \to H \otimes \mathbb{R}$ is therefore $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, with eigenvalues $2 \pm \sqrt{3}$. Hence $\varphi_{ab} : H \to H$ is not positively triangular, whereas $\varphi_{ab} \otimes Id : H \otimes \mathbb{R} \to H \otimes \mathbb{R}$ is positively triangular, because the matrix A is diagonalisable over $\mathbb{R}$. Note the map φ is thus order-preserving by Proposition 3·2; we can explicitly define a bi-ordering on $H \otimes \mathbb{R}$ which is invariant under $\varphi_{ab} \otimes Id$: $\overline{x_1} \otimes a + \overline{x_2} \otimes b > 0$ if and only if $a + b\sqrt{3} > 0$ for $a, b \in \mathbb{R}$.

We can check that the construction of $<$ in the proof of Proposition 3·2 doesn't depend on the homomorphism φ, and instead depends only on the underlying choice of ordered basis.

In fact, the ordering is constructed as in our next definition, where (3·4) involves Definition 2·1, Definition 2·6 and the diagram in Lemma 3·1.

Definition 3·4. Let F be a free group and $\mathcal{B}$ be an ordered basis of $(F/[F,F]) \otimes \mathbb{R}$. Let $<_k$ denote the ordering of F_k/F_{k+1} defined by $a <_k b$ if and only if

$$j_k(i_k(\psi_k(a))) <_{\mathcal{B}^{\otimes k}} j_k(i_k(\psi_k(b))). \tag{3·4}$$

Let $<$ be the bi-ordering of F induced by the orderings $<_k$ as in the proof of Proposition 3·2. We call $<$ the *bi-ordering of* F *associated with* $\mathcal{B}$ and denote it by $<_{\mathcal{B}}$.

We can thus, upon reviewing the proof, give a version of Proposition 3·2 that makes explicit the invariant bi-ordering.

THEOREM 3·5. *Let F be a free group and $\mathcal{B}$ an ordered basis of $(F/[F,F]) \otimes \mathbb{R}$. Let $<_{\mathcal{B}}$ be the bi-ordering of F associated with $\mathcal{B}$. If $\varphi : F \to F$ is an automorphism such that*

$$\varphi_{ab} \otimes Id : (F/[F,F]) \otimes \mathbb{R} \to (F/[F,F]) \otimes \mathbb{R}$$

is positively triangular with respect to $\mathcal{B}$, then $<_{\mathcal{B}}$ is invariant under φ.

As an application of Theorem 3·5, we are now ready to prove Theorem 3·6, our main result from the introduction.

THEOREM 3·6. *Let F be the free group generated by $\{x_i \mid i \in I\}$ where I is a nonempty set. Suppose that $\varphi : F \to F$ is an automorphism satisfying*

$$\varphi(x_i) = w_i x_{\sigma(i)} w_i^{-1}, \forall i \in I,$$

where $w_i \in F$ and σ is a bijection on I. Assume that σ fixes $i_0 \in I$. Define homomorphism $h : F \to \mathbb{Z}$ by $h(x_{i_0}) = 1$ and $h(x_j) = 0$ for all $j \neq i_0$. Assume that for each finite orbit $\mathcal{O}$ of σ, we have

$$\gcd\left(|\mathcal{O}|, \sum_{i \in \mathcal{O}} h(w_i)\right) = 1.$$

Then there exists a bi-ordering of F invariant under φ.

Proof. Let K be the kernel of h. We see that $\varphi(K) = K$ and thus the restriction $\varphi|_K$ is an automorphism of K. Let $\psi : F \to F$ be the conjugation $\alpha \to x_{i_0} \alpha x_{i_0}^{-1}$. Similarly, we can verify that $\psi|_K$ is an automorphism on K. We now proceed in two steps.

First, we assume that there is a bi-ordering $<_K$ on K which is invariant under both $\varphi|_K$ and $\psi|_K$. Under this assumption, we now show that the lexicographic left-ordering on F coming from the short exact sequence

$$1 \to K \xrightarrow{i} F \xrightarrow{h} \mathbb{Z} \to \{0\}$$

is in fact a bi-ordering invariant under φ. Here, i is the inclusion of K in F. The positive cone of F is defined to be

$$P_F := i(P_K) \cup h^{-1}(\mathbb{Z}_{>0}) = P_K \cup \{\alpha \in F_n \mid h(\alpha) > 0\},$$

where P_K is the positive cone of K corresponding to $<_K$. To verify that it gives a bi-ordering, we only need to verify that $x_{i_0}^k P_K x_{i_0}^{-k} = P_K$ for all $k \in \mathbb{Z}$. This is true because $x_{i_0} P_K x_{i_0}^{-1} = \psi|_K(P_K) = P_K$ as $<_K$ is invariant under $\psi|_K$. To verify that it is invariant under φ, we want to show that $\varphi(P_F) \subseteq P_F$. This is true, because $\varphi(P_K) = \varphi|_K(P_K) = P_K$ using that $<_K$ is invariant under $\varphi|_K$; and if $\alpha \in F$ satisfies $h(\alpha) > 0$, then have $h(\varphi(\alpha)) = h(\alpha) > 0$.

So, to prove the theorem it suffices to prove that there is a bi-ordering on K which is invariant under both $\varphi|_K$ and $\psi|_K$, which is our second step in the proof.

It can be shown that K has a free basis:

$$\left\{ x_{i_0}^j x_i x_{i_0}^{-j} \mid i \in I\backslash\{i_0\}, j \in \mathbb{Z} \right\}.$$

For $\alpha \in K$, we write $\overline{\alpha}$ for the image of α in the quotient $H := K/[K,K]$, which is the abelianisation of K. Then H is a free abelian group and

$$\mathfrak{B} := \{A_{i,j} \mid i \in I\backslash\{i_0\}, j \in \mathbb{Z}\} \text{ with } A_{i,j} := \overline{x_{i_0}^j x_i x_{i_0}^{-j}} \tag{3·5}$$

is a basis of H.

We now define two bijections on $\eta, \zeta : (I\backslash\{i_0\}) \times \mathbb{Z} \to (I\backslash\{i_0\}) \times \mathbb{Z}$ by

$$\zeta(i,j) = (i, j+1), \quad \eta(i,j) = (\sigma(i), j + h(w_i)), \quad \forall i \in I\backslash\{i_0\}, j \in \mathbb{Z}.$$

Claim: The abelianisations $(\psi|_K)_{ab}$ of $\psi|_K$ and $(\varphi|_K)_{ab}$ of $\varphi|_K$ satisfy

$$(\psi|_K)_{ab}(A_{i,j}) = A_{i,j+1} = A_{\zeta(i,j)}, \quad (\varphi|_K)_{ab}(A_{i,j}) = A_{\sigma(i),j+h(w_i)} = A_{\eta(i,j)} \tag{3·6}$$

for all $i \in I\backslash\{i_0\}$ and $j \in \mathbb{Z}$. (We will usually omit parentheses in the subscripts and identify $A_{i,j}$ with $A_{(i,j)}$.)

Proof of claim. The first equality follows easily by definition, so we focus on the second. Note that we have $\overline{x_{i_0}^r} A_{i,j} \overline{x_{i_0}^{-r}} = A_{i,j+r}$ and that for $k \neq i_0$ we have $\overline{x_k} = A_{k,0}$ and therefore $\overline{x_k^r} A_{i,j} \overline{x_k^{-r}} = A_{i,j}$ since H is abelian. Therefore for $\alpha \in F$, we have $\overline{\alpha} A_{i,j} \overline{\alpha^{-1}} = A_{i,j+h(\alpha)}$. Now for the abelianisation $(\varphi|_K)_{ab}$ of $\varphi|_K$ on H, $i \in I\backslash\{i_0\}$ and $j \in \mathbb{Z}$, we have

$$\begin{aligned}
(\varphi|_K)_{ab}(A_{i,j}) &= \overline{\varphi(x_{i_0}^j x_i x_{i_0}^{-j})} \\
&= \overline{(w_{i_0} x_{i_0} w_{i_0}^{-1})^j w_i} A_{\sigma(i),0} \overline{w_i^{-1} (w_{i_0} x_{i_0} w_{i_0}^{-1})^{-j}} \\
&= A_{\sigma(i),j+h(w_i)} = A_{\eta(i,j)}.
\end{aligned}$$

By Theorem 3·5 and the second sentence of Remark 3·3, we only need to prove that there is an ordered basis of H, such that with respect to this ordered basis, both $(\varphi|_K)_{ab}$ and $(\psi|_K)_{ab}$ are positively triangular. Let $\mathbb{O}(\sigma)$ be the set of orbits of σ which are not equal to $\{i_0\}$. For each $\mathcal{O} \in \mathbb{O}(\sigma)$, we let $H_{\mathcal{O}}$ be the subgroup of H generated by $\mathfrak{B}_{\mathcal{O}} := \{A_{i,j} \mid i \in \mathcal{O}, j \in \mathbb{Z}\}$. By (3·6), $(\varphi|_K)_{ab}(H_{\mathcal{O}}) \subset H_{\mathcal{O}}$ and $(\psi|_K)_{ab}(H_{\mathcal{O}}) \subset H_{\mathcal{O}}$. Hence by Lemma 2·4, we only need to find an ordered basis of $H_{\mathcal{O}}$ with respect to which $(\varphi|_K)_{ab}$ and $(\psi|_K)_{ab}$ (when restricted on $H_{\mathcal{O}}$) are positively triangular. By (3·6) again, to do this we only need to find a total order on $\mathcal{O} \times \mathbb{Z}$ which is preserved by both ζ and η, for each $\mathcal{O} \in \mathbb{O}(\sigma)$.

First, we consider infinite orbits. Fix an infinite orbit $\mathcal{O}$ of σ. We define a total order $<$ on $\mathcal{O} \times \mathbb{Z}$: $(i,j) < (i',j')$ if and only if $i' \in \{\sigma^k(i) \mid k \geq 1\}$ or $(i',j') \in \{(i, j+k) \mid k \geq 1\}$. This order is easily seen to be preserved by ζ and η, as desired.

Second, we consider finite orbits. Fix a finite orbit $\mathcal{O}$ which is not equal to the orbit $\{i_0\}$. We also fix a tuple $(k_1, \ldots, k_r)$ such that $\mathcal{O} = \{k_1, \ldots, k_r\}$ with $r = |\mathcal{O}|$, $\sigma(k_i) = k_{i+1}$ for $1 \leq i \leq r-1$ and $\sigma(k_r) = k_1$. Let $h_{\mathcal{O}} = \sum_{i=1}^{r} h(w_{k_i})$, then by assumption $\gcd(h_{\mathcal{O}}, r) = 1$. Let $y_i = \sum_{1 \leq j \leq i-1} -h(w_{k_j})$ for $i = 1, 2, \ldots, r+1$. (Then we have, for example, $y_1 = 0$, $y_{r+1} = -h_{\mathcal{O}}$ and $y_{i+1} = y_i - h(w_{k_i})$ for $1 \leq i \leq r$.) We now proceed by the following two sub-steps:

(1) We define a map $W : \mathcal{O} \times \mathbb{Z} \to \mathbb{Z}$ by

$$W(k_i, j) = r(y_i + j) + ih_{\mathcal{O}}, \forall (k_i, j) \in \mathcal{O} \times \mathbb{Z}.$$

Since $\gcd(h_{\mathcal{O}}, r) = 1$, given $t \in \mathbb{Z}$, there is a unique $i \in \{1, 2, \ldots, r\}$ and a unique $j \in \mathbb{Z}$ such that $t = r(y_i + j) + ih_{\mathcal{O}}$. Therefore the map W is a bijection from $\mathcal{O} \times \mathbb{Z}$ to $\mathbb{Z}$. So we define a total order $<$ on $\mathcal{O} \times \mathbb{Z}$ by: $(k_i, j) < (k_{i'}, j')$ if and only if $W(k_i, j) < W(k_{i'}, j')$.

(2) Now we only need to verify that this order $<$ is preserved by ζ and η. This is true because we have, for every $(k_i, j) \in \mathcal{O} \times \mathbb{Z}$,

$$W(\zeta(k_i, j)) = W(k_i, j+1) = r(y_i + j + 1) + ih_{\mathcal{O}} = W(k_i, r) + r$$

and, letting $k_{r+1} = k_1$ and recalling that $y_{r+1} = h_{\mathcal{O}}$ (for the case that $i = r$),

$$\begin{aligned} W(\eta(k_i, j)) = W(k_{i+1}, j + h(w_{k_i})) &= r(y_{i+1} + j + h(w_{k_i})) + (i+1)h_{\mathcal{O}} \\ &= r(y_i + j) + ih_{\mathcal{O}} + h_{\mathcal{O}} \\ &= W(k_i, j) + h_{\mathcal{O}}. \end{aligned}$$

Theorem 3·6 says that if an automorphism as in the statement of the theorem satisfies the coprime condition: $\gcd(h_{\mathcal{O}}, |\mathcal{O}|) = 1$ for each finite orbit $\mathcal{O}$ of σ, then the automorphism preserves a bi-ordering of the free group. If we replace this coprime condition by the (weaker) non-vanishing condition: $h_{\mathcal{O}} \neq 0$ for each finite orbit of σ with $|\mathcal{O}| \geq 2$, we can still prove that the automorphism preserves a left-ordering.

The proof of this fact remains almost the same; except that we forget about the maps ψ, $(\psi|_K)_{ab}$ and ζ (which were used to prove the bi-orderabiliy of the ordering constructed) and changed the definition of the map W to the following.

Fix a finite orbit $\mathcal{O}$. Write $\mathcal{O} = \{k_1, \ldots, k_r\}$ with $r = |\mathcal{O}|$, $\sigma(k_i) = k_{i+1}$ for $1 \leq i \leq r-1$. Let $h_j = \sum_{1 \leq p \leq j-1} h(w_{k_p})$ for $j = 1, 2, \ldots, r+1$ and so $h_{\mathcal{O}} = h_{r+1}$. For every $(k_i, j) \in \mathcal{O} \times \mathbb{Z}$, there are unique $m, t \in \mathbb{Z}$ such that $j = mh_{\mathcal{O}} + h_i + t$ and $0 \leq t \leq |h_{\mathcal{O}}| - 1$, and now we define

$$W(k_i, j) = (mr + i)h_{\mathcal{O}} + t.$$

(We then verify that we still have $W(\eta(k_i, j)) = W(k_i, j) + h_{\mathcal{O}}$.)

Example 3·7. Consider the free group F with basis $\{x_1, x_2, x_3\}$ and $\varphi : F \to F$ given by $x_1 \mapsto x_3 x_2 x_3^{-1}$, $x_2 \mapsto x_3 x_1 x_3^{-1}$, $x_3 \mapsto x_3$. With $i_0 = 3$ we see $h_{\mathcal{O}} = 2$ for the orbit $\mathcal{O} = \{1, 2\}$, and so φ preserves a left-ordering, although it clearly does not preserve a bi-ordering.

Example 3·8. Theorem 3·6 gives us an example of automorphism φ on free group F which is order preserving but $\varphi_{ab} \otimes Id$ is not positively triangular on $F/[F, F] \otimes \mathbb{R}$. For instance,

let F be free with basis $\{x_1, x_2, x_3\}$ and let φ be the automorphism of F defined by

$$x_1 \mapsto x_3 x_2 x_3^{-1}, x_2 \mapsto x_1, x_3 \mapsto x_3.$$

Then φ is order-preserving by Theorem 3·6. However $\varphi_{ab} \otimes Id$ is not positively triangular on $F/[F,F] \otimes \mathbb{R}$.

To see this, suppose that $\varphi_{ab} \otimes Id : F/[F,F] \otimes \mathbb{R} \to F/[F,F] \otimes \mathbb{R}$ is positively triangular, then by Lemma 2·5 $\varphi_{ab} \otimes Id$ preserves a bi-ordering on $F/[F,F] \otimes \mathbb{R}$. But this is impossible as $\varphi_{ab} \otimes Id$ swaps the two elements $\overline{x_1} \otimes 1$ and $\overline{x_2} \otimes 1$.

4. *Order-preserving braids, bi-orderable link groups and* 3*-manifolds*

Recall from the introduction that the braid group B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$ subject to $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \le i \le n-2$, and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$; and that the Artin action of B_n on F_n is determined by the following action of the generators σ_i of B_n on the generators x_j of F_n:

$$x_j^{\sigma_i} = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } j = i \\ x_i & \text{if } j = i+1 \\ x_j & \text{if } j \neq i, i+1. \end{cases} \tag{3·6}$$

We will call a braid $\beta \in B_n$ order-preserving if its action on F_n induces an order-preserving automorphism $F \to F$.

Every braid has an associated permutation, which is its image under the homomorphism $B_n \to S_n$ given by sending σ_i to the transposition that exchanges i and $i+1$. We will often refer to the image of β under this map as "*the underlying permutation* of β." The kernel of this homomorphism is PB_n, the subgroup of pure braids.

As our convention is that the action of B_n on F_n is a right action (consistent with [**11**]), for this section our convention for composition of permutations is to do the leftmost first. For example, $\sigma\tau$ acts on elements first by σ. We therefore write the action of a permutation σ on an element i as exponentiation, i^σ.

4·1. *Minimal volume* 3*-manifolds*

In [**11**], the authors initiated a study of bi-orderability of the fundamental groups of minimal volume cusped hyperbolic 3-manifolds. For such manifolds, they determined whether or not the fundamental group is bi-orderable for all examples with five or fewer cusps, with one exception. The fundamental group of a three-cusped manifold, called the "magic manifold", could not be addressed by the techniques of [**11**].

PROPOSITION 4·1. *The fundamental group of the magic manifold is bi-orderable.*

Proof. By [**11**, Section 5·5], the fundamental group of the magic manifold is bi-orderable if and only if the braid $\sigma_1^2\sigma_2^{-1}$ is order-preserving. Let F_3 denote the free group on generators x_1, x_2, x_3. One can verify that $\sigma_1^2\sigma_2^{-1}$ induces a map $\varphi : F_3 \to F_3$ given by

$$\varphi(x_1) = x_1 x_3 x_1 x_3^{-1} x_1^{-1},\ \varphi(x_2) = x_1 x_3 x_1^{-1}\ \varphi(x_3) = x_3^{-1} x_2 x_3,$$

the underlying permutation σ is the transposition (2 3).

In order to apply Theorem 3·6, we define $h : F_3 \to \mathbb{Z}$ by $h(x_1) = 1, h(x_2) = 0, h(x_3) = 0$. Then observe that σ has a single orbit $\mathcal{O} = \{2, 3\}$ that does not contain 1, and that $h_{\mathcal{O}} = h(x_1) + h(x_3^{-1}) = 1$. Since $\gcd(2, 1) = 1$, the braid $\sigma_1^2\sigma_2^{-1}$ is order-preserving.

This resolves [**11**, Question 5·6] in the affirmative. Moreover, from the proof, we see that the braid $\beta = \sigma_1^2\sigma_2^{-1}$ is order-preserving. Note that multiplication of this by σ_2^2 does not change the underlying permutation and also does not alter the value of $h_{\mathcal{O}}$ in the above discussion. The same is true of multiplication by σ_2^{-2}. Therefore the braid $\sigma_1^2\sigma_2^{2n-1}$ is also order-preserving. On the other hand $\sigma_1^2\sigma_2^{2n}$, being a pure braid, is also order-preserving. Therefore we have the following.

THEOREM 4·2. *For every integer n, the braid $\sigma_1^2\sigma_2^n$ is order-preserving.*

We note that it was shown by Johnson, Scherich and Turner that the braids $\sigma_1\sigma_2^{2k+1}$ are not order-preserving for any integer k [**10**]. Such 3-braids have underlying permutation a single cycle. We do not know if there exists an order-preserving braid whose permutation is a single cycle.

4·2. *Dehn surgery and sublinks of bi-orderable links*

We can also improve upon [**11**, Theorem 5·12], which says that every link L in S^3 is a sublink of a link L' such that $\pi_1(S^3 \setminus L')$ is bi-orderable. We provide a construction of such an L' by adding only two components, and show that a similar result holds for all links in arbitrary 3-manifolds.

Recall that the pure braid group PB_n is generated by

$$A_{i,j} = \sigma_{j-1} \dots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1},$$

where $1 \le i < j \le n$ [**3**, Lemma 1·8·2]. For example, we have $A_{i,i+1} = \sigma_i^2$ for $1 \le i \le n-1$ and $A_{i,i+2} = \sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}$ for $1 \le i \le n-2$.

LEMMA 4·3. *Fix i, j with $1 \le i < j \le n$. Then we have the following*:

$$x_k^{A_{i,j}} = \begin{cases} (x_ix_j)x_i(x_j^{-1}x_i^{-1}) & \text{if } k = i \\ (x_ix_jx_i^{-1}x_j^{-1})x_k(x_jx_ix_j^{-1}x_i^{-1}) & \text{if } i+1 \le k \le j-1 \\ x_ix_jx_i^{-1} & \text{if } k = j \\ x_k & \text{otherwise.} \end{cases} \tag{3·6}$$

Proof. First, if $k \notin \{i, i+1, \dots, j\}$ then $x_k^{A_{i,j}} = x_k$ follows from the observation that $x_k^{\sigma_\ell} = x_k$ for all $\ell \in \{i, i+1, \dots, j-1\}$.

Next consider $x_i^{A_{i,j}}$. We compute

$$\begin{aligned} x_i^{A_{i,j}} &= x_i^{\sigma_i^2\sigma_{i+1}^{-1}\dots\sigma_{j-1}^{-1}} \\ &= (x_ix_{i+1}x_i^{-1}x_ix_ix_{i+1}^{-1}x_i^{-1})^{\sigma_{i+1}^{-1}\dots\sigma_{j-1}^{-1}} \\ &= (x_ix_j)x_i(x_j^{-1}x_i^{-1}). \end{aligned}$$

One similarly computes that $x_j^{A_{ij}} = x_ix_jx_i^{-1}$.

On the other hand, if $k \in \{i+1, \ldots, j-1\}$ then

$$\begin{aligned} x_k^{A_{i,j}} &= x_k^{\sigma_k \ldots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-1}^{-1}} \\ &= (x_k x_{k+1} x_k^{-1})^{\sigma_{k-1}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-1}^{-1}} \\ &= (x_i x_{k+1} x_i^{-1})^{\sigma_i\sigma_{i+1}^{-1}\ldots\sigma_{j-1}^{-1}} \\ &= ((x_i x_{i+1} x_i^{-1}) x_{k+1} (x_i x_{i+1}^{-1} x_i^{-1}))^{\sigma_{i+1}^{-1}\ldots\sigma_{j-1}^{-1}} \\ &= ((x_i x_k x_i^{-1}) x_{k+1} (x_i x_k^{-1} x_i^{-1}))^{\sigma_k^{-1}\ldots\sigma_{j-1}^{-1}} \\ &= ((x_i x_{k+1} x_i^{-1} x_{k+1}^{-1}) x_k (x_{k+1} x_i x_{k+1}^{-1} x_i^{-1}))^{\sigma_{k+1}^{-1}\ldots\sigma_{j-1}^{-1}} \\ &= (x_i x_j x_i^{-1} x_j^{-1}) x_k (x_j x_i x_j^{-1} x_i^{-1}). \end{aligned}$$

LEMMA 4·4. *Let $\beta \in B_n$ with underlying permutation σ and fix i with $1 \le i < n$. Suppose that β yields the automorphism $\phi : F_n \to F_n$ with $\phi(x_k) = w_k x_j w_k^{-1}$ where $j = k^\sigma$, and let $\psi : F_n \to F_n$ given by $\psi(x_k) = u_k x_j u_k^{-1}$ denote the automorphism of F_n induced by $\beta A_{i,n}$. Define homomorphism $h : F_n \to \mathbb{Z}$ by $h(x_n) = 1$ and $h(x_k) = 0$ for all $k = 1, \ldots, n-1$. For each cycle $c = (k_1, \ldots, k_r)$ in the cycle decomposition of σ, set $h_c = \sum_{i=1}^r h(w_{k_i})$ and $\ell_c = \sum_{i=1}^r h(u_{k_i})$. Then $\ell_c = h_c + 1$ if $i \in \{k_1, \ldots, k_r\}$ and $\ell_c = h_c$ otherwise.*

Proof. Given $k \in \{1, \ldots, n\}$ we write $j = k^\sigma$ and compute that $u_k x_j u_k^{-1} = \psi(x_k) = x_k^{\beta A_{i,n}} = (w_k x_j w_k^{-1})^{A_{i,n}}$. Thus we have

$$u_k = \begin{cases} w_k^{A_{i,n}} x_i x_n & \text{if } j = i \\ w_k^{A_{i,n}} (x_i x_n x_i^{-1} x_n^{-1}) & \text{if } j \in \{i+1, \ldots, n-1\} \\ w_k^{A_{i,n}} x_i & \text{if } j = n \\ w_k^{A_{i,n}} & \text{otherwise,} \end{cases} \tag{3·6}$$

by appealing to Lemma 4·3.

Note that by Lemma 4·3 we have $h(w_k^{A_{i,n}}) = h(w_k)$, and so we conclude that $h(u_k) = h(w_k) + 1$ if $k^\sigma = i$, and $h(u_k) = h(w_k)$ otherwise. The conclusion of the lemma follows immediately from this observation.

In the proposition below, for $g_1, \ldots, g_n \in G$ we use the notation $sg\{g_1, \ldots, g_n\}$ to denote the subsemigroup of G generated by $\{g_1, \ldots, g_n\}$.

PROPOSITION 4·5. *Suppose $n \ge 3$. For all $\beta \in B_{n-1}$ there exists $\alpha \in sg\{A_{1,n}, \ldots, A_{n-1,n}\}$ such that $\beta\alpha$ is order-preserving.*

Proof. Let σ denote the underlying permutation of the braid β, and observe that $n^\sigma = n$. Since every $\alpha \in sg\{A_{1,n}, \ldots, A_{n-1,n}\}$ is a pure braid, the underlying permutation of the braid $\beta\alpha$ is also σ and so satisfies $n^\sigma = n$. So, our task is to choose a braid α such that the relative primeness condition of Theorem 3·6 is satisfied by $\beta\alpha$.

For this, we inductively apply Lemma 4·4, right-multiplying β by braids $A_{i,n}$ as needed to guarantee that the relative primeness condition holds.

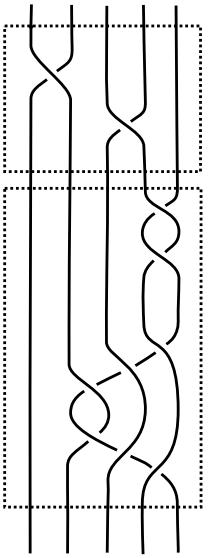

Fig. 2. The braid $\beta = \sigma_1\sigma_3$ in the dashed box on top, and the braid $\alpha = A_{4,5}A_{2,5}$ in the dashed box on the bottom.

Example 4·6. As an example of how to construct α as in Proposition 4·5, consider the braid $\beta = \sigma_1\sigma_3 \in B_5$ as in Figure 2. Since the generators σ_i are not order-preserving [**11**], it follows that β is not order-preserving by [**16**, Corollary 9]. In particular, the underlying permutation of β is (1 2)(3 4), and each cycle fails the coprime condition of Theorem 3·6 with $i_0 = 5$. Yet, as we see in Figure 2, we can multiply the braid β by $\alpha = A_{4,5}A_{2,5}$ in order to produce a braid in B_5 that is order-preserving. The factors of $A_{4,5}$ and $A_{2,5}$ guarantee that the comprime condition holds for the cycles (3 4) and (1 2) respectively.

In fact, the previous proposition holds for every braid $\beta \in B_n$ whose underlying permutation fixes n. However, the statement of Proposition 4·5 is sufficient for our next theorem.

THEOREM 4·7. *Every n-component link L in S^3 is a sublink of an $(n+2)$-component link L' in S^3 such that $\pi_1(S^3 \setminus L')$ is bi-orderable.*

Proof. By Alexander's Theorem [**1**], the n-component link L can be written as the closure of a braid $\beta \in B_k$ for some k. By Proposition 4·5, there is a braid $\alpha \in sg\{A_{1,k+1}, \ldots, A_{k,k+1}\} \subset B_{k+1}$ such that $\beta\alpha$ is order-preserving. The closure of $\beta\alpha$, together with the braid axis, gives a link L' having $n+2$ components such that $\pi_1(S^3 \setminus L')$ is bi-orderable.

COROLLARY 4·7. *Every link L in a compact, connected, closed orientable 3-manifold M is a sublink of a link L' in M such that $\pi_1(M \setminus L')$ is bi-orderable. Moreover, if n denotes the minimal number of surgery curves in in S^3 needed to produce M, then L' is obtained from L by adding $n+2$ components.*

Proof. By the Lickorish–Wallace theorem, M can be obtained by surgery on S^3 [**13, 17**], so there are n-component links $L_1 \subset M$ and $L_2 \subset S^3$ and a homeomorphism

$h : S^3 \setminus L_2 \to M \setminus L_1$. We may assume that L_1 and L are disjoint. By Theorem 4·7, there is a link $L_3 \subset S^3$ with $L_2 \cup h^{-1}(L) \subset L_3$ and $\pi_1(S^3 \setminus L_3)$ bi-orderable; moreover, L_3 has two more components than $L_2 \cup h^{-1}(L)$. Define $L' = h(L_3 \setminus L_2) \cup L_1$. Then $h(S^3 \setminus L_3) = M \setminus L'$, so $\pi_1(M \setminus L')$ is bi-orderable.

4·3. *Multiplication by braids in B_{n-1}*

We can also generalise the observations made following Proposition 4·1. There, we observed that it is possible to change a braid in B_3 which is *not* order preserving into an order-preserving braid by multiplying by powers of a single generator of B_3.

PROPOSITION 4·8. *Given $\beta \in B_3$, there exists k with $0 \le k \le 3$ such that $\beta\sigma_1^k$ is order-preserving.*

Proof. Let $\beta \in B_3$ be given. If β is pure then $k = 0$ suffices, and if the underlying permutation of β is (1 2), then $\beta\sigma_1$ is pure, and therefore $k = 1$ suffices.

On the other hand, suppose the underlying permutation of β is $\tau = (1\,3)$, and define $h : F_3 \to \mathbb{Z}$ by $h(x_1) = h(x_3) = 0$ and $h(x_2) = 1$. If β yields an automorphism $\phi : F_3 \to F_3$ given by

$$\phi(x_1) = w_1 x_3 w_1^{-1},\ \phi(x_2) = w_2 x_2 w_2^{-1},\ \phi(x_3) = w_3 x_1 w_3^{-1}$$

and $h(w_1) + h(w_3)$ is odd, then β is order-preserving by Theorem 3·6. On the other hand if $h(w_1) + h(w_3)$ is even, then consider the braid $\beta\sigma_1^2$, also with underlying permutation (1 3) which acts by the permutation

$$\psi(x_1) = w_1^{\sigma_1^2} x_3 (w_1^{-1})^{\sigma_1^2},\ \psi(x_2) = w_2^{\sigma_1^2} x_1 x_2 x_1^{-1} (w_2^{-1})^{\sigma_1^2},\ \psi(x_3) = w_3^{\sigma_1^2} x_1 x_2 x_1 x_2^{-1} x_1^{-1} (w_3^{-1})^{\sigma_1^2}.$$

We see that $h(w_1^{\sigma_1^2}) + h(w_3^{\sigma_1^2} x_1 x_2) = h(w_1) + h(w_3) + 1$ is odd, and so $\beta\sigma_1^2$ satisfies the coprime condition of Theorem 3·6 and so is order-preserving. When the underlying permutation is (2 3), we similarly prove that β or $\beta\sigma_1^2$ is order-preserving.

Last, if the underlying permutation of β is a 3-cycle, then the braid $\beta\sigma_1$ has underlying permutation a single transposition. Then $(\beta\sigma_1)\sigma_1^k$ is order-preserving by the previous paragraphs, for some k with $0 \le k \le 2$.

We can deal with $n \ge 4$ by a similar, general argument.

LEMMA 4·9. *Suppose that $\sigma \in S_n$ is a permutation with $n^\sigma \ne n$ where $n \ge 4$. Then there exists $\tau \in S_{n-1}$ such that $\sigma\tau$ has disjoint cycle decomposition $c_1 c_2$, where $c_1 = (i)$ for some $i \in \{1, \dots, n-2\}$ and c_2 is an $(n-1)$-cycle.*

Proof. To complete the proof, we need to find two disjoint cycles c_1 and c_2 with $c_1 = (i)$ for $1 \le i < n-2$ and c_2 a $(n-1)$-cycle in S_n, such that $\tau := \sigma^{-1} c_1 c_2$ fixes n. Assume that $n^{\sigma^{-1}} = j \ne n$. Arrange the elements of the set $\{1, 2, \dots, n\} \setminus \{j, n\}$ in any order $i_1, i_2, \dots, i_{n-2}$ with $i_{n-2} \ne n-1$. Now $c_1 = (i_{n-2})$ and $c_2 = (j\, n\, i_1\, i_2\, \dots\, i_{n-3})$ are the desired cycles.

PROPOSITION 4·10. *If $n \ge 3$ and $\beta \in B_n$, then there exists $\alpha \in B_{n-1}$ such that $\beta\alpha$ is order-preserving.*

Proof. The case $n = 3$ is given by Proposition 4·8, thus we assume $n \geq 4$. Suppose that β has underlying permutation σ and that $\sigma(n) = n$. Then we can choose $\alpha \in B_{n-1}$ such that the underlying permutation of $\beta\alpha$ is trivial. Therefore $\beta\alpha$ is a pure braid, and so is order-preserving.

Now consider the case that $\sigma(n) \neq n$. Choose $\tau \in S_{n-1}$ such that the cycle decomposition of $\sigma\tau$ is c_1c_2 where $c_1 = (i_0)$ with $i_0 < n-1$ and c_2 is an $(n-1)$-cycle, which we can do by Lemma 4·9. Choose a braid $\alpha \in B_{n-1}$ having underlying permutation τ.

Suppose that $\beta\alpha$ yields an automorphism $\phi : F_n \to F_n$ given by

$$\phi(x_i) = w_i x_j w_i^{-1},$$

where $j = i^{\sigma\tau}$. Define $h : F_n \to \mathbb{Z}$ by $h(x_{i_0}) = 1$ and $h(x_i) = 0$ for all $i \neq i_0$ and set $h_{c_2} = \sum_{i \neq i_0} h(w_i)$. If $\gcd(h_{c_2}, n-1) = 1$ then $\beta\alpha$ is order-preserving by Theorem 3·6.

If $\gcd(h_{c_2}, n-1) \neq 1$, then consider the product $\beta\alpha\sigma_{i_0}^2$. This braid gives rise to an automorphism $\psi : F_n \to F_n$ given by

$$\psi(x_i) = v_i x_j v_i^{-1},$$

where $j = i^{\sigma\tau}$, as the underlying permutation is unchanged. We compute:

$$(w_i x_j w_i^{-1})^{\sigma_{i_0}^2} = \begin{cases} (w_i)^{\sigma_{i_0}^2} x_{i_0} x_{i_0+1} x_{i_0}^{-1} x_{i_0+1}^{-1} (w_i^{-1})^{\sigma_{i_0}^2} & \text{if } j = i_0 \\ (w_i)^{\sigma_{i_0}^2} x_{i_0} x_{i_0+1} x_{i_0}^{-1} (w_i^{-1})^{\sigma_{i_0}^2} & \text{if } j = i_0 + 1 \\ (w_i)^{\sigma_{i_0}^2} x_j (w_i^{-1})^{\sigma_{i_0}^2} & \text{if } j \neq i_0, i_0+1. \end{cases} \tag{3·6}$$

Consequently $v_i = w_i^{\sigma_1^2}$ whenever $i^{\sigma\tau} \neq i_0, i_0+1$ and $v_i = w_i^{\sigma_{i_0}^2} x_{i_0}$ when $i^{\sigma\tau} = i_0 + 1$. Therefore we can compute the quantity $h'_{c_2} = \sum_{i \neq i_0}^{n} h(v_i)$ by observing that $h'_{c_2} = 1 + \sum_{i \neq i_0}^{n} h(w_i^{\sigma_1^2}) = 1 + h_{c_2}$, where the final equality follows from observing that $h(w_i^{\sigma_1^2}) = h(w_i)$ for all i. It follows that if $\gcd(h'_{c_2}, n-1) = 1$, then $\beta\alpha\sigma_1^2$ is order-preserving by Theorem 3·6.

If $\gcd(h'_{c_2}, n-1) \neq 1$, we may continue inductively to add powers of $\sigma_{i_0}^2$, until we arrive at a braid $\beta\alpha\sigma_{i_0}^{2\ell}$ which satisfies the coprime condition of Theorem 3·6, and is therefore order-preserving.

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